Approximation of Sobolev functions in Jordan domains

John L. Lewis

1. Introduction

Let $\Omega \subset \mathbf{C}$ be a bounded simply connected domain and let $L_p(\Omega)$, $1 \leqq p \leqq \infty$, be the usual space Lebesgue measurable functions f on Ω which are pth power integrable with norm denoted by $\|f\|_p$. For k a positive integer and $1 \leqq p \leqq \infty$ let $W_{k,p}(\Omega)$ denote the Sobolev space whose functional elements f and distributional partial derivations $D^\alpha f$, $|\alpha| \leqq k$, satisfy

$$\|f\|_{k,p} = \|f\|_p + \sum_{|\alpha| \leqq k} \|D^\alpha f\|_p < +\infty.$$

Let $\bar{E}$, ∂E, and E^c denote the closure, boundary and complement of the set E. In [9], P. Jones proved an extension theorem for (ε, δ) domains and showed that in two dimensions his theorem implied:

Theorem A. *Let k be a positive integer, $1 \leqq p \leqq \infty$, and suppose that Ω is a bounded simply connected domain. Then each function in $W_{k,p}(\Omega)$ extends to a function in $W_{k,p}(\mathbf{C})$ if and only if $\partial\Omega$ is a quasi-circle.*

Recall that a quasi-circle is the image of the unit circle under a quasiconformal mapping of $\mathbf{C}$ onto $\mathbf{C}$. In this note we shall be concerned with when the space of infinitely differentiable functions on $\mathbf{C}$, $C^\infty(\mathbf{C})$, is dense in $W_{k,p}(\Omega)$ for $k=1, 2, \ldots,$ and $1 \leqq p \leqq \infty$. If $\partial\Omega$ is a quasi-circle, it follows from Theorem A in a well known way that $C^\infty(\mathbf{C})$ is dense in $W_{k,p}(\Omega)$ for $k=1, 2, \ldots,$ and $1 \leqq p \leqq \infty$. On the other hand, there are simple examples which show there is a Jordan domain Ω (i.e., a domain bounded by a Jordan curve) and a function $f \in W_{1,p}(\Omega)$ which can be approximated arbitrarily closely by $C^\infty(\mathbf{C})$ functions for $1 \leqq p \leqq \infty$, but does not extend for all p. A standard example is the function $f(x, y) = y^{-1}$ in the domain bounded by the line $y=1$ and

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the curve

$$y = -1/[\ln(|x|)], \quad 0 < |x| \leqq 1/e.$$

Thus, extension and approximation problems are in general different. In this note we prove the following theorem which partially answers the question raised in [7, problem 8.2].

Theorem 1. *Let Ω be a Jordan domain. Then $C^\infty(\mathbf{C})$ is dense in $W_{1,p}(\Omega)$ for $1<p<\infty$.*

As for our proof of Theorem 1 we note that the case $p=2$ follows easily from conformal mapping. Our proof for $1<p<\infty$ is motivated by this case. To be more specific, let (Ω_n) be a sequence of bounded Jordan domains with

$$\bar{\Omega} \subseteqq \Omega_{n+1} \subseteqq \Omega_n, \quad n = 1, 2, \ldots,$$

and such that $\partial\Omega_n$ converges to $\partial\Omega$ in the sense of Hausdorff distance as $n\to\infty$. That is, given $\varepsilon>0$, there exists N such that each point of $\partial\Omega_n$ lies within ε of a point of $\partial\Omega$ for $n\geqq N$ and vice versa. Take, for example, Ω_n, $n=1, 2, \ldots,$ to be the domains bounded by certain levels of the Green's function for $\bar{\Omega}^c$ with pole at ∞. Given p, $1<p<\infty$, let $q=p/(p-1)$ be the conjugate exponent to p, and suppose that

$$\Delta(a, 4r) = \{z: |z-a| < 4r\} \subseteqq \Omega.$$

Consider the problem of minimizing the Euler functional,

$$\int_{\mathbf{C}} |\nabla\psi|^q \, dA$$

where $\nabla\psi$ denotes the gradient of ψ and the minimum is taken over all functions ψ in $\mathring{W}_{1,q}(\Omega_n)$ with $\psi\equiv 1$ on $\bar{\Delta}(a, r)$. Here dA denotes two dimensional Lebesgue measure and $\mathring{W}_{1,q}(\Omega_n)$ is the closure of the infinitely differentiable functions with compact support in Ω_n under the $W_{1,q}(\Omega_n)$ norm. It is well known that this functional attains its minimum at a unique $U_n\in\mathring{W}_{1,q}(\Omega_n)$ with $0\leqq U_n\leqq 1$, and $U_n\equiv 1$ on $\bar{\Delta}(a, r)$. Moreover, if $D_n=\Omega_n-\bar{\Delta}(a, r)$, then

$$(1.1) \qquad \int_{\mathbf{C}} |\nabla U_n|^{(q-2)} \nabla U_n \cdot \nabla\varphi \, dA = 0,$$

whenever $\varphi\in\mathring{W}_{1,q}(D_n)$. Observe from (1.1) and the divergence theorem that U_n is a weak solution to

$$(1.2) \qquad \nabla\cdot[|\nabla U_n|^{(q-2)} \nabla U_n] = 0$$

in D_n. In § 2 we show that given a compact set K in Ω there exists $N=N(K)$ a positive integer and $t=t(K)>0$ such that

$$(1.3) \qquad \{z: \; t/2 \leqq U_N(z) \leqq t\} \subseteqq \Omega - K.$$

Next in § 2 we show that

$$\text{(1.4)} \qquad \nabla U_n \neq 0 \quad \text{and} \quad U_n \text{ is real analytic in } D_n,$$

for $n=1, 2, \ldots$. From (1.4) and standard ordinary differential equation theory (see [3, Ch. 5]) or by constructing a "conjugate" to U_n as in § 3, it follows that for given $z_0 \in D_n$ there exists an open real analytic arc J_0 containing z_0 with ∇U_n tangent to J_0. The theory also guarantees that if J_1 is another such arc through $z_1 \in \Omega_n$, then J_0, J_1 are locally close together in the sense of Hausdorff distance when z_1, z_0 are near each other. We observe that if the parametrization of J_0 is properly chosen, then U_n decreases along J_0. Thus J_0 is a Jordan arc. A maximal Jordan arc containing J_0 for which ∇U_n is tangent will be called an orthogonal trajectory. It follows from the above discussion and the maximum principle for elliptic equations of the form (1.2) that there is a unique orthogonal trajectory through z_0 which must approach ∂D_n as one proceeds along it in either direction from z_0.

It suffices to prove Theorem 1 for fixed p, $1<p<\infty$, and

$$\text{(1.5)} \qquad f \in W_{1,p}(\Omega) \cap C^\infty(\Omega),$$

since this space is dense in $W_{1,p}(\Omega)$ (see [12]). Given $\varepsilon>0$ choose $M>0$ so large that if $E=\{z\in\Omega\colon |f(z)| \geqq M\}$, then

$$\int_E (|f|^p + |\nabla f|^p)\, dA \leqq \varepsilon^p.$$

Put $f_1=f$ in $\Omega-E$ and $f_1=(\operatorname{Sgn} f)M$ in E. Then $f_1\in W_{1,p}(\Omega)$,

$$\text{(1.6)} \qquad \|f-f_1\|_{1,p} \leqq 3\varepsilon,$$

and f_1 is locally Lipschitz in Ω. With M now fixed, choose K a compact set so that

$$\text{(1.7)} \qquad M \cdot \text{area}\, [\Omega-K]^{1/p} \leqq \varepsilon,$$

$$\text{(1.8)} \qquad \|(\nabla f_1)\chi_{\Omega-K}\|_p \leqq \varepsilon.$$

From (1.3), (1.4), and the coarea formula (see [4, 3.2.12]) we find

$$\varepsilon^p \geqq \int_{\Omega-K} |\nabla f_1|^p\, dA \geqq \int_{\{t/2 \leqq U_N \leqq t\}} |\nabla f_1|^p\, dA = \int_{t/2}^{t} \Big[\int_{\{U_N=\tau\}} |\nabla f_1|^p/|\nabla U_N|\, ds\Big]\, d\tau,$$

for some positive integer N and $t>0$. Here ds denotes arc length. Hence, there exists τ, $t/2 \leqq \tau \leqq t$, such that

$$\text{(1.9)} \qquad \int_{\{U_N=\tau\}} |\nabla f_1|^p/|\nabla U_N|\, ds \leqq 2\varepsilon^p/\tau.$$

Given $z\in D_N$ let z^* denote the point in $\{W\colon U_N(W)=\tau\}$ which lies on the same othogonal trajectory as z. Define f_2 in Ω_N by $f_2=f_1$ in $\{W\colon U_N(W)\geqq\tau\}$ and $f_2(z)=f_1(z^*)$ when $z\in\{W\colon 0<U_N(W)<\tau\}$. In § 3 we show that f_2 is Lipschitz in Ω_N and

$$\text{(1.10)} \qquad \int_{\{0<U_N\leqq\tau\}} |\nabla f_2|^p\, dA \leqq \tau \int_{\{U_N=\tau\}} |\nabla f_1|^p/|\nabla U_N|\, ds \leqq 2\varepsilon^p.$$

Note that the last inequality follows from (1.9). Using (1.6)—(1.8) and (1.10) we deduce that $\|f-f_2\|_{1,p}\leqq 8\varepsilon$. Finally, convoluting f_2 with a suitable approximate identity we obtain a C^∞ function on $\mathbf{C}$ which approximates f in $W_{1,p}(\Omega)$ within 9ε. Thus to prove Theorem 1 it remains to prove (1.3)—(1.4) and (1.10).

Next, let $A_{1,p}(\Omega)$ denote the space of analytic functions in Ω whose real and imaginary parts are in $W_{1,p}(\Omega)$ with norm induced from this space. In § 4 we easily obtain from (1.10) and Theorem 1.

Corollary 1. *Polynomials in z are dense in* $A_{1,p}(\Omega)$.

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2. Preliminary reductions

Let (Ω_n) and Ω be as in § 1. Recall that

$$\Delta(a, 4r) \subseteqq \Omega \subseteqq \Omega_n \subseteqq \Delta(a, R),$$

$n=1, 2, \ldots$ for $R>0$ large enough. For fixed q, $1<q<\infty$, let U_n be defined relative to q as in § 1. We first prove (1.3). To do this, we note from a Harnack inequality of Serrin (see [13, Thm. 1.1]) that for a given compact set $K\subseteqq\Omega$ there exists $t=t(K)$ such that $U_n\geqq 2t$ on K for $n=1, 2, \ldots$. To prove (1.3) for this t, it clearly suffices to show

$$U_n(z) \leqq k_1 |d_n(z)/r|^k, \quad z\in\Omega_n, \tag{2.1}$$

where $k_1=k_1(q)$ and $k=k(q)$ are positive constants independent of n, since $\partial\Omega_n$ converges to $\partial\Omega$ in the sense of Hausdorff distance. In (2.1), $d_n(z)$ denotes the distance from z to $\partial\Omega_n$. If $q>2$, then (2.1) is a consequence of the fact that functions g in $W_{1,q}(\mathbf{C})$ are Hölder continuous with exponent $1-2/q$ and norm bounded by a constant times $\|g\|_{1,q}$. If $q=2$, then (2.1) is a consequence of the Milloux—Schmidt inequality (in this case $k=1/2$). If $1<q<2$, $z_1\in\Omega-\bar{\Delta}(a, 2r)$, and $0<\varrho<r/2$ we use the inequalities;

$$M(\varrho)^q \leqq k\varrho^{-2}\int_{\Delta(z_1, 2\varrho)} U_n^q\, dA \quad \text{(see [13, Thm. 1.3])}, \tag{2.2}$$

$$\varrho^{(q-2)}\int_{\Delta(z_1, 2\varrho)} |\nabla U_n|^q\, dA \leqq kM(4\varrho)[M(4\varrho)-M(\varrho)]^{(q-1)} \quad \text{(see [5, Thm. 2.1])}, \tag{2.3}$$

where $k=k(q)$.

Here, for $t>0$

$$M(t) = M(t, z_1) = \operatorname*{ess.\,sup}_{\Delta(z_1, t)} U_n.$$

Now if $d_n(z_0) \leqq \sigma < r/4$ it follows from Poincaré's inequality applied to U_n restricted to $\partial \Delta(z_0, t)$ $\sigma < t < 2\sigma$, that

$$(2.4) \qquad \sigma^q \int_{\Delta(z_0, 2\sigma)} |\nabla U_n|^q(z)\, dA \geqq 2^{-q} \int_\sigma^{2\sigma} \Big[\int_0^{2\pi} |\partial U_n/\partial\theta|^q (z_0 + te^{i\theta})\, d\theta\Big]\, t\, dt$$

$$\geqq k \int_{\{z:\, \sigma < |z - z_0| < 2\sigma\}} U_n^q\, dA.$$

Again $k = k(q)$ is a positive constant not necessarily the same at each occurrence. From the weak maximum principle implied by (1.1) we see that $U_n(z_2) \geqq \frac{1}{2} M(\sigma, z_0)$ for some z_2 with $\frac{5}{4}\sigma < |z_2 - z_0| < \frac{7}{4}\sigma$. Using this inequality, (2.2) with $\varrho = \sigma/8$, $z_1 = z_2$; (2.4), and (2.3) with $\varrho = \sigma$, $z_1 = z_0$, we get

$$M(\sigma, z_0)^q = M(\sigma)^q \leqq k\sigma^{-2} \int_{\{z:\, \sigma < |z - z_0| < 2\sigma\}} U_n^q\, dA$$

$$\leqq k\sigma^{(q-2)} \int_{\Delta(z_0, 2\sigma)} |\nabla U_n|^q\, dA \leqq kM(4\sigma)[M(4\sigma) - M(\sigma)]^{(q-1)}.$$

Hence,

$$[M(\sigma)/M(4\sigma)]^q \leqq k\left[1 - \frac{M(\sigma)}{M(4\sigma)}\right]^{(q-1)}$$

from which it follows that

$$(2.5) \qquad M(\sigma) \leqq \gamma M(4\sigma),$$

for some $\gamma = \gamma(q)$, $0 < \gamma < 1$. Iterating (2.5) with $\sigma = 4^j d_n(z_0)$, $j = 0, 1, \dots, m$, where m is the least integer such that $4^{m+1} d_n(z_0) < r$, we get

$$M[d_n(z_0)] \leqq \gamma^m \leqq k_1 [d_n(z_0)/r]^k,$$

where $k_1 = k_1(q)$, $k = k(q)$. Hence (2.1) is valid when $d_n(z_0) < r/4$. If $d_n(z_0) \geqq r/4$, then clearly (2.1) holds for k_1 large enough, and k as previously. We conclude from (2.1) that (1.3) is true.

Finally in § 2 we prove (1.4). To do so we cannot apply elliptic regularity theory directly since (1.2) is degenerate elliptic. Instead, given $\varepsilon > 0$ and n a positive integer, let $V = V(\cdot, \varepsilon, n)$ be the minimizer of the Euler functional

$$\int_{\mathbf{C}} [\varepsilon + |\nabla \psi|^2]^{q/2}\, dA$$

where the minimum is taken over $\psi \in \dot{W}_{1,q}(\Omega_n)$ with $\psi \equiv 1$ on $\bar{\Delta}(a, r)$.

As in § 1 we see that V is a weak solution in D_n to

$$(2.6) \qquad 0 = \nabla \cdot [(\varepsilon + |\nabla V|^2)^{((q/2)-1)} \nabla V] = \sum_{i,j} a_{ij} V_{x_i x_j},$$

where at $z = x_1 + ix_2 \in D_n$,

$$(2.7) \qquad a_{ij} = (\varepsilon + |\nabla V|^2)^{q/2-1} [(q-2) V_{x_i} V_{x_j} (\varepsilon + |\nabla V|^2)^{-1} + \delta_{ij}] = (\varepsilon + |\nabla V|^2)^{q/2-1} b_{ij},$$

$1 \leqq i, j \leqq 2$. In (2.7), δ_{ij} denotes the Kronecker delta. It is easily seen for $z \in D_n$ and $\xi = \xi_1 + i\xi_2$ that

$$\alpha_1 |\xi|^2 \leqq \sum_{i,j} b_{ij} \xi_i \xi_j \leqq \alpha_2 |\xi|^2, \tag{2.8}$$

where $\alpha_1 = \min(q-1, 1)$, $\alpha_2 = \max(q-1, 1)$. It follows from (2.6) and (2.8) (see [10, Ch. 4]) that V is infinitely differentiable in D_n. Hence V is a strong solution to (2.6). Moreover, from (2.8) it follows as in [6, 11.20] that if $z \in D_n$, $V_{zz} \neq 0$, and $\lambda = V_z$, then

$$\lambda_{\bar{z}} = \mu(z) \lambda_z, \quad |\mu(z)| \leqq \beta < 1, \tag{2.9}$$

where $\beta = \beta(q)$ is independent of n and ε. Thus λ is quasiregular in D_n. Put $\nu = \mu$ in D_n when $V_{zz} \neq 0$ and let $\nu = 0$ on the rest of $\mathbf{C}$. Let τ be the unique quasiconformal solution to the Beltrami equation satisfying

$$\tau_{\bar{z}} = \nu(z) \tau_z, \quad z \in \mathbf{C},$$

for almost every z in $\mathbf{C}$ with respect to two dimensional Lebesgue measure and $\tau(0) = 1$, $\tau(1) = 1$, $\tau(\infty) = \infty$ (see [2, Ch. 5]). Then

$$\lambda = g \circ \tau \tag{2.10}$$

where g is analytic in $\tau(D_n)$. It follows from (2.10) and (2.6) (see [6, Thm. 11.4]) that for a given compact subset $E \subseteqq \Omega_n$ there exists $\alpha = \alpha(q) > 0$ and $k = k(q, E)$ (independent of ε) such that

$$\max_{z \in E} |\lambda| \leqq k, \tag{2.11}$$

$$|\lambda(z) - \lambda(w)| \leqq k |z - w|^\alpha, \quad z, w \in E. \tag{2.12}$$

From (2.10) it is clear that λ has only isolated zeros in D_n. We in fact show

$$\lambda \neq 0 \quad \text{in} \quad D_n. \tag{2.13}$$

Indeed, if $\lambda(z_0) = 0$ for some $z_0 \in D_n$, then from (2.7) we see that $\Delta V(z_0) = 0$. Since V is real analytic in D_n it follows from (2.6), (2.7) that for some positive integer $m \geqq 2$,

$$V(z) - V(z_0) = P(z - z_0) + O(|z - z_0|^{m+1})$$

in a neighborhood of z_0, where P is a homogeneous, harmonic polynomials of degree m. Hence for some $\eta \in \mathbf{C}$

$$V(z) - V(z_0) = \operatorname{Re}[\eta (z - z_0)^m] + O(|z - z_0|^{m+1}) \tag{2.14}$$

in a neighborhood of z_0.

Now from the maximum principle for solutions to (2.6) we see that $\{z \colon V(z) > V(z_0)\}$ has exactly one component. However, (2.14) implies that this set has more than one component. To see why, choose rays l_1, l_2, l_3, l_4, beginning at z_0 such that

$$\operatorname{Re}[\eta(z-z_0)^m] < 0, \quad z \in l_1 \cup l_2 - \{z_0\},$$
$$\operatorname{Re}[\eta(z-z_0)^m] > 0, \quad z \in l_3 \cup l_4 - \{z_0\}.$$

We also choose l_3, l_4, so that l_3 lies in one of the sectors determined by l_1, l_2, and l_4 lies in the other sector. Then from (2.14) it is clear that there are segments $\sigma_1 \subseteqq l_1$, $\sigma_2 \subseteqq l_2$, each containing z_0 with $V > V(z_0)$ on $(\sigma_1 \cup \sigma_2) - \{z_0\}$. Since $\{z\colon V(z) > V(z_0)\}$ is connected, there exists a curve σ contained in this set joining σ_1 to σ_2. If $\gamma = \sigma_1 \cup \sigma_2 \cup \sigma$, then an argument similar to [1, Ch. 4, Lemma 2] shows that $V(z) < V(z_0)$ at some point z in one of the bounded components K of $\mathbf{C} - \gamma$. But from the minimum principle for solutions to (2.6) it would follow first that $(\mathbf{C} - \Omega_n) \cap K \neq \emptyset$ and thereupon from connectivity of $\mathbf{C} - \Omega_n$ that $\mathbf{C} - \Omega_n \subseteqq K$. From this contradiction we conclude that (2.13) is valid.

Let (ε_j) be such that $0 < \varepsilon_j < 1$, $j = 1, 2, \ldots$ and $\lim_{j \to \infty} \varepsilon_j = 0$. With n still fixed, let $V_j = V(\cdot, \varepsilon_j, n)$, $\lambda_j = (V_j)_z$, $j = 1, 2, \ldots$. Then from (2.10) we see there exists τ_j quasiconformal in $\mathbf{C}$ with $\tau_j(0) = 0$, $\tau_j(1) = 1$, $\tau_j(\infty) = \infty$, and g_j analytic on $\tau_j(D_n)$, $j = 1, 2, \ldots$ such that $\lambda_j = g_j \circ \tau_j$. From the normalization on $(\tau_j)_1^\infty$ and (2.9) it follows (see [2, Ch. 3, Thm. 2]) that a subsequence of $(\tau_j)_1^\infty$ (also denoted $(\tau_j)_1^\infty$) converges uniformly on compact subsets of $\mathbf{C}$ to a quasiconformal function τ. From (2.11)—(2.12) and the fact that $V_j \to U_n$ in $W_{1,p}(\Omega_n)$ (see, for example, [11, (1.6)]) it follows that λ_j converges uniformly to $(U_n)_z$ on compact subsets of D_n. Thus (g_j) converges uniformly to an analytic function g on compact subsets of $\tau(D_n)$ and $(U_n)_z = g \circ \tau$. It follows from this representation of $(U_n)_z$, (2.13), and Hurwitz's theorem that $(U_n)_z \neq 0$ in D_n. Also $(U_n)_z$ is Hölder continuous as follows from (2.11)—(2.12) or the fact that τ is Hölder continuous. Using these facts we see that U_n is a weak solution to a locally uniformly elliptic equation with Hölder continuous coefficients. From a slight generalization of Schauder's Theorem and a "bootstrap" method, we conclude that $U_n \in C^\infty(D_n)$ (see [10, Ch. 4, Thm. 6.3]). Applying a Theorem of E. Hopf [8], it then follows that U_n is real analytic in D_n. Hence (1.4) is valid.

3. Proof of Theorem 1

Given p, $1 < p < \infty$,

$$f \in W_{1,p}(\Omega) \cap C^\infty(\Omega),$$

and $\varepsilon > 0$ we choose M as in § 1 and define f_1 relative to M as in § 1. Using (1.3)—(1.4) and proceeding as in § 1, we obtain ((1.7)—(1.9)) for some $\tau > 0$. In (1.9) it is understood of course that $\tau > 0$ is chosen so that ∇f_1 exists almost everywhere with respect to arc length measure on $\{z\colon U_N(z) = \tau\}$. Next we define f_2 relative to τ and f_1 as in § 1.

We claim that f_2 is Lipschitz in D_N. To prove this claim, let $z_0 \in D_N$ and suppose $\delta > 0$ is so small that $\tau+\delta<1$. Let γ be an orthogonal trajectory in D_N with $z_0 \notin \gamma$ and put

$$G = \{z \in D_N\colon\ 0 < U_N(z) < \tau+\delta\} - \gamma.$$

Clearly G is a simply connected domain. Put $U=U_N$ and consider the differential

$$|\nabla U|^{(q-2)}[-U_y\,dx + U_x\,dy].$$

From (1.2) it follows that this differential is exact. Hence there exists a real analytic function V in G with

$$V_x = -|\nabla U|^{(q-2)} U_y, \quad V_y = |\nabla U|^{(q-2)} U_x.$$

Note that V is constant on arcs of orthogonal trajectories and is increasing on an arc contained in a level of U provided the parametrization is properly chosen. It follows that $W=U+iV$ maps G one-one into a square S in the W plane. If φ denotes the inverse of W, then φ is real analytic on S and by construction for $W=U+iV \in S$

$$\text{(3.1)} \qquad \begin{aligned} f_2 \circ \varphi(W) &= f_1(\varphi(W)), \quad U > \tau, \\ f_2 \circ \varphi(W) &= f_1(\varphi(\tau, V)), \quad U \leqq \tau. \end{aligned}$$

Since f_1 is Lipschitz and $\varphi \in C^\infty(S)$, it follows that $f_2 \circ \varphi$ is Lipschitz in a neighborhood of $W(z_0)$. Hence f_2 is Lipschitz in a neighborhood of z_0. Applying Rademacher's theorem, we see that f_2 is differentiable for almost every z in Ω_n with respect to two dimensional Lebesque measure.

Let $F=\{z\colon 0<U(z)\leqq\tau\}$ $z_0 \in E$, and suppose f_2 is differentiable at z_0. Since f_2 is constant on arcs of orthogonal trajectories in F, we see that $|\nabla f_2(z_0)| = |df_2(z_0)/ds|$, where $df_2(z)/ds$ denotes the directional derivative of f_2 in a tangential direction to $\{\xi\colon U(\xi)=U(z)\}$ at z. Let z_0^* be the point in $\{z\colon U(z)=\tau\}$ which lies on the same orthogonal trajectory as z_0. Then from the above remarks and (3.1) we deduce for almost every $z_0 \in F$,

$$|\nabla f_2(z_0)| \left|\frac{\partial \varphi}{\partial V}(W(z_0))\right| = \left|\frac{\partial}{\partial V}(f \circ \varphi)(W(z_0))\right|$$

$$= \left|\frac{\partial}{\partial V}(f \circ \varphi)(W(z))\right| = \left|\frac{d}{ds} f_1(z_0^*)\right| \left|\frac{\partial \varphi}{\partial V}(W(z_0^*))\right|.$$

Since $|\partial\varphi(W)/\partial V| = |\nabla U|^{(1-q)}(z)$, we conclude that

$$\text{(3.2)} \qquad |\nabla f_2(z_0)| = \left|\frac{d}{ds} f_1(z_0^*)\right| |\nabla U(z_0)|^{(q-1)} |\nabla U(z_0^*)|^{(1-q)}$$

for almost every $z_0 \in F$. Observe that $p=q/(q-1)$ and that the Jacobian of W at z_0 in F is $|\nabla U(z_0)|^q$. Raising (3.2) to the pth power, integrating over F, and changing variables, we get

$$\int_F |\nabla f_2|^p\,dA = \int_F |\nabla U(z)|^q \left|\frac{d}{ds} f_1(z^*)\right|^p |\nabla U(z^*)|^{-q}\,dA \tag{3.3}$$

$$= \left(\int_0^\theta \left|\left(\frac{d}{ds} f_1\right)(\varphi(\tau, V))\right|^p |\nabla U(\varphi(\tau, V))|^{-q}\,dV\right)\int_0^\tau dU = \tau \int_{\{U=\tau\}} \left|\frac{d}{ds} f_1\right| |\nabla U|^{-1}\,ds.$$

Here,

$$\theta = \int_{\{U=\tau\}} |\nabla U|^{(q-1)}\,ds$$

and we have used the fact that $dV=|\nabla U|^{(q-1)}\,ds$ on $\{z\colon U(z)=\tau\}$. From this equality and (1.9) we conclude that (1.10) is valid.

4. Proof of Corollary 1

Let $g=h+il\in A_{1,q}(\Omega)$. Arguing as in §§ 1—2 with f replaced by h, l, respectively, we get h_2, l_2, defined on $\Omega_N\supseteqq\bar\Omega$ which approximate h, l, respectively in the norm of $W_{1,q}(\Omega)$ within 8ε. Moreover from (1.10), it is clear that

$$\int_{\Omega_N-\Omega} (|\nabla h_2|^p+|\nabla l_2|^p)\,dA \leqq 4\varepsilon^p. \tag{4.1}$$

Let $g_2=h_2+il_2$ in Ω_N. We write $g_2=\psi+Pg_2$ in Ω_N where

$$Pg_2(z) = \frac{1}{\pi}\int_{\Omega_N} (g_2)_{\bar\zeta}[(z-\zeta)^{-1}-(a-\zeta)^{-1}]\,dA, \quad z\in\Omega_N.$$

Since g_2-g has norm in $W_{1,p}(\Omega)$ at most 16ε and (4.1) holds, we deduce that $\|(g_2)_{\bar\zeta}\|_p\leqq 24\varepsilon$, where the norm is in Ω_N. Now (see [2, Ch. 5]), $(Pg_2)_{\bar\zeta}=(g_2)_{\bar\zeta}$ when $\zeta\in\Omega_N$ and $(Pg_2)_{\bar\zeta}=0$ in $\mathbf{C}-\Omega_N$ almost everywhere with respect to two dimensional Lebesgue measure. Thus from Weyl's lemma, ψ is analytic in Ω_N. Using Calderón—Zygmund theory it follows that

$$\|(Pg_2)_\zeta\|_p \leqq k\|(g_2)_{\bar\zeta}\|_p \leqq k\varepsilon, \tag{4.2}$$

$$\|Pg_2\|_p \leqq k_1\|(g_2)_{\bar\zeta}\|_p \leqq k_1\varepsilon, \tag{4.3}$$

where $k_1=k_1(p, \Omega)$ and $k=k(p)$. Also, all norms in (4.2)—(4.3) are taken relative to Ω_N. From (4.1)—(4.3) we conclude that ψ is analytic in Ω_N and $\|g-\psi\|_{1,p}\leqq k_2\varepsilon$ where $k_2=k_2(\Omega, p)$ and the norm is in Ω. Since $\bar\Omega\subseteqq\Omega_N$ it follows from Runge's theorem that there is a polynomial Q which approximates ψ within ε in $W_{1,p}(\Omega)$. Thus,

$$\|Q-g\|_{1,p} \leqq (k_2+1)\varepsilon,$$

and the proof of Corollary 1 is complete.

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Department of Mathematics
University of Kentucky
Lexington, KY 40506—0027
U.S.A.