An ill-posed moving boundary problem for doubly-connected domains

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0. Introduction

Let, for Ω any bounded doubly-connected domain in $\mathbf{R}^N$ with smooth boundary, u_Ω denote the harmonic function in Ω with boundary values one and zero on the inner and outer components of $\partial\Omega$ respectively. We consider the problem: given Ω as above find a one-parameter family $\{\Omega(t)\}$ of doubly-connected domains for $t\in\mathbf{R}$ (t=time) in some interval around $t=0$ such that $\Omega(0)=\Omega$ and such that the outward normal velocity of $\partial\Omega(t)$ equals $-\partial u_{\Omega(t)}/\partial n$ on $\partial\Omega(t)$ ($\partial/\partial n$=the outward normal derivative).

This problem may arise e.g. in electro-chemistry (for $N\leqq 3$). Then $\Omega(t)^e$, the unbounded component of $\Omega(t)^c=\mathbf{R}^N\setminus\Omega(t)$, is an anode, $\Omega(t)^i=\Omega(t)^c\setminus\Omega(t)^e$ is a cathode and $\Omega(t)$ itself is an electrolyte. Finally, $u_{\Omega(t)}$ will be (proportional to) minus the electric potential. Under suitable assumptions the anode will dissolve and the dissolved material will be taken up by the cathode according to the law above. See [4], [6] for a related problem, with $\Omega(t)^i$ fixed. Actually, a slightly more general problem may be more adequate for this electro-chemical model. See § 3.

For $N=2$ there is also an interpretation of the problem in terms of Hele Shaw flow, i.e. for the flow of an incompressible viscous fluid in the narrow region between two slightly separated and infinitely extended parallel surfaces. Then $\Omega(t)$ is (the two-dimensional picture of) the region of fluid, $\Omega(t)^i$ is occupied by e.g. air of some positive pressure and $\Omega(t)^e$ is empty or is occupied by air at a lower pressure. The pressure of the air in $\Omega(t)^i$ will make $\Omega(t)$ move according to the stated law. Cf. e.g. [5], [13], [16].

Our problem may also be regarded as a kind of degenerate Stefan problem with three phases. In fact, if u_Ω is extended to all $\mathbf{R}^N$ by $u_\Omega=0$ in Ω^e, $u_\Omega=1$ in Ω^i then the law for the motion of $\Omega(t)$ can be expressed

$$\frac{\partial}{\partial t}H(u)=\Delta u \tag{0.1}$$

(cf. equation (1.4)′), where $u(x,t)=u_{\Omega(t)}(x)$ and H is the function

$$H(u)=\begin{cases}0 & \text{for} \quad u\leqq 0,\\ 1 & \text{for} \quad 0<u<1,\\ 0 & \text{for} \quad u\geqq 1.\end{cases}$$

(Thus $H(u_\Omega)=\chi_\Omega$.)

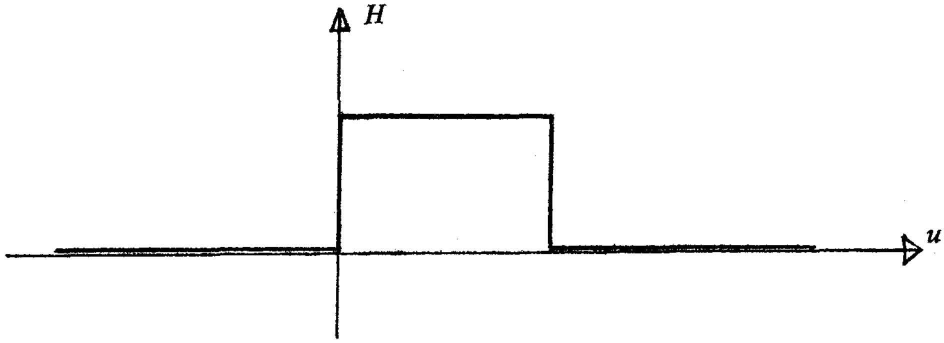

Fig. 1
$H(u)$ (completed to a connected graph)

(0.1) may be viewed upon as an enthalpy formulation of a Stefan problem. Then u is the temperature and H the enthalpy. In our case the enthalpy function $H=H(u)$ is of a quite unusual form (Fig. 1). In particular, it is non-monotone and it has two jump discontinuities, to mean that phase transitions occur at two different temperature levels. See e.g. [5] for enthalpy formulations of the Stefan problem.

The fact that $H(u)$ is non-monotone is related to the ill-posedness of our problem. E.g., as t increases the inner boundary $\partial\Omega(t)^i$ becomes more and more irregular and this corresponds to the fact that $H(u)$ makes a negative jump at $u=1$ (i.e. on $\partial\Omega(t)^i$). On the other hand $\partial\Omega(t)^e$ becomes more and more regular as t increases, corresponding to the positive jump of $H(u)$ at $u=0$.

It turns out that a necessary condition for our problem to have a solution is that $\partial\Omega$ is real analytic. The main result in this paper Theorem 2.1, states that, conversely, if $\partial\Omega$ is real analytic then there does exist a solution $\{\Omega(t)\}$, at least in a certain weak sense (and for t in some small interval around $t=0$). Our weak solution is constructed by first solving a Cauchy problem for the Laplacian with Cauchy data on $\partial\Omega$. This takes care of the ill-posedness. Then $\Omega(t)$ are obtained from solutions of certain elliptic variational inequalities. It is plausible that the weak solution constructed in this way actually is a classical solution, but we do not prove this in the present paper.

An interesting property of our problem is that if e.g. $N=2$ then

$$\frac{d}{dt}\int_{\Omega(t)} f=0$$

for every function f analytic in a neighbourhood of $\overline{\Omega(t)}$ whenever $\{\Omega(t)\}$ is a solution.

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Some notation

$|E|=N$-dimensional Lebesgue measure of $E\subset\mathbf{R}^N$.

$\int_E \varphi=\int_E \varphi\, dx=$integral of φ with respect to N-dimensional Lebesgue measure ($E\subset\mathbf{R}^N$).

$A\triangle B$ symmetric difference between the sets A and B.

$$\mathcal{D} = C_0^\infty(\mathbf{R}^N).$$

$H(\Omega)=\{$harmonic functions in $\Omega\}$ ($\Omega\subset\mathbf{R}^N$ open).
$H(\bar\Omega)=\{$functions harmonic in a neighbourhood of $\bar\Omega\}$.

$$HL^1(\Omega) = H(\Omega)\cap L^1(\Omega).$$

$SL^1(\Omega)=\{$subharmonic functions in $\Omega\}\cap L^1(\Omega)$.
$-SL^1(\Omega)=\{-u\colon u\in SL^1(\Omega)\}=\{$superharmonic functions in $\Omega\}\cap L^1(\Omega)$.

1. Classical solutions

In this section we introduce a precise concept of a (classical) solution of our problem and derive some properties of such solutions.

Let $I\subset\mathbf{R}$ be an open interval containing zero. A one-parameter family $\{\Omega(t)\colon t\in I\}$ of bounded domains in $\mathbf{R}^N$ will be called *smooth* if there exists a C^2 function $g(x,t)$ ($x\in\mathbf{R}^N$, $t\in I$) such that

$$\begin{aligned}
\Omega(t) &= \{x\in\mathbf{R}^N\colon g(x,t)<0\},\\
\partial\Omega(t) &= \{x\in\mathbf{R}^N\colon g(x,t)=0\},\\
\nabla_x g(x,t) &\neq 0 \quad \text{for} \quad x\in\partial\Omega(t).
\end{aligned} \tag{1.1}$$

If Ω is any bounded doubly-connected domain in $\mathbf{R}^N$ ($N\geqq 2$) we denote by Ω^i and Ω^e the bounded and unbounded components of $\Omega^c=\mathbf{R}^N\setminus\Omega$ respectively (doubly-connected means that Ω^c has exactly two components). Set $\partial\Omega^i=\partial(\Omega^i)$, $\partial\Omega^e=\partial(\Omega^e)$ so that $\partial\Omega=\partial\Omega^i\cup\partial\Omega^e$ or, taking orientation into account, $\partial\Omega=-\partial\Omega^i-\partial\Omega^e$.

When Ω is a bounded doubly-connected domain u_Ω denotes the harmonic function in Ω with boundary values

$$u_\Omega=\begin{cases}0 & \text{on } \partial\Omega^e,\\ 1 & \text{on } \partial\Omega^i\end{cases}$$

(whenever this function exists). If $\partial\Omega$ is smooth (C^2) then certainly u_Ω exists and even $u_\Omega\in C^1(\bar\Omega)$. See [21] for the latter.

A smooth family $\{\Omega(t)\colon t\in I\}$ of doubly-connected domains will be called a *(classical) solution* of our problem if

$$\frac{d}{dt}\int_{\Omega(t)}\varphi=-\int_{\partial\Omega(t)}\varphi\frac{\partial u_{\Omega(t)}}{\partial n}\,ds \tag{1.2}$$

for all $\varphi\in\mathcal{D}=C_0^\infty(\mathbf{R}^N)$ and all $t\in I$. Here ds denotes the $(N-1)$-dimensional area measure on $\partial\Omega(t)$ and $\dfrac{\partial}{\partial n}$ the outward normal derivative on $\partial\Omega(t)$.

Formula (1.2) expresses that $\partial\Omega(t)$ moves with the velocity $-\partial u_{\Omega(t)}/\partial n$ measured in the direction of the outward normal of $\partial\Omega(t)$. Indeed, the smoothness assumption on $\{\Omega(t)\}$ implies the existence of a continuous function $\beta=\beta(t)$ on $\partial\Omega(t)$ such that

$$\frac{d}{dt}\int_{\Omega(t)}\varphi=\int_{\partial\Omega(t)}\varphi\beta\,ds \tag{1.3}$$

for all $\varphi\in\mathcal{D}$. β is the outward normal velocity of $\partial\Omega(t)$ and is given, in terms of $g(x,t)$ in (1.1), by $\beta=-\dfrac{\partial g/\partial t}{\partial g/\partial n}$. Hence (1.2) asserts that $\beta=-\dfrac{\partial u_{\Omega(t)}}{\partial n}$.

We extend u_Ω to all $\mathbf{R}^N$ by

$$u_\Omega=\begin{cases}0 & \text{in } \Omega^e,\\ 1 & \text{in } \Omega^i.\end{cases}$$

Then, assuming that $\partial\Omega$ is smooth, u_Ω is continuous in $\mathbf{R}^N$ and we have, for $\varphi\in\mathcal{D}$,

$$\int_{\partial\Omega}\varphi\frac{\partial u_\Omega}{\partial n}\,ds=\int_{\partial\Omega}u_\Omega\frac{\partial\varphi}{\partial n}\,ds+\int_\Omega\varphi\Delta u_\Omega-\int_\Omega u_\Omega\Delta\varphi$$

$$=-\int_{\partial\Omega^i}\frac{\partial\varphi}{\partial n}\,ds-\int_\Omega u_\Omega\Delta\varphi=-\int_{\Omega^i}\Delta\varphi-\int_\Omega u_\Omega\Delta\varphi=-\int_{\mathbf{R}^N}u_\Omega\Delta\varphi.$$

Thus, for $\{\Omega(t)\colon t\in I\}$ smooth (1.2) is equivalent to

$$\frac{d}{dt}\int_{\Omega(t)}\varphi=\int_{\mathbf{R}^N}u_{\Omega(t)}\Delta\varphi \tag{1.4}$$

for all $\varphi\in\mathcal{D}$. (1.4) expresses that

$$\frac{d}{dt}\chi_{\Omega(t)}=\Delta u_{\Omega(t)} \tag{1.4$'$}$$

in some weak sense.

The following lemma follows easily from the fact (cf. [21]) that $\partial u_{\Omega(t)}/\partial n$ is strictly positive (negative) on $\partial\Omega(t)^i$ $(\partial\Omega(t)^e)$ for $\{\Omega(t)\}$ smooth. (The proof is omitted.)

Lemma 1.1. *Any solution $\{\Omega(t)\colon t\in I\}$ of our problem is strictly monotone in the sense that*

$$\Omega(\tau)^i\subset\subset\Omega(t)^i\quad\textit{and}\quad\Omega(\tau)^e\supset\supset\Omega(t)^i$$

for $\tau<t$ $(\tau, t\in I)$. Moreover, $u_{\Omega(\tau)}\leqq u_{\Omega(t)}$ $(\tau<t)$.

Example 1.1. There exists a $T>0$ $(T=+\infty$ if $N=2$, $T<\infty$ if $N\geqq 3)$ and a positive increasing function $r(t)$ for $0<t<T$ with $\lim_{t\to 0}r(t)=0$, $\lim_{t\to T}r(t)=\infty$ such that, for any $A>0$ and any $t_0<0$ the annuli (or shell-domains)

$$\Omega(t)=\{x\in\mathbf{R}^N\colon\ r(t-t_0)^N<|x|^N<r(t-t_0)^N+A\} \tag{1.5}$$

are a classical solution defined for $t_0<t<t_0+T$. Observe that $|\Omega(t)|=\text{constant}$, in accordance with e.g. Corollary 1.1 below (choose $f\equiv 1$). Implicit formulas for $r(t)$ may be found in [15].

An interesting question is whether the above solutions (and translations of them) are the only solutions which exhaust all $\mathbf{R}^N$.

The solution (1.5) possesses a number of extremal properties, e.g. the following. If $\{\Omega'(t)\}$ is any other solution such that $|\Omega(0)^i|=|\Omega'(0)^i|$ and $|\Omega(0)|=|\Omega'(0)|$ then

$$|\Omega(t)^i|\leqq|\Omega'(t)^i|,\quad\frac{d}{dt}|\Omega(t)^i|\leqq\frac{d}{dt}|\Omega'(t)^i|,\quad-\int_{\partial\Omega(t)^i}\frac{\partial u_{\Omega(t)}}{\partial n}ds\leqq-\int_{\partial\Omega'(t)^i}\frac{\partial u_{\Omega'(t)}}{\partial n}ds$$

for every $t\geqq 0$ for which $\Omega'(t)$ is defined. See [15]. □

Our main concern in this paper is the local initial value problem associated with (1.2), i.e. the problem of finding a solution $\{\Omega(t)\colon t\in I\}$ on some small interval I around $t=0$ when $\Omega(0)$ is prescribed. There is then no loss of generality in assuming that I is so small so that there exists a fixed closed oriented hypersurface $\Gamma\subset\mathbf{R}^N$ such that, for all $t\in I$, $\Gamma\subset\Omega(t)$ and Γ separates the two components of $\Omega(t)^c$.

More specifically, when considering a smooth family $\{\Omega(t)\colon t\in I\}$ of doubly-connected domains we shall henceforth (in § 1) assume that there exists a bounded domain $\omega\subset\mathbf{R}^N$ with $\Gamma=\partial\omega$ smooth such that $\Omega(t)^i\subset\omega$ and $\Gamma\subset\Omega(t)$ for all $t\in I$. Observe that then $\Omega(t)^i=\Omega(t)^c\cap\omega$ and $\Omega(t)^e=\Omega(t)^c\setminus\omega$.

Proposition 1.1. *For a smooth family* $\{\Omega(t)\colon t\in I\}$ *with* I *and* Γ *as above the following conditions are equivalent.*

(i) $\displaystyle \frac{d}{dt}\int_{\Omega(t)}\varphi = -\int_{\partial\Omega(t)}\varphi\frac{\partial u_{\Omega(t)}}{\partial n}\,ds$

for all $\varphi\in\mathcal{D}$, *i.e.* $\{\Omega(t)\}$ *is a solution.*

(ii) $\displaystyle \frac{d}{dt}\int_{\Omega(t)}\varphi = \int_{\Gamma}\frac{\partial\varphi}{\partial n}\,ds$

for all $\varphi\in\mathcal{D}$ *harmonic in* $\Omega(t)$.

(iii) $\displaystyle \int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi = (t-\tau)\int_{\Gamma}\frac{\partial\varphi}{\partial n}\,ds$

for all $\varphi\in\mathcal{D}$ *harmonic in* $\Omega(\tau)\cup\Omega(t)$.

(iv) $\displaystyle \int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi \geqq (t-\tau)\int_{\Gamma}\frac{\partial\varphi}{\partial n}\,ds$

for all $\varphi\in\mathcal{D}$ *superharmonic in* $\Omega(\tau)$ *and subharmonic in* $\Omega(t)$ *(and all* $\tau, t\in I$*).*

(v) *For every* $\tau, t\in I, \tau<t$ *there is a function* $v=v_{\tau,t}\in C^1(\mathbf{R}^N)$ *satisfying*

$$\chi_{\Omega(t)}-\chi_{\Omega(\tau)}=\Delta v,\quad 0\leqq v\leqq t-\tau, \tag{1.6}$$

$$v=\begin{cases}0 & \text{in } \Omega(t)^e,\\ t-\tau & \text{in } \Omega(\tau)^i.\end{cases}$$

Remark 1.1. The strenghtened form of (ii) corresponding to (iv) reads

(vi) $\displaystyle \frac{d}{dt}\int_{\Omega(t)}\varphi \geqq \int_{\Gamma}\frac{\partial\varphi}{\partial n}\,ds$

for all $\varphi\in\mathcal{D}$ satisfying $\Delta\varphi\leqq 0$ in $\Omega(t)\cap\omega$ and $\Delta\varphi\geqq 0$ in $\Omega(t)\setminus\omega$ $(\Gamma=\partial\omega)$.

It is easy to check that e.g. (iv) implies (vi).

In (v) the inequalities can be strengthened to

$$0<v<t-\tau\quad\text{in}\quad\Omega(\tau)\cup\Omega(t).$$

A more complete version of the equivalence between (iv) and (v) is given in Proposition 2.1. □

Proof. (i) implies (v): Define, for $\tau<t$,

$$v=v_{\tau,t}=\int_\tau^t u_{\Omega(r)}\,dr.$$

Then it follows from (1.4) (which is equivalent to (i)) that (1.6) holds in the distribution sense. In particular $\Delta v \in L^\infty$ so that $v \in C^1$ (even $v \in C^{1,\alpha}$ for any $\alpha<1$). The remaining assertions follow immediately from the definition of u_Ω, $0 \leqq u_\Omega \leqq 1$ and Lemma 1.1.

(v) implies (iv): Assuming (v) we have, for $\varphi \in \mathscr{D}$ satisfying $\Delta\varphi \leqq 0$ in $\Omega(\tau)\cap\omega$ and $\Delta\varphi \geqq 0$ in $\Omega(t)\setminus\omega$,

$$\begin{aligned}\int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi &= \langle \Delta v, \varphi\rangle = \int_{\mathbf{R}^N} v\Delta\varphi = \int_{\Omega(\tau)^i} v\Delta\varphi+\int_{\Omega(\tau)\cap\omega} v\Delta\varphi \\ &+\int_{\Omega(t)\setminus\omega} v\Delta\varphi+\int_{\Omega(t)^e} v\Delta\varphi \geqq (t-\tau)\int_{\Omega(\tau)^i}\Delta\varphi \\ &+\int_{\Omega(\tau)\cap\omega}(t-\tau)\Delta\varphi+\int_{\Omega(t)\setminus\omega} 0\Delta v+0 \\ &= (t-\tau)\int_\omega \Delta\varphi = (t-\tau)\int_\Gamma \frac{\partial\varphi}{\partial n}\,ds,\end{aligned}$$

proving (iv).

(iv) implies (iii): This is obvious.

(iii) implies (ii): Differentiating (iii) gives

$$\frac{d}{dt}\int_{\Omega(t)}\varphi = \int_\Gamma \frac{\partial\varphi}{\partial n}\,ds, \tag{1.7}$$

at least for all $\varphi \in \mathscr{D}$ harmonic in a neighbourhood of $\overline{\Omega(t)}$. But every $\varphi \in \mathscr{D}$ harmonic in $\Omega(t)$ can be approximated uniformly on $\Omega(t)$ (hence on $\partial\Omega(t)$) by functions $\varphi \in \mathscr{D}$ harmonic in a neighbourhood of $\overline{\Omega(t)}$ ([3], [11], [14, Ch. 5, § 5]) and from this (ii) follows, using the fact that the left hand side of (1.7) actually is of the form $\int_{\partial\Omega(t)}\varphi\beta\,ds$ (see (1.3)).

(ii) implies (i): Assuming (ii) Green's formula gives

$$\frac{d}{dt}\int_{\Omega(t)}\varphi = -\int_{\partial\Omega(t)}\varphi\frac{\partial u_{\Omega(t)}}{\partial n}\,ds \tag{1.8}$$

for every $\varphi \in \mathscr{D}$ harmonic in $\Omega(t)$ and we have to show that (1.8) actually holds for all $\varphi \in \mathscr{D}$. But this also follows from (1.3) and [3], [11], [14] by approximating the function $\psi \in C(\overline{\Omega(t)})\cap H(\Omega(t))$ which equals $\varphi \in \mathscr{D}$ on $\partial\Omega(t)$ by functions harmonic in a neighbourhood of $\overline{\Omega(t)}$ (for which (1.8) clearly holds). □

From (ii) or (iii) in the proposition we see that if $\{\Omega(t)\}$ is a solution and φ is a harmonic function in (say) a neighbourhood of $\overline{\Omega(t)}$ ($\varphi \in H(\overline{\Omega(t)})$) such that

$$\int_\Gamma \frac{\partial\varphi}{\partial n}\,ds = 0 \tag{1.9}$$

holds, then $\frac{d}{dt}\int_{\Omega(t)}\varphi=0$ (or, if $\varphi \in H(\overline{\Omega(\tau)\cup\Omega(t)})$, $\int_{\Omega(\tau)}\varphi=\int_{\Omega(t)}\varphi$). In two dimensions (1.9) holds whenever φ is a component of an analytic function. In fact, in this case φ has a single-valued harmonic conjugate $\psi=\varphi^*$ and (1.9) follows directly from the Cauchy—Riemann equations:

$$\int_\Gamma \frac{\partial\varphi}{\partial n}\,ds = \int_\Gamma \frac{\partial\psi}{\partial s}\,ds = 0.$$

The higher dimensional analogue of the analytic functions (or, more precisely, the anti-analytic functions) is, at least from one point of view, the *harmonic vector fields*. A harmonic vector field in a domain $\Omega\subset\mathbf{R}^N$ is a (smooth) vector field $f=(f_1, \dots, f_N)$ in Ω satisfying

$$\frac{\partial f_k}{\partial x_j}-\frac{\partial f_j}{\partial x_k} = 0 \quad (\text{all } k, j), \tag{1.10}$$

$$\sum_{j=1}^N \frac{\partial f_j}{\partial x_j} = 0. \tag{1.11}$$

(1.10) and (1.11) are equivalent to the existence, locally, of a harmonic function u such that $f=\operatorname{grad} u$.

In terms of the corresponding one-form $\omega=f_1\,dx_1+\dots+f_N\,dx_N$ (1.10) and (1.11) say that $d\omega=0$ and $d^*\omega=0$ respectively, where $*$ is the Hodge's star operator (see e.g. [20]).

Now, if φ is a component of a harmonic vector field in Ω, say $\varphi=f_1$ with f as above, then φ is harmonic in Ω (since locally $\varphi=\partial u/\partial x_1$ with u harmonic) and for any closed oriented smooth hypersurface Γ in Ω

$$\int_\Gamma \frac{\partial\varphi}{\partial n}\,ds = 0.$$

In fact, using (1.10), (1.11) and Stokes' theorem,

$$\int_\Gamma \frac{\partial f_1}{\partial n}\,ds = \int_\Gamma \sum_{j=1}^N (-1)^{j-1}\frac{\partial f_1}{\partial x_j}\,dx_1\dots d\hat{x}_j\dots dx_N$$

$$= \int_\Gamma \frac{\partial f_1}{\partial x_1}\,dx_2\dots dx_N + \sum_{j=2}^N (-1)^{j-1}\int_\Gamma \frac{\partial f_j}{\partial x_1}\,dx_1\dots d\hat{x}_j\dots dx_N$$

$$= \int_\Gamma \frac{\partial f_1}{\partial x_1}\,dx_2\dots dx_N + \sum_{j=2}^N (-1)^{j-1}\int_\Gamma d(f_j\,dx_2\dots d\hat{x}_j\dots dx_N)$$

$$-\sum_{j=2}^N (-1)^{j-1}\int_\Gamma (-1)^j\frac{\partial f_j}{\partial x_j}\,dx_2\dots dx_N = 0$$

($dx_1\dots d\hat{x}_j\dots dx_N$ means $dx_1\dots dx_{j-1}\,dx_{j+1}\dots dx_N$).

Thus we obtain

Corollary 1.1. *If* $\{\Omega(t)\colon t\in I\}$ *is a solution then*

$$\frac{d}{dt}\int_{\Omega(t)} f = 0$$

for every harmonic vector field f defined in a neighbourhood of $\overline{\Omega(t)}$, *and, in case* $N=2$, *also for every analytic function f in a neighbourhood of* $\overline{\Omega(t)}$.

Example 1.2. If, in case $N=2$, $0\in\operatorname{int}\Omega(t)^i$ for all $t\in I$ Corollary 1.1 shows that

$$\int_{\Omega(t)} z^n = \text{constant in } t$$

for every $n\in\mathbf{Z}$ $(z=x_1+ix_2)$ whenever $\{\Omega(t)\}$ is a solution.

Corollary 1.2. *If* $\{\Omega(t)\colon t\in I\}$ *is a solution then* $\partial\Omega(t)$ *is real analytic for every* $t\in I$. *In particular, a solution of the initial value problem with* $\Omega(0)=\Omega$ *prescribed can only exist if* $\partial\Omega$ *is real analytic.*

Remark 1.2. By "$\partial\Omega$ real analytic" we mean that for every $y\in\partial\Omega$ there is a neighbourhood U of y in $\mathbf{R}^N$ and a real analytic function g in U such that

$$\partial\Omega\cap U=\{x\in U\colon g(x)=0\},\quad \nabla g\neq 0 \quad\text{on}\quad \partial\Omega.$$

Proof. Consider e.g. $\partial\Omega(t)^e$ for a fixed value of t. Choose $\tau\in I$, $\tau<t$. Then, by (v) of Proposition 1.1 we have a function v which in a neighbourhood D of $\partial\Omega(t)^e$ (e.g. $D=\mathbf{R}^N\setminus\overline{(\Omega(\tau)\cup\omega)}$) satisfies

$$\begin{aligned} \Delta v &= \chi_{\Omega(t)} \quad\text{in}\quad D,\\ v &> 0 \quad\text{in}\quad D\cap\Omega(t),\\ v &= 0 \quad\text{in}\quad D\setminus\Omega(t). \end{aligned}$$

By the $W^{2,\infty}$-regularity theory for variational inequalities [22], [23], [7] we get $v\in C^{1,1}(D)$. Then a theorem of Caffarelli [1] (see also [7, Ch. 2, § 5]) shows that $v\in C^2\big(\overline{\Omega(t)\cap D}\big)$ and finally [12, Thm. 1], [7, Thm. 1.1, Ch. 2, § 1] can be applied, showing that $\partial\Omega(t)$ is analytic. □

Remark 1.3. Notice that in order to prove that $\partial\Omega(t)^e$ was analytic we had to make use of the existence of a $\tau<t$ with $\Omega(\tau)^e\supset\supset\Omega(t)^e$. This is an indication of the fact that, as t increases, $\partial\Omega(t)^e$ becomes more and more regular in some sence. Similarly, $\partial\Omega(t)^i$ becomes more and more irregular as t increases.

2. Weak solutions

Let I be an open interval containing the origin and choose a bounded domain $\omega\subset\mathbf{R}^N$ with smooth boundary $\Gamma=\partial\omega$. For any bounded domain Ω in $\mathbf{R}^N$ such that $\Gamma\subset\Omega$ we set

$$\Omega^i = \Omega^c\cap\omega,$$

$$\Omega^e = \Omega^c\setminus\omega.$$

Thus if Ω is doubly-connected and ω contains the "hole" in Ω then Ω^i and Ω^e will be the bounded and unbounded components of Ω^c respectively, as in § 1. In general, ω has the role of dividing the components of Ω^c into two groups, the interior and the exterior ones.

In view of Proposition 1.1 the following definition seems reasonable. A family $\{\Omega(t)\colon t\in I\}$ of bounded domains in $\mathbf{R}^N$ such that $\Gamma\subset\Omega(t)$ for all $t\in I$ is called a *weak solution* of our problem if, for every $\tau, t\in I$, $\tau<t$, there exists a function $v=v_{\tau,t}\in C^1(\mathbf{R}^N)$ satisfying

$$\chi_{\Omega(t)}-\chi_{\Omega(\tau)} = \Delta v, \tag{2.1}$$

$$0\leqq v\leqq t-\tau, \tag{2.2}$$

$$v = \begin{cases} t-\tau & \text{in} \quad \Omega(\tau)^i, \\ 0 & \text{in} \quad \Omega(t)^e. \end{cases} \tag{2.3}$$

Example 2.1. The solutions (1.5) in Example 1.1 are, strictly speaking, not weak solutions since for technical reasons weak solutions can be defined only on small intervals I (so that $\Gamma\subset\Omega(t)$ for all $t\in I$). However, restricted to suitable small time intervals (containing $t=0$) they obviously are weak solutions if ω is chosen so that it covers the "holes" (otherwise they will not be weak solutions). Moreover, they can be extended as weak solutions for $t\leqq t_0$ by

$$\Omega(t) = \{x\in\mathbf{R}^N\colon\ |x|^N < A\}. \tag{2.4}$$

Here $\Omega(t_0)$ may be replaced by $\Omega(t_0)\setminus\{0\}$.

It follows that weak solutions are not uniquely determined by $\Omega(0)$ in general. In fact, choose $\Omega=\{x\in\mathbf{R}^N\colon |x|^N<A\}$ and $\omega=\{x\in\mathbf{R}^N\colon |x|^N<B\}$ for some $0<B<A$. Then we have the following sets of weak solutions $\{\Omega(t)\}$ with $\Omega(0)=\Omega$:

(i) $\Omega(t)$ defined by (2.4) for all $t\in\mathbf{R}$.

(ii) $\Omega(t)$ defined by (2.4) for $t\leqq t_0$ and by (1.5) for $t_0<t<t_0+\varepsilon$, where $t_0>0$ and $\varepsilon>0$ are sufficiently small.

(iii) As in (ii) but with $\Omega(t_0)$ replaced by $\Omega(t_0)\setminus\{0\}$.

Although this example is very special (e.g. in the sense that $\Omega(0)^c$ is connected) the phenomenon of creation of new holes (in $\Omega(t)\cap\omega$ as t increases) is completely general. See Remark 2.1. □

In this section we give a fairly elementary proof of the existence of a weak solution with $\Omega(0)$ prescribed ($\partial\Omega(0)$ analytic) for I small. We also prove some regularity and uniqueness results.

Lemma 2.1. *Any weak solution is monotone in the sense that, for* $\tau<t$,

$$\Omega(\tau)^i \subset \Omega(t)^i, \quad \Omega(\tau)^e \supset \Omega(t)^e \quad \text{a.e.}$$

(the latter means that $|\Omega(t)^e \setminus \Omega(\tau)^e|=0$). If $\Omega(\tau)^i \neq \emptyset$ then actually $\Omega(\tau)^e \supset \Omega(t)^e$.

Proof. In $\Omega(t)$ $\Delta v \geqq 0$. Thus either $v<t-\tau$ in $\Omega(t)$ or $v\equiv t-\tau$ in $\Omega(t)$. But the latter alternative cannot occur since v is continuous and $v=0$ in $\Omega(t)^e \neq \emptyset$. Thus $v<t-\tau$ in $\Omega(t)$ showing that $\Omega(t)\cap\Omega(\tau)^i=\emptyset$, hence that $\Omega(\tau)^i \subset \Omega(t)^i$. Similarly, $\Delta v \leqq 0$ in $\Omega(\tau)$ implies that either $v>0$ in $\Omega(\tau)$ or $v\equiv 0$ in $\Omega(\tau)$. In the first case we have $\Omega(\tau)\cap\Omega(t)^e=\emptyset$, i.e. $\Omega(t)^e \subset \Omega(\tau)^e$.

Since v is continuous the second case can occur only if $\Omega(\tau)^i=\emptyset$. Then $v=0$ in $\omega\subset\Omega(\tau)^i\cup\Omega(\tau)$ so that $v=0$ on all $\partial\Omega(t)$. Hence $v=0$ in all $\mathbf{R}^N$ since $\Delta v\geqq 0$ in $\Omega(t)$. Now (2.1) shows that $|\Omega(t)\triangle\Omega(\tau)|=0$. □

Proposition 2.1. *Let* $\{\Omega(t)\colon t\in I\}$ *be a family of bounded domains in* $\mathbf{R}^N$ *such that* $\Gamma\subset\Omega(t)$ *for all* $t\in I$. *Then* $\{\Omega(t)\colon t\in I\}$ *is a weak solution if and only if*

$$\int_{\Omega(t)} \varphi - \int_{\Omega(\tau)} \varphi \geqq (t-\tau)\int_\Gamma \frac{\partial\varphi}{\partial n}\,ds \tag{2.5}$$

for all $\varphi\in SL^1(\Omega(t))\cap -SL^1(\Omega(\tau))$ *and all* $\tau, t\in I$.

Proof. Suppose (2.5) and let

$$E(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{if } N=2 \\ \dfrac{C_N}{|x|^{N-2}} & \text{if } N\geqq 3 \end{cases}$$

so that $-\Delta E=\delta$. Set

$$v(x) = -\int_{\Omega(t)} E(x-y)\,dy + \int_{\Omega(\tau)} E(x-y)\,dy.$$

Then clearly $v\in C^1(\mathbf{R}^N)$ and $\Delta v=\chi_{\Omega(t)}-\chi_{\Omega(\tau)}$.

Since $\varphi(y)=-E(x-y)$ is allowed in (2.5) whenever $x\notin\Omega(\tau)$ and

$$-\int_{\partial\omega} \frac{\partial E(x-y)}{\partial n}\,ds(y) = \begin{cases} 1 & \text{if } x\in\omega \\ 0 & \text{if } x\notin\bar\omega \end{cases}$$

(2.5) shows that $v \geqq t-\tau$ in $\Omega(\tau)^i$ and $v \geqq 0$ in $\Omega(\tau)^e$. Similarly, $\varphi(y)=E(x-y)$ is allowed in (2.5) whenever $x \notin \Omega(t)$ showing that $v \leqq t-\tau$ in $\Omega(t)^i$, $v \leqq 0$ in $\Omega(t)^e$.

In $\Omega(t)$, $\Delta v \geqq 0$. Hence $\max_{\mathbf{R}^n} v = \max_{\Omega(t)^c} v \leqq t-\tau$ by the maximum principle and the above. Similarly, $\min_{\mathbf{R}^n} v = \min_{\Omega(\tau)^c} v \geqq 0$. Hence $0 \leqq v \leqq t-\tau$ in $\mathbf{R}^N$. Furthermore, we get $v=t-\tau$ in $\Omega(\tau)^i$ (since we had $v \geqq t-\tau$ in $\Omega(\tau)^i$ above) and $v=0$ in $\Omega(t)^e$ (similarly). Thus $\{\Omega(t)\}$ is a weak solution.

To prove the converse we shall use an approximation device due to Sakai (see e.g. [18, proof of Lemma 7.3]). Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of C^∞-functions in $\mathbf{R}^N$ such that $0 \leqq \psi_n \leqq 1$, $\psi_n = 0$ in a neighbourhood of $(\Omega(\tau) \cup \Omega(t))^c$, $\lim_{n\to\infty} \psi_n(x)=1$ for every $x \in \Omega(\tau) \cup \Omega(t)$ and

$$|D^\alpha \psi_n(x)| \leqq \frac{A_\alpha}{n\delta(x)^{|\alpha|} \log 1/\delta(x)} \tag{2.6}$$

for all $x \in \Omega(\tau) \cup \Omega(t)$ and all multi-index α. Here $\delta(x) = \operatorname{dist}(x, (\Omega(\tau) \cup \Omega(t))^c)$ and A_α are constants. The existence of such a sequence is proved in [10].

Suppose $\{\Omega(t)\}$ is a weak solution. Since the function v in (2.1)—(2.3) is a constant multiple of the Newtonian potential of $\chi_{\Omega(t)} - \chi_{\Omega(\tau)}$ elementary estimates show that

$$\nabla v(x) - \nabla v(y) = O\left(|x-y| \log \frac{1}{|x-y|}\right)$$

for $|x-y|$ small. Thus, since $v \in C^1(\mathbf{R}^N)$ and $v=0$ in $\Omega(t)^e$

$$\nabla v(x) = O\left(\delta(x) \log \frac{1}{\delta(x)}\right), \quad v(x) = O\left(\delta(x)^2 \log \frac{1}{\delta(x)}\right) \tag{2.7}$$

in a neighbourhood of $\partial\Omega(t)^e$ and, similarly,

$$\nabla(v(x)-(t-\tau)) = O\left(\delta(x) \log \frac{1}{\delta(x)}\right), \quad v(x)-(t-\tau) = O\left(\delta(x)^2 \log \frac{1}{\delta(x)}\right) \tag{2.8}$$

in a neighbourhood of $\partial\Omega(\tau)^i$.

(2.6)—(2.8) show that

$$\begin{aligned}
\lim_{n\to\infty} \left[\int_\omega \varphi \Delta(\psi_n(v-(t-\tau))) + \int_{\omega^c} \varphi \Delta(\psi_n v)\right] &= \lim_{n\to\infty} \left[\int_\omega \varphi\psi_n \Delta(v-(t-\tau)) + \int_{\omega^c} \varphi\psi_n \Delta v\right] \\
&+ \lim_{n\to\infty} \left[\int_\omega \varphi 2\nabla\psi_n \nabla(v-(t-\tau)) + \int_{\omega^c} \varphi 2\nabla\psi_n \nabla v\right] \\
&+ \lim_{n\to\infty} \left[\int_\omega \varphi \Delta\psi_n (v-(t-\tau)) + \int_{\omega^c} \varphi \Delta\psi_n v\right] \\
&= \int_{\Omega(\tau)\cap\omega} \varphi \Delta(v-(t-\tau)) + \int_{\Omega(t)\setminus\omega} \varphi \Delta v + 0 + 0 \quad \text{for} \quad \varphi \in L^1(\mathbf{R}^N).
\end{aligned}$$

Thus, for $\varphi\in SL^1(\Omega(t))\cap -SL^1(\Omega(\tau))$ and using also Lemma 2.1

$$\begin{aligned}
\int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi &= \int_{\Omega(\tau)\cup\Omega(t)}\varphi\Delta v = \int_{\Omega(\tau)\cap\omega}\varphi\Delta\big(v-(t-\tau)\big)+\int_{\Omega(t)\setminus\omega}\varphi\Delta v \\
&=\lim_{n\to\infty}\Big[\int_{\Omega(\tau)\cap\omega}\varphi\Delta\big(\psi_n(v-(t-\tau))\big)+\int_{\Omega(t)\setminus\omega}\varphi\Delta(\psi_n v)\Big] \\
&= \lim_{n\to\infty}\Big[\int_{\Omega(\tau)\cap\omega}\psi_n(v-(t-\tau))\Delta\varphi+\int_{\Omega(t)\setminus\omega}\psi_n v\Delta\varphi\Big] \\
&\quad+\lim_{n\to\infty}\Big[\int_{\partial\omega}\varphi\frac{\partial}{\partial n}\big(\psi_n(v-(t-\tau))\big)\,ds-\int_{\partial\omega}\varphi\frac{\partial}{\partial n}(\psi_n v)\,ds\Big] \\
&\quad-\lim_{n\to\infty}\Big[\int_{\partial\omega}\psi_n(v-(t-\tau))\frac{\partial\varphi}{\partial n}\,ds-\int_{\partial\omega}\psi_n v\frac{\partial\varphi}{\partial n}\,ds\Big] \\
&= \lim_{n\to\infty}\Big[\int_{\Omega(\tau)\cap\omega}\psi_n(v-(t-\tau))\Delta\varphi+\int_{\Omega(t)\setminus\omega}\psi_n v\Delta\varphi\Big] \\
&\quad+(t-\tau)\int_{\partial\omega}\frac{\partial\varphi}{\partial n}\,ds \geqq (t-\tau)\int_{\Gamma}\frac{\partial\varphi}{\partial n}\,ds,
\end{aligned}$$

completing the proof of the proposition. □

Theorem 2.1. *Let Ω be a bounded domain in $\mathbf{R}^N$ with $\partial\Omega$ real analytic. Choose a bounded domain ω with smooth boundary $\Gamma=\partial\omega$ such that $\Gamma\subset\Omega$. Then there exists a weak solution $\{\Omega(t)\colon t\in I\}$ (I some open interval containing the origin) with $\Omega(0)=\Omega$.*

Proof. Let $B\subset\mathbf{R}^N$ be a large ball so that (in particular) $\Omega\subset\subset B$. We shall work in the Sobolev spaces $H_0^1(B)$ and $H^{-1}(B)$. $H_0^1(B)$ is provided with the inner product

$$(u, v) = \int_B \nabla u\cdot\nabla v$$

and the corresponding norm $\|u\|=\sqrt{(u, u)}$. $H^{-1}(B)$ is provided with that inner product and norm (also denoted $(\cdot,\cdot)$ and $\|\cdot\|$ respectively) which make the Laplacian operator $\Delta\colon H_0^1(B)\to H^{-1}(B)$ an isometric isomorphism.

Let $F\colon H^{-1}(B)\to H^{-1}(B)$ denote the orthogonal projection onto the closed and convex set $\{f\in H^{-1}(B)\colon f\leqq 1\}$. As is easily seen (see [9] for details) F can also be described as

$$F(f) = f+\Delta u \tag{2.9}$$

where $u\in H_0^1(B)$ is the unique solution of the complementary problem

$$\begin{cases}\Delta u+f\leqq 1,\\ u\geqq 0,\\ \langle\Delta u+f-1, u\rangle = 0.\end{cases} \tag{2.10}$$

Here $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $H^{-1}(B)$ and $H_0^1(B)$. (2.10) is equivalent to the variational inequality $u\in K=\{v\in H_0^1(B)\colon f+\Delta v\leqq 1\}$, $(u, v-u)\geqq 0$ for all $v\in K$. The latter is the variational formulation of the minimum norm problem $u\in K$, $\|u\|\leqq\|v\|$ for all $v\in K$.

If, in (2.9), (2.10), $f\in L^\infty(B)$ and there exists a domain $D\subset\subset B$ such that $f\geqq 1$ in D, $f=0$ outside D and $\int_D f>|D|$ then

$$F(f)=\chi_\Omega, \tag{2.11}$$

where $\Omega=\{x\in B\colon u(x)>0\}$ (u is a continuously differentiable function here, since $f\in L^\infty$). Moreover, Ω is connected and, if B is large enough, $\Omega\subset\subset B$. In that case Ω does not depend upon B. See e.g. [9] for the above matters.

Now, with Ω and Γ as in the theorem, choose a neighbourhood G of Γ such that $G\subset\subset\Omega$ and choose $w\in C_0^\infty(B)$ such that $0\leqq w\leqq 1$ and

$$w=\begin{cases}0 & \text{in } G^e,\\ 1 & \text{in } G^i.\end{cases}$$

We claim, first of all, that there exists a domain D, $G\subset\subset D\subset\subset\Omega$, with smooth boundary and a function $\varrho\in L^\infty(B)$ with

$$\begin{cases}\varrho=0 & \text{in } B\setminus D,\\ \varrho\geqq 1+\varepsilon\chi_G & \text{in } D\end{cases} \tag{2.12}$$

(for some $\varepsilon>0$) satisfying

$$F(\varrho)=\chi_\Omega. \tag{2.13}$$

More precisely, it is required that

$$\Omega=\{x\in B\colon u(x)>0\}, \tag{2.14}$$

where u is the function occurring in the definition of F.

Secondly we claim that, for $|t|$ small enough,

$$F(\varrho+t\Delta w)=\chi_{\Omega(t)}, \tag{2.15}$$

where $\Omega(t)=\{x\in B\colon u_t(x)>0\}$ are the desired domains.

To prove the first claim we construct D and ϱ and a function $u\in H_0^1(B)$ satisfying $F(\varrho)=\varrho+\Delta u$ as follows. Set $u=0$ in $B\setminus\Omega$ and continue u into Ω by solving the (ill-posed) Cauchy problem

$$\begin{cases}\Delta u=1 \text{ in some neighbourhood, in } \Omega, \text{ of } \partial\Omega,\\ u=0,\ \nabla u=0 \text{ on } \partial\Omega.\end{cases} \tag{2.16}$$

The analyticity of $\partial\Omega$ guarantees, by the Cauchy—Kovalevskaya theorem, that (2.16) has a solution u. Clearly, u will be (strictly) positive in some neighbourhood (in Ω) of $\partial\Omega$.

Choose a domain D with analytic boundary and $G \subset\subset D \subset\subset \Omega$ such that u is defined by the above and positive in $\Omega \setminus D$ and constant on ∂D. Finally, continue u as a bounded positive function into all of D such that $\Delta u \in L^\infty(B)$, u is smooth in D and such that, for some $\varepsilon > 0$, $\Delta u \leqq -\varepsilon \chi_G$ in D. All this is easily seen to be possible. See [9, proof of Lemma 2] for more details.

Set $\varrho = \chi_\Omega - \Delta u$. Then D and ϱ satisfy (2.12) and $u \in H_0^1(B)$ satisfies $\Delta u + \varrho = \chi_\Omega$, $u > 0$ in Ω and $u = 0$ in $B \setminus \Omega$, showing (by (2.10), (2.11)) that (2.13), (2.14) hold.

To prove the second claim (2.15) just define $\Omega(t)$ by

$$\Omega(t) = \{x \in B \colon u_t(x) > 0\},$$

where u_t is the function u occurring in (2.9) for $f = f_t = \varrho + t\Delta w$. Then $f = 0$ outside D and, for $|t| < \delta := \varepsilon / \|\Delta w\|_\infty$, $f \geqq 1$ on D and $\int_D f > |D|$. Thus it follows from (2.11) that $\Omega(t)$ is a domain satisfying $F(\varrho + t\Delta w) = \chi_{\Omega(t)}$ for $|t| < \delta$. Also, by (2.14) $\Omega(0) = \Omega$.

We now assume that B was chosen so large so that $\Omega(t) \subset\subset B$ for $|t| < \delta$. (It is easy to estimate apriori a sufficient radius for B.)

Set, for $\tau < t$, $|\tau|, |t| < \delta$ $v = v_{\tau,t} = u_t - u_\tau + (t - \tau) w$. Then $v \in H_0^1(B)$ and

$$\chi_{\Omega(t)} - \chi_{\Omega(\tau)} = \varrho + t\Delta w + \Delta u_t - (\varrho + \tau \Delta w + \Delta u_\tau) = \Delta v,$$

proving (2.1).

(2.2) follows easily from the maximum/minimum principles, using that u_τ, $u_t \geqq 0$ and $u_\tau = 0$ in $B \setminus \Omega(\tau)$, $u_t = 0$ in $B \setminus \Omega(t)$. Infact, we have $v \leqq t - \tau$ in $B \setminus \Omega(t)$, $v = 0$ on ∂B and $\Delta v \geqq 0$ in $\Omega(t)$ showing that $v \leqq t - \tau$ in B. Similarly, $v \geqq 0$ in B.

In $\Omega(\tau)^i$ $u_\tau = 0$ so that $v \geqq t - \tau$ there. Similarly, $v \leqq 0$ in $\Omega(t)^e$. Hence (2.3) follows, using (2.2).

Observe finally that $v = 0$ in a neighbourhood of ∂B since $\Omega(t) \subset\subset B$ so that v extends to a function in $C^1(\mathbf{R}^N)$ by setting $v = 0$ outside B. This finishes the proof of the theorem. □

Remark 2.1. Theorem 2.1 also applies when $\partial\Omega$ contains certain kinds of singular points, e.g. when some of the (finitely many) components of $\partial\Omega$ consist of just one point (and the other are real analytic curves). Infact, the hypotheses on $\partial\Omega$ are needed only to guarantee the existence of a solution, positive in Ω, of (2.16) and if $y \in \partial\Omega$ is an isolated point in $\partial\Omega$ then, in a neighbourhood of y, (2.16) has the solution $u(x) = (1/2N)|x-y|^2$. As an example, take $\Omega = \{x \in \mathbf{R}^N \colon 0 < |x|^N < A\}$ and $\omega = \{x \in \mathbf{R}^N \colon |x|^N < B\}$ where $0 < B < A$. Then Theorem 2.1 will produce the weak solution (iii) in Example 2.1 (with $t_0 = 0$). In general, if $y \in \partial\Omega$ is an isolated point in $\partial\Omega$ and $y \in \omega$ then, if $\{\Omega(t)\}$ is a solution produced by Theorem 2.1, $y \in \Omega(t)$ for $t < 0$ and $y \in \Omega(t)^i$ for $t \geqq 0$. If $y \notin \bar{\omega}$ then $y \in \Omega(t)^e$ for $t \leqq 0$ and $y \in \Omega(t)$ for $t > 0$ (assuming that $\Omega(t)^i \neq 0$).

The above shows that weak solutions are far from being unique (given $\Omega(0)$). For given a weak solution $\{\Omega(t): t\in I\}$ we can initiate the creation of a new hole at an arbitrary point $y\in\Omega(t_0)\cap\omega$ for $t=t_0>0$ by applying Theorem 2.1 with $\Omega'=\Omega(t_0)\setminus\{y\}$ as initial domain. Then we get a weak solution $\{\Omega'(t)\}$ with the property that $\Omega'(t-t_0)=\Omega(t)$ for $t<t_0$ but not for $t\geqq t_0$. Similar statements, with reversed inequalities, hold if $y\in\Omega(t_0)\setminus\bar{\omega}$.

This phenomenon, the possibility of spontaneous creation of new holes in $\Omega(t)\cap\omega$ as t increases (and in $\Omega(t)\setminus\bar{\omega}$ as t decreases) when $\{\Omega(t)\}$ is a weak solution, is an indication of extreme unstability of $\partial\Omega(t)^i$ as t increases (and of $\partial\Omega(t)^e$ as t decreases). It perhaps also suggests that our notion of weak solution is "too weak". □

Remark 2.2. In Theorem 2.1 $\Omega(t)$ is obtained as the non-coincidence set for the variational inequality (in complementarity form)

$$\begin{cases} \Delta u \leqq 1-\varrho-t\Delta w, \\ u \geqq 0, \\ \langle \Delta u-1+\varrho+t\Delta w, u\rangle = 0 \end{cases} \tag{2.17}$$

$(u=u_t\in H_0^1(B))$. It may be interesting to see what (2.17) means when interpreted as an obstacle problem.

Define an obstacle function $\psi\in H_0^1(B)$ by $\Delta\psi=\varrho-1$ (corresponding to $t=0$) and let (for arbitrary t) $v=u+\psi+tw$ $(v=v_t\in H_0^1(B))$. Then (2.17) is equivalent to

$$\begin{cases} \Delta v \leqq 0, \\ v \geqq \psi+tw, \\ \langle \Delta v, v-\psi-tw\rangle = 0 \end{cases} \tag{2.18}$$

and $\Omega(t)=\{x\in B: v>\psi+tw\}$. (2.18) is the variational inequality (in complementarity form) of the usual obstacle problem with obstacle function $\psi+tw$ and $\Omega(t)$ is the non-coincidence set for its solution.

In view of our particular forms of ψ and w the picture will be that of Fig. 2 (one-dimensional section). The middle hump is raised (lowered) according as t increases (decreases).

Theorem 2.2. *The weak solution constructed in Theorem 2.1 has the following additional properties.*

(i) $0<v<t-\tau$ *in* $\Omega(\tau)\cup\Omega(t)(\tau<t)$ *unless* $v\equiv 0$. *In the latter case* $\Omega(\tau)^i=\emptyset$, $\Omega(t)\subset\Omega(\tau)$ *and* $|\Omega(\tau)\setminus\Omega(t)|=0$.

(ii) $\partial\Omega(t)$ *is real analytic for* $|t|$ *sufficiently small.*

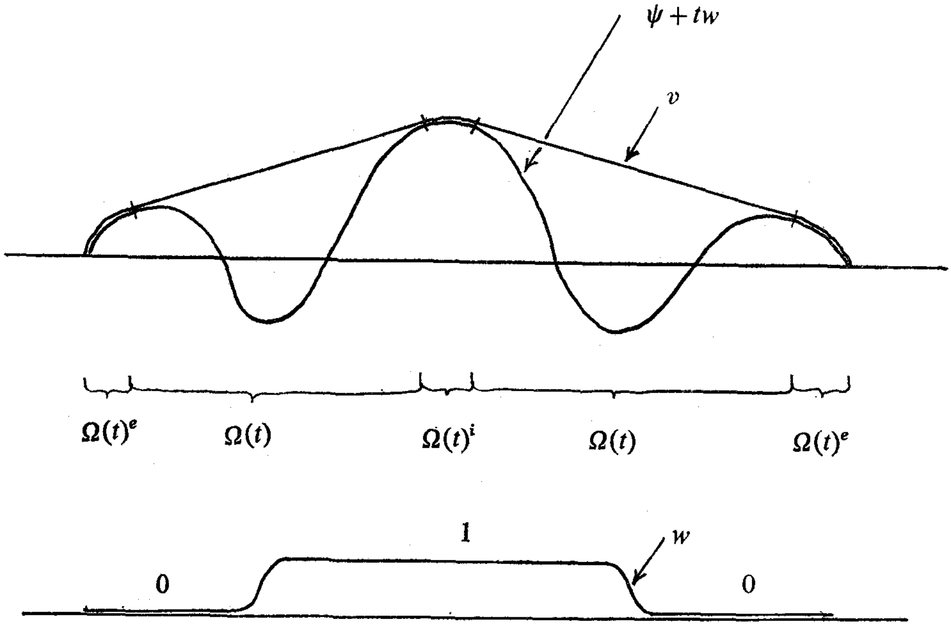

Fig. 2

(iii) $\Omega(t)$ *depends continuously on t e.g. in the following senses.*

$$\|\chi_{\Omega(t)}-\chi_{\Omega(\tau)}\|_{H^{-1}(B)} \leqq C|t-\tau|$$

if B is any ball containing $\overline{\Omega(t)}$ *for all* $t\in I$;

$$\|\chi_{\Omega(t)}-\chi_{\Omega(\tau)}\|_{L^p(\mathbf{R}^N)} \to 0 \quad as \quad t\to\tau \quad for\ all \quad p<\infty.$$

(iv) $|\Omega(t)^i|>0$ *for* $|t|$ *small enough. In particular* $\Omega(t)$ *has connectivity at least two (for* $|t|$ *small).*

Proof. (i) For a weak solution in general we have $v<t-\tau$ in $\Omega(t)$ and, unless $v\equiv 0$, $v>0$ in $\Omega(\tau)$. In fact, by the definition of a weak solution $\Omega(t)$ is connected, $v\leqq t-\tau$ and $\Delta v\geqq 0$ in $\Omega(t)$. Hence either $v\equiv t-\tau$ in $\Omega(t)$ or $v<t-\tau$ in $\Omega(t)$. But the first alternative is impossible since v is continuous and $v=0$ in $\Omega(t)^e$. Hence $v<t-\tau$ in $\Omega(t)$. Similarly, $v>0$ in $\Omega(\tau)$ if $v\not\equiv 0$.

If now $v=u_t-u_\tau+(t-\tau)w$ as in the proof of Theorem 2.1 we also have $v=-u_\tau+(t-\tau)w<t-\tau$ in $\Omega(\tau)\setminus\Omega(t)$ since $u_t=0$ outside $\Omega(t)$, $u_\tau>0$ in $\Omega(\tau)$ and $w\leqq 1$. Thus $v<t-\tau$ in all $\Omega(\tau)\cup\Omega(t)$. Similarly, $v>0$ in $\Omega(t)\setminus\Omega(\tau)$. Hence $v>0$ in all $\Omega(\tau)\cup\Omega(t)$ unless $v\equiv 0$. In the latter case necessarily $\Omega(t)\setminus\Omega(\tau)=\emptyset$ and, since $|\Omega(t)|=|\Omega(\tau)|$ by Proposition 2.1, $|\Omega(\tau)\setminus\Omega(t)|=0$.

(ii) We have $\Omega(t)=\{x\in B\colon u(x)>0\}\subset\subset B$, where $u\in H_0^1(B)$ solves

$$\begin{cases}\Delta u \leqq f,\\ u \geqq 0,\\ \langle \Delta u - f, u\rangle = 0,\end{cases}$$

with $f=1-\varrho-t\Delta w$ (and $\varrho+t\Delta w\geqq 1$ in D, $=0$ in $B\setminus D$, $|t|<\delta$). Thus $u>0$ and $\Delta u\leqq 0$ in $D\subset\Omega(t)$. From this and the fact that ∂D is smooth (real analytic) and f is smooth in D it follows that $D\subset\subset\Omega(t)$. Indeed, $u=0$, $\nabla u=0$ in $B\setminus\Omega(t)$ (the latter because $u\geqq 0$, $u\in C^1(B)$) so at a point $x\in\partial\Omega(t)\cap\partial D$ we would have a contradiction to the Hopf maximum principle (applied to $-u$ in D). Hence $D\subset\subset\Omega(t)\subset\subset B$ for $|t|<\delta$.

In particular, $f=1$ in a neighbourhood of $\partial\Omega(t)$ Also, by the $W^{2,\infty}$-regularity of solutions of variational inequalities [22], [23], [7] $u\in C^{1,1}$ in a neighbourhood of $\partial\Omega(t)$. Finally, recall that $\partial\Omega(0)$ is smooth (real analytic).

Now a stability result of Caffarelli [2, Corollary 6] (see also [7, Ch 2 § 10]) can be applied to show that $\partial\Omega(t)$ is smooth (e.g. C^1) for $|t|$ sufficiently small and then also the analyticity of $\partial\Omega(t)$ follows (since f is real analytic in a neighbourhood of $\partial\Omega(t)$) as in [12] or [7, Ch 2 § 1].

(iii) The first estimate follows from (2.15) and the fact that F is an orthogonal projection in $H^{-1}(B)$: $\|\chi_{\Omega(t)}-\chi_{\Omega(\tau)}\|\leqq\|\varrho+t\Delta w-(\varrho+\tau\Delta w\|=|t-\tau|\ \|\Delta w\|_{H^{-1}(B)}$. The second assertion follows from [17, Thm 2].

(iv) It follows from (iii), choosing the L^1-norm, that $|\Omega(t)^i|$ depends continuously on t. Since $|\Omega(0)^i|>0$ the conclusion follows. □

Remark 2.3. Probably the solution constructed in Theorem 2.1 is actually a classical solution for $|t|$ small enough. One reason for believing that is the fact that, under certain assumptions, obstacle problems with smooth data in a neighbourhood of the data of a known solution for which the boundary of the coincidence set is a smooth hypersurface can be solved also by means of the Nash—Moser implicit function theorem. See [19]. This kind of method gives additional information on the smoothness of the solution and applied to our variational inequality (e.g. (2.17)) it should show that $\{\Omega(t)\}$ actually is a classical solution for $|t|$ small. □

It is obvious from the definitions that a classical solution always is a weak solution when restricted to a sufficiently small interval and with ω chosen in the obvious way. However, weak solutions (with $\Omega(0)$ prescribed) are not unique (Remark 2.1) and it is not immediately clear that, if a classical solution (with $\Omega(0)$ prescribed) exists, the weak solution constructed in Theorem 2.1 really coincides with this classical solution. The next theorem shows at least that if the weak solution constructed in Theorem 2.1 turns out to be doubly-connected (which is very plausible for $|t|$ small, by Remark 2.3) then it is the only possible classical solution. The theorem is related to [18, Cor. 4.8].

Theorem 2.3. *Let* $\{\Omega'(t)\colon t\in I\}$ *be a classical solution and let* $\{\Omega(t)\colon t\in I\}$ *be a weak solution with* $\Omega(0)=\Omega'(0)$ *and constructed as in the proof of Theorem* 2.1 *with* $\omega\supset\Omega'(0)^i$. *Then, for each particular* $t\in I$ *small enough, either* $\Omega'(t)=\Omega(t)$ *or* $\partial\Omega'(t)\subset\Omega(t)$. *In particular,* $\Omega'(t)=\Omega(t)$ *if* $\Omega(t)$ *is doubly-connected (in such a way that* $\Omega(t)^i$ *and* $\Omega(t)^e$ *are both connected).*

Proof. Let B, D, ϱ, w and u_t be as in the construction of $\{\Omega(t)\}$ in the proof of Theorem 2.1. Thus

$$\begin{aligned} F(\varrho+t\Delta w) = \varrho+t\Delta w+\Delta u_t &= \chi_{\Omega(t)}, \\ u_t > 0 \quad &\text{in} \quad \Omega(t), \\ u_t = 0 \quad &\text{in} \quad B\setminus\Omega(t). \end{aligned} \tag{2.19}$$

We first consider the case that $t>0$ and assume that $|t|$ is so small so that $D\subset\subset \Omega'(t)\subset\subset B$. From Proposition 1.1 we obtain $v=v_{0,t}\in C^1(\mathbf{R}^N)$ such that

$$\begin{gathered} \chi_{\Omega'(t)}-\chi_{\Omega'(0)} = \Delta v, \\ 0\leqq v\leqq t, \\ v=0 \quad \text{in} \quad \Omega'(t)^e, \\ v=t \quad \text{in} \quad \Omega'(0)^i. \end{gathered}$$

Set $u'_t=v+u_0-tw$. Then, by (2.19) for $t=0$ and $\Omega'(0)=\Omega(0)$,

$$\Delta u'_t = \chi_{\Omega'(t)}-\varrho-t\Delta w. \tag{2.20}$$

Further, since $u_0=0$ outside $\Omega(0)=\Omega'(0)$, $\Omega'(t)^e\subset\Omega'(0)^e$ and $v-tw=0$ in $\Omega'(0)^i\cup \Omega'(t)^e$

$$u'_t=0 \quad \text{in} \quad \Omega'(0)^i\cup\Omega'(t)^e.$$

Even more, we get

$$u'_t=0 \quad \text{in} \quad \Omega'(t)^c=\Omega'(t)^i\cup\Omega'(t)^e, \tag{2.21}$$

since $\Omega'(t)^i$ is the closure of its interior which is a (connected) domain containing $\Omega'(0)^i$ (which has nonempty interior) and, by (2.20), $\Delta u'_t=0$ in $\operatorname{int}(\Omega'(t)^i)$.

Now consider $\varphi=u'_t-u_t=v-tw-u_t+u_0$ in $\Omega'(t)$. Since, by (2.19), (2.20), $\Delta\varphi=\chi_{\Omega'(t)}-\chi_{\Omega(t)}$

$$\Delta\varphi\geqq 0 \quad \text{in} \quad \Omega'(t).$$

On $\partial\Omega'(t)$ $u'_t=0$ and $\nabla u'_t=0$ by (2.21) and using that $u'_t\in C^1$ and that $\Omega'(t)^c$ is the closure of its interior. Since $u_t>0$ in $\Omega(t)$ and $u_t=0$, $\nabla u_t=0$ in $\Omega(t)^c$ (the last assertion depends on $u_t \geqq 0$, $u_t \in C^1$), we obtain

$$\varphi < 0 \quad \text{on} \quad \partial\Omega'(t) \cap \Omega(t),$$

$$\begin{cases} \varphi = 0 \\ \nabla\varphi = 0 \end{cases} \text{on} \quad \partial\Omega'(t) \setminus \Omega(t).$$

Suppose now that $\partial\Omega'(t)\setminus\Omega(t)$ is non-empty. Then it follows from the Hopf maximum principle that $\varphi \equiv 0$ in $\Omega'(t)$. (The fact the φ is not necessarily of class C^2 in $\Omega'(t)$ as required in the maximum principle can easily be handled by considering, instead of φ, the harmonic function ψ in $\Omega'(t)$ with $\psi=\varphi$ on $\partial\Omega'(t)$; then $\psi \geqq \varphi$ and hence $\partial\psi/\partial n \leqq \partial\varphi/\partial n$ at any point of $\partial\Omega'(t)\setminus\Omega(t)$.)

Thus $\partial\Omega'(t)\cap\Omega(t)=\emptyset$, i.e. $\partial\Omega'(t)\subset\Omega(t)^c$, i.e., since $\Omega(t)$ is connected and $\Omega'(t)\cap\Omega(t)\supset D\neq\emptyset$, $\Omega(t)\subset\Omega'(t)$. Finally, $\Omega(t)=\Omega'(t)$ since $\partial\Omega(t)$ is smooth (Theorem 2.2) and $|\Omega(t)|=|\Omega'(t)|$ (choose $\varphi=\pm 1$ and $\tau=0$ in Propositions 1.1 and 2.1).

To prove the final claim suppose that $\partial\Omega'(t)\subset\Omega(t)$. Then $\Omega(t)^i\subset\Omega'(t)^i$ by the connectedness of $\Omega(t)^i$ and the fact that $\Omega(t)^i\cap\Omega'(t)^i\supset\Omega(0)^i\neq\emptyset$. Similarly, $\Omega(t)^e\subset\Omega'(t)^e$. It follows that $\Omega'(t)\subset\subset\Omega(t)$ which contradicts $|\Omega'(t)|=|\Omega'(0)|=|\Omega(t)|$.

The case $t<0$ is handled similarly. In the final part one can use the continuity of $|\Omega(t)^i|$ ((iii) of Theorem 2.2) to infer that $\Omega(t)^i\cap\Omega'(t)^i\neq\emptyset$ (the argument for $t>0$ does not work for $t<0$). □

3. Generalizations

Theorem 2.1 generalizes without troubles to domains Ω of arbitrary finite connectivity as follows.

Theorem 3.1. *Let $\Omega\subset\mathbf{R}^N$ be a bounded domain of finite connectivity $m+1$ ($m\geqq 1$) and with real analytic boundary. Let $\omega_1, \ldots, \omega_m$ be bounded domains in $\mathbf{R}^N$ with smooth boundaries $\Gamma_i=\partial\omega_i$ such that $\{\overline{\omega}_i\}$ are pairwise disjoint, $\Gamma_i\subset\Omega$ and ω_i contains the i: th bounded component of Ω^c for each $i=1, \ldots, m$. Then there is a neighbourhood I of the origin in $\mathbf{R}^N$ and a family $\{\Omega(t)\colon t=(t_1, \ldots, t_m)\in I\}$ of perturbations of Ω satisfying $\Gamma_i\subset\Omega(t)$ for all i and t, $\Omega(0)=\Omega$ and*

$$\int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi = \sum_{i=1}^N (t_i-\tau_i)\int_{\Gamma_i}\frac{\partial\varphi}{\partial n}ds$$

for every φ harmonic and integrable in $\Omega(\tau)\cup\Omega(t)$ ($t=(t_1, \ldots, t_m)$, $\tau=(\tau_1, \ldots, \tau_m)$).

In particular

$$\int_{\Omega(\tau)}\varphi = \int_{\Omega(t)}\varphi$$

if φ is a component of a harmonic vector field or, in the case $N=2$, φ is an analytic function (defined and integrable in $\Omega(\tau)\cup\Omega(t)$).

Remark 3.1. This theorem is related to some matters in [18, § 5]. See also [8, § 7].

Proof. Choose small (and disjoint) neighbourhoods G_j of Γ_j and functions $w_j\in C_0^\infty(\mathbf{R}^N)$ satisfying $w_j=0$ and 1 in $G_j^c\setminus\omega_j$ and $G_j^c\cap\omega_j$ respectively $(j=1, \ldots, m)$. Construct a domain D and $\varrho\in L^\infty(B)$ as in the proof of Theorem 2.1, now with $G=G_1\cup\ldots\cup G_m$. Thus $F(\varrho)=\chi_\Omega$. Now the desired domains $\Omega(t)=\Omega(t_1, \ldots, t_m)$ are obtained as

$$F\big(\varrho+\textstyle\sum_{j=1}^N t_j\Delta w_j\big)=\chi_{\Omega(t)},\quad \Omega(t)=\{x\in B\colon\ u_t(x)>0\}$$

for $|t|$ small enough and B large enough. (u_t is the function u in (2.9), (2.10) for $f=\varrho+\sum t_j\Delta w_j$.) Then, for (say) $\varphi\in H_0^1(B)$ harmonic in $\Omega(\tau)\cup\Omega(t)$

$$\begin{aligned}\int_{\Omega(t)}\varphi-\int_{\Omega(\tau)}\varphi &= \langle\chi_{\Omega(t)}-\chi_{\Omega(\tau)}, \varphi\rangle\\ &= \langle\varrho+\Sigma t_j\Delta w_j+\Delta u_t-(\varrho+\Sigma\tau_j\Delta w_j+\Delta u_\tau), \varphi\rangle\\ &= \langle u_t, \Delta\varphi\rangle-\langle u_\tau, \Delta\varphi\rangle+\Sigma(t_j-\tau_j)\langle\Delta w_j, \varphi\rangle = \Sigma(t_j-\tau_j)\int_{\Gamma_j}\frac{\partial\varphi}{\partial n}\,ds.\end{aligned}$$

For general $\varphi\in HL^1(\Omega(\tau)\cup\Omega(t))$ one has to use an approximation argument as in the proof of Proposition 2.1. □

Another generalization is that, in the classical formulation (1.2) of the problem, we may allow the velocities of the two components of $\partial\Omega(t)$ to be proportional to $-\partial u_{\Omega(t)}/\partial n$ with different constants of proportionality on the two components. More generally, we may replace (1.2) by

$$\frac{d}{dt}\int_{\Omega(t)}\varphi=-\int_{\partial\Omega(t)}\varphi\alpha\frac{\partial u_{\Omega(t)}}{\partial n}\,ds,$$

where $\alpha=\alpha(x)$ is a given smooth function in $\mathbf{R}^N$ satisfying $0<c_1\leqq\alpha\leqq c_2<\infty$. (1.4) then takes the form

$$\frac{d}{dt}\chi_{\Omega(t)}=\alpha\Delta u_{\Omega(t)}.$$

This generalization may be of significance e.g. in the electro-chemical interpretation of the problem since the material transported between the anode and the cathode may contribute differently to the volume change at the two electrodes. Also, part of the material may be transported away by the electrolyte.

For the Hele Shaw model the above generalization means that the distance between the two surfaces may be nonconstant (proportional to $1/\alpha$).

Our method of constructing weak solutions applies to this more general problem. In the definition of a weak solution (2.1) is changed to $\chi_{\Omega(t)}-\chi_{\Omega(\tau)}=\alpha\Delta v$, in the construction of weak solutions as in the proof of Theorem 2.1 the operator $F\colon H^{-1}(B)\to H^{-1}(B)$ now will the orthogonal projection onto $\{f\in H^1(B)\colon f\leqq 1/\alpha\}$ and the Cauchy problem to be solved initially will be $\Delta u=1/\alpha$ in some neighbourhood in $\Omega(0)$ of $\partial\Omega(0)$,

$$\begin{cases} u=0 \\ \nabla u=0 \end{cases} \text{ on } \partial\Omega(0). \tag{3.1}$$

The solvability of (3.1) (with $u>0$ in $\Omega(0)$) is a hypothesis on $\partial\Omega(0)$ which is necessary e.g. for a classical solution to exist. If α is real analytic in a neighbourhood of $\partial\Omega(0)$ this hypothesis reduces to that $\partial\Omega(0)$ shall be real analytic, as in Corollary 1.2 and Theorem 2.1.

As a final generalization, the concept of a weak solution also makes sense e.g. for time intervals of the kind

$$I=[0,\delta)=\{t\in\mathbf{R}\colon 0\leqq t<\delta\}\quad(\delta\geqq 0)\quad\text{or}\quad I=(-\delta,0]$$

and we then have the following result.

Theorem 3.2. *Let Ω be as in Theorem 2.1 except that we require only $\partial\Omega^i$ to be real analytic; about $\partial\Omega^e$ we make no regularity assumptions at all. Then there exists a weak solution $\{\Omega(t)\colon t\in I\}$ on some interval $I=[0,\delta)$ satisfying $\Omega(0)=\Omega$. Similarly for intervals $I=(-\delta,0]$ if instead $\partial\Omega^e$ is real analytic.*

Proof (outline): If $\partial\Omega^i$ is real analytic choose a domain E with $\omega\subset\subset E\subset\Omega\cup\Omega^i$ and with ∂E real analytic. Then $\Omega\cap E$ satisfies the hypotheses of Theorem 2.1. Choose ϱ and w as in the proof of Theorem 2.1 but now with $\Omega\cap E$ as initial domain. Thus $F(\varrho)=\chi_{\Omega\cap E}$. Define $\Omega(t)$ by

$$\Omega(t)=E\cup\{x\in B\colon u_t(x)>0\} \tag{3.2}$$

where u_t is the function u in (2.9) for $f=\varrho+\chi_{\Omega\setminus E}+t\Delta w$. The term E in (3.2) is needed only when $t=0$. Compare [9, Prop. 1 (h)]. Then $F(\varrho+\chi_{\Omega\setminus E}+t\Delta w)=\chi_{\Omega(t)}$ for $0\leqq t<\delta$, $\delta>0$ sufficiently small, and it is straightforward to check that $\{\Omega(t)\colon t\in[0,\delta)\}$ is the required weak solution. Similarly for $I=(-\delta,0]$ if $\partial\Omega^e$ is real analytic.

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