Partial regularity for minima of variational integrals

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The purpose of this paper is to study regularity properties of vector-valued functions minimizing variational integrals of the form

$$\mathcal{F}(u) = \int_{\Omega} F\big(x, u(x), Du(x)\big)\, dx$$

where Ω is a domain in $\mathbf{R}^n$ and $F(x, u, p)$ is a continuous function, convex in p and growing, for $|p|$ large, like $|p|^m$, $m \geqq 2$.

We will derive partial regularity, i.e. continuity except on a closed set of measure zero, for the derivatives of the minima of $\mathcal{F}$ under the assumption that F is twice continuously differentiable in p but only Hölder continuous in x and u, which means that the functional $\mathcal{F}$ is in general non-differentiable. This extends previous results of Giaquinta—Giusti [5] and Ivert [7], where the case $m=2$ is treated.

Although the techniques employed are much in the same spirit as the ones used in [5] and [7], the additional difficulties which arise for $m>2$ require some technical adjustments which may be of some independent interest.

Let us state our assumptions precisely:

General assumptions. *Let Ω be a domain in euclidean n-space $\mathbf{R}^n$, $n \geqq 3$, let N be a positive integer and let F: $\Omega \times \mathbf{R}^N \times \frown \mathbf{R}^{Nn}$ be a function satisfying for all $x, y \in \Omega$, $u, v \in \mathbf{R}^N$ and $p, q \in \mathbf{R}^{Nn}$:*

(i) $|p|^m \leqq F(x, u, p) \leqq c_0(1+|p|^2)^{m/2}$

(ii) $|F(x, u, p) - F(y, v, p)| \leqq c_0(1+|p|^2)^{m/2}(|x-y|^\sigma + |u-v|^\sigma)$

(iii) $|F_p(x, u, p)| \leqq c_0(1+|p|^2)^{(m-1)/2}$

(iv) $c_0^{-1}(1+|p|^2)^{(m-2)/2}|q|^2 \leqq F_{p^i_\alpha p^j_\beta}(x, u, p)\, q^i_\alpha q^j_\beta \leqq c_0(1+|p|^2)^{(m-2)/2}|q|^2$

(v) $|F_{pp}(x, u, p) - F_{pp}(x, u, q)| \leqq (1+|p|^2+|q|^2)^{(m-2)/2}\omega(|p-q|^2)$.

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Here m, σ *and* c_0 *are constants with* $m \geqq 2$, $0<\sigma\leqq 1$ *and* $c_0>0$, *and* ω *is a non-decreasing concave function on* $\mathbf{R}_+$ *with* $\lim_{t\to 0}\omega(t)=0$.

We use subscripts to denote differentiation, i.e.

$$F_{p_\alpha^i p_\beta^j} = \frac{\partial^2 F}{\partial p_\alpha^i \partial p_\beta^j},$$

and the summation convention is used, meaning that summation is to be understood over repeated indices, from 1 to n for Greek letters and from 1 to N for Latin letters.

Let $H^{1,m}(\Omega;\mathbf{R}^N)$ denote the Sobolev space of $\mathbf{R}^N$-valued functions in $L^m(\Omega)$, having first-order distributional derivatives in $L^m(\Omega)$.

We define the functional $\mathscr{F}\colon H^{1,m}(\Omega;\mathbf{R}^N)\curvearrowright\mathbf{R}$ by

$$\mathscr{F}(z) = \int_\Omega F(x, z(x), Dz(x))\,dx$$

and assume that u is a *local minimum* for $\mathscr{F}$, more precisely that

$$\mathscr{F}(u) \leqq \mathscr{F}(u+\varphi) \quad \text{for all} \quad \varphi\in H_0^{1,m}(\Omega;\mathbf{R}^N),$$

where $H_0^{1,m}$ denotes the closure in $H^{1,m}$ of the set of continuously differentiable functions with compact support.

In the following, the letter c will denote a constant, changing its value from time to time, but at each occurrence it will depend only on the parameters n, N, m and c_0, unless otherwise indicated. Moreover, we use the notation $B_r(x_0)$ for the ball in $\mathbf{R}^n$ with center at x_0 and radius r, and $(z)_{x_0,R}$ denotes the mean value $⨍_{B_R(x_0)} z(x)\,dx$ of the function z over the ball $B_R(x_0)$. We often write B_R and $(z)_R$ instead of $B_R(x_0)$ and $(z)_{x_0\,R}$ when there is no fear of confusion.

1. A special case

In this section we consider the case when the function F does not depend on x and u. We shall derive an estimate, Proposition 1.1, which is of essentially the same type as the one obtained in [6]. It gives partial regularity of the local minima in this case, and it will be useful when studying the general case. Because of the simple proof and of its interest by itself, we treat this case separately.

We thus assume, in this section, that $u\in H^{1,m}(\Omega;\mathbf{R}^N)$ satisfies

$$\int_\Omega F(Du(x))\,dx \leqq \int_\Omega F(Du(x)+D\varphi(x))\,dx \quad \text{for all} \quad \varphi\in H_0^{1,m}(\Omega;\,\mathbf{R}^N),$$

where the function $F(p)$ satisfies (i), (iii), (iv) and (v) of the General assumptions.

Fix a point $x_0 \in \Omega$ and a positive number $R < \operatorname{dist}(x_0, \partial\Omega)$. Put $p_0 = (Du)_{x_0, R}$. Define the function G: $\mathbf{R}^{Nn} \to \mathbf{R}$ to be the Taylor polynomial of second degree of F around p_0:

$$G(p) = F(p_0) + F_{p_\alpha^i}(p_0)(p_\alpha^i - p_{0_\alpha}^i) + \frac{1}{2} F_{p_\alpha^i p_\beta^j}(p_0)(p_\alpha^i - p_{0_\alpha}^i)(p_\beta^j - p_{0_\beta}^j).$$

Comparing with the Taylor expansion of F, we get from (v):

$$|F(p) - G(p)| \leqq c(1 + |p_0|^2 + |p|^2)^{(m-2)/2} |p - p_0|^2 \omega(|p - p_0|^2). \tag{1.1}$$

Now let $v \in H^{1,2}(B_{1/2R}(x_0); \mathbf{R}^N)$ be the unique function with

$$\int_{B_{1/2R}} G(Dv(x))\,dx \leqq \int_{B_{1/2R}} G(Dv(x) + D\varphi(x))\,dx \quad \text{for all} \quad \varphi \in H_0^{1,2}(B_{1/2R};\, \mathbf{R}^N)$$

$$v - u|_{B_{1/2R}} \in H_0^{1,2}(B_{1/2R};\, \mathbf{R}^N). \tag{1.2}$$

The existence and uniqueness of v follows from elementary Hilbert space theory. v is the solution of an elliptic second-order system with constant coefficients, and from the L^p-theory for elliptic systems we get the estimate

$$⨍_{B_{1/2R}} |Dv - p_0|^s\,dx \leqq c_s ⨍_{B_{1/2R}} |Du - p_0|^s\,dx \quad \text{if} \quad 1 < s < \infty.$$

Moreover it is not difficult to prove (see [2], [3] p. 78)

$$⨍_{B_\varrho} |Dv - (Dv)_\varrho|^k\,dx \leqq c_k \left(\frac{\varrho}{R}\right)^k ⨍_{B_{1/2R}} |Du - \lambda|^k\,dx \tag{1.3}$$

$$\text{for all} \quad \lambda \in \mathbf{R}^{Nn}, \quad \varrho < \tfrac{1}{2}R \quad \text{and} \quad k \geqq 2.$$

Now the function $h(t) = \int_{B_{1/2R}} F(Du + t(Dv - Du))\,dx$ is twice continuously differentiable and attains its minimum for $t = 0$, i.e.

$$\int_{B_{1/2R}} [F(Dv) - F(Du)]\,dx = h(1) - h(0) = \int_0^1 (1-t) h''(t)\,dt.$$

Performing the differentiations of h and considering (iv) of the General assumptions we get the estimate

$$⨍_{B_{1/2R}} (1 + |Du|^2 + |Dv|^2)^{(m-2)/2} |Du - Dv|^2\,dx \leqq c ⨍_{B_{1/2R}} [F(Dv) - F(Du)]\,dx.$$

To the integral on the right-hand side we add the integral $⨍_{B_{1/2R}} [G(Du) - G(Dv)]\,dx$, which is certainely nonnegative due to the minimizing property of v. Then we apply (1.1) and arrive at

$$\begin{aligned} &⨍_{B_{1/2R}} (1 + |Du|^2 + |Dv|^2)^{(m-2)/2} |Du - Dv|^2\,dx \\ &\leqq c ⨍_{B_{1/2R}} (1 + |p_0|^2 + |Dv|^2)^{(m-2)/2} |Dv - p_0|^2 \omega(|Dv - p_0|^2)\,dx \\ &+ c ⨍_{B_{1/2R}} (1 + |p_0|^2 + |Du|^2)^{(m-2)/2} |Du - p_0|^2 \omega(|Du - p_0|^2)\,dx. \end{aligned}$$

The first term on the right-hand side can be estimated by

$$c \fint_{B_{1/2R}} [(1+|p_0|^2)^{(m-2)/2}|Dv-p_0|^2+|Dv-p_0|^m]\omega(|Dv-p_0|^2)\,dx$$

$$\leqq c\Big\{\fint_{B_{1/2R}} [(1+|p_0|^2)^{(m-2)/2}|Dv-p_0|^2+|Dv-p_0|^m]^{n/(n-2)}\,dx\Big\}^{(n-2)/n}$$

$$\times\Big\{\fint_{B_{1/2R}} \omega(|Dv-p_0|^2)^{n/2}\,dx\Big\}^{2/n}$$

$$\leqq c\Big\{\fint_{B_{1/2R}} [(1+|p_0|^2)^{(m-2)/2}|Du-p_0|^2+|Du-p_0|^m]^{n/(n-2)}\,dx\Big\}^{(n-2)/n}$$

$$\times\Big\{\omega\Big(\fint_{B_{1/2R}} |Dv-p_0|^2\,dx\Big)\Big\}^{2/n}$$

$$\leqq c\Big\{\fint_{B_{1/2R}} [1+|p_0|^2+|Du|^2)^{(m-2)/2}|Du-p_0|^2]^{n/(n-2)}\,dx\Big\}^{(n-2)/n}$$

$$\times\Big\{\omega\Big(c\fint_{B_R} |Du-p_0|^2\,dx\Big)\Big\}^{2/n},$$

where we used the above-mentioned estimates for v as well as the boundedness and concavity of ω. The second term can be estimated in the same way, and thus

$$\fint_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}\,|Du-Dv|^2\,dx \tag{1.4}$$

$$\leqq c\omega\Big(c\fint_{B_R}|Du-p_0|^2\,dx\Big)^{2/n}\Big\{\fint_{B_{1/2R}}[(1+|p_0|^2+|Du|^2)^{(m-2)/2}|Du-p_0|^2]^{n/(n-2)}\,dx\Big\}^{(n-2)/n}$$

Now, since u satisfies the Euler equation

$$\int F_{p^i_\beta}(Du(x))D_\beta\varphi^i(x)\,dx = 0 \quad \text{for all} \quad \varphi\in H_0^{1,m}(\Omega),$$

it is easy to derive by the difference quotient method (see e.g. [8]) that it has weak second order derivatives, satisfying

$$\fint_{B_{1/2R}} (1+|Du|^2)^{(m-2)/2}|D^2u|^2\,dx \leqq cR^{-2}\fint_{B_R}(1+|Du|^2)^{(m-2)/2}|Du-p_0|^2\,dx. \tag{1.5}$$

We introduce the function $S(p)=(1+|p|^2)^{(m-2)/4}$, for which we have the elementary inequalities

$$2^{2-m}(1+|p|^2+|q|^2)^{(m-2)/2}|p-q|^2 \leqq |S(p)p-S(q)q|^2 \tag{1.6}$$

$$\leqq \frac{m^2}{4}(1+|p|^2+|q|^2)^{(m-2)/2}|p-q|^2.$$

Put $w(x)=S(Du(x))Du(x)$. It is easily seen that

$$|Dw(x)| \leqq \frac{m}{2}S(Du(x))|D^2u(x)|.$$

Now we can continue the estimate from (1.4), using the Sobolev—Poincaré inequality as well as the inequalities just mentioned:

$$\left\{\fint_{B_{1/2R}} [(1+|p_0|^2+|Du|^2)^{(m-2)/2}|Du-p_0|^2]^{n/(n-2)}\,dx\right\}^{(n-2)/n}$$

$$\leqq c\left\{\fint_{B_{1/2R}} |w(x)-S(p_0)p_0|^{2n/(n-2)}\,dx\right\}^{(n-2)/n}$$

$$\leqq c\left\{\fint_{B_{1/2R}} |w(x)-(w)_{1/2R}|^{2n/(n-2)}\,dx\right\}^{(n-2)/n}+c|(w)_{1/2R}-S(p_0)p_0|^2$$

$$\leqq cR^2\fint_{B_{1/2R}} |Dw|^2\,dx+c\fint_{B_{1/2R}} |S(Du(x))Du(x)-S(p_0)p_0|^2\,dx,$$

so, from (1.4), (1.5) and (1.6) we get

$$\fint_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\,dx \tag{1.7}$$

$$\leqq c\omega\left(c\fint_{B_R} |Du-(Du)_R|^2\,dx\right)^{2/n}\fint_{B_R} (1+|(Du)_R|^2+|Du|^2)^{(m-2)/2}|Du-(Du)_R|^2\,dx.$$

After these preparations, we are ready to prove

Lemma 1.1. *Let* $u\in H^{1,m}(\Omega;\mathbf{R}^N)$ *be a minimum for* $\int_\Omega F(Du(x))\,dx$, F *satisfying* (i), (iii), (iv) *and* (v) *of General assumptions. Put*

$$U(x_0,r)=\fint_{B_r(x_0)}|Du-(Du)_r|^2\,dx+\fint_{B_r(x_0)}|Du-(Du)_r|^m\,dx.$$

Then there is a constant A, depending only on n, N, m and c_0 and a bounded function $\tilde\omega$ with $\lim_{t\to 0+}\tilde\omega(t)=0$, *depending only on n, N, m, c_0 and ω, such that*

$$U(x_0,\varrho)\leqq A\left[\left(\frac{\varrho}{R}\right)^2+\left(\frac{R}{\varrho}\right)^n(1+|(Du)_R|^2)^{(m-2)/2}\tilde\omega\big(U(x_0,R)\big)\right]U(x_0,R)$$

if $0<\varrho<R<\operatorname{dist}(x,\partial\Omega)$.

Proof. Let v be the function defined by (1.2). Using the estimates (1.3) and (1.7) we get

$$U(x_0,\varrho)\leqq c\left[\fint_{B_\varrho}|Du-(Dv)_\varrho|^2\,dx+\fint_{B_\varrho}|Du-(Dv)_\varrho|^m\,dx\right]$$

$$\leqq c\left[\fint_{B_\varrho}|Dv-(Dv)_\varrho|^2\,dx+\fint_{B_\varrho}|Dv-(Dv)_\varrho|^m\,dx\right]$$

$$+c\left(\frac{R}{\varrho}\right)^n\left[\fint_{B_{1/2R}}|Du-Dv|^2\,dx+\fint_{B_{1/2R}}|Du-Dv|^m\,dx\right]$$

$$\leqq c\left(\frac{\varrho}{R}\right)^2U\left(x_0,\frac{1}{2}R\right)+c\left(\frac{R}{\varrho}\right)^n\fint_{B_{1/2R}}(1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\,dx$$

$$\leqq c\left(\frac{\varrho}{R}\right)^2U(x_0,R)+c\left(\frac{R}{\varrho}\right)^n\omega(cU(x_0,R))^{2/n}(1+|(Du)_R|^2)^{(m-2)/2}U(x_0,R),\qquad\text{q.e.d.}$$

Now we can state

Proposition 1.1. *Under the assumptions of Lemma 1.1 there is for each $L>0$ a positive number $\varepsilon=\varepsilon(L; n, N, m, c_0, \omega)$ such that if for some $x_0\in\Omega$ and some $R<\operatorname{dist}(x_0, \partial\Omega)$*

$$|(Du)_{x_0,R}| < L \quad \text{and} \quad U(x_0, R) < \varepsilon(L) \tag{1.8}$$

then, for all $\varrho\in(0, R)$

$$U(x_0, \varrho) \leqq B\left(\frac{\varrho}{R}\right)^{\mu} U(x_0, R) \tag{1.9}$$

where B and μ are positive constants depending only on n, N, m, c_0 and ω.

Proof. The proof proceeds by an elementary iteration argument, used many times, see e.g. [3], p. 174:

Fix a number $\tau\in(0, 1)$ such that $A\tau^2\leqq 1/4$, and let ε be such that

$$A\tau^{-n}(1+|\tilde{L}|^2)^{(m-2)/2}\tilde{\omega}(\varepsilon) \leqq \frac{1}{4}, \quad \text{where} \quad \tilde{L} = L+\frac{1+\tau^{-n/2}\sqrt{2}}{\sqrt{2}-1}\varepsilon^{1/2}.$$

A and $\tilde{\omega}$ are here taken from Lemma 1.1, which then tells us that

$$U(\tau^k R) \leqq \tfrac{1}{2}\, U(\tau^{k-1}R) \quad \text{if} \quad |(Du)_{\tau^{k-1}R}| \leqq \tilde{L} \quad \text{and} \quad U(\tau^{k-1}R) < \varepsilon.$$

Now, using the inequality

$$|(Du)_{\tau^k R}-(Du)_{\tau^{k-1}R}| \leqq U(\tau^k R)^{1/2}+\tau^{-n/2}U(\tau^{k-1}R)^{1/2},$$

it is easily seen by induction that if $|(Du)_R|\leqq L$ and $U(R)<\varepsilon$, then for every $k\in\mathbf{N}$ we have $|(Du)_{\tau^k R}|\leqq\tilde{L}$ and $U(\tau^k R)\leqq\frac{1}{2}U(\tau^{k-1}R)\leqq 2^{-k}U(R)$, which proves the proposition with $\mu=\log 2/\log 1/\tau$.

If (1.8) holds for some x_0 and R, then $|(Du)_{x,R}|<L$ and $U(x, R)<\varepsilon$ for all x in some neighbourhood of x_0, and thus, from Proposition 1.1, $U(x, \varrho)\leqq B(\varrho/R)^{\mu}\, U(x, R)$ for all x in this neighbourhood.

This implies, see Campanato [1], that Du is Hölder continuous in a neighbourhood of x_0. Now we note that (1.8) holds for almost all x_0 in Ω, so we can state

Theorem 1.1. *Let u be a minimum for the functional $\int_\Omega F(Du(x))\,dx$, F satisfying the General assumptions. Then there exists an open set Ω_0 with* meas $(\Omega\setminus\Omega_0)=0$, *such that u has Hölder continuous first-order derivatives in Ω_0.*

Actually, one can show that the Hausdorff dimension of $\Omega\setminus\Omega_0$ is less than $n-2$, see [6] and [3] for a discussion.

2. The general case

We now consider the general case $F=F(x,u,p)$ as it is described in the General assumptions. First we state our main theorem:

Theorem 2.1. *Let* $u\in H^{1,m}(\Omega;\mathbf{R}^N)$ *be a minimum for the functional* $\int_\Omega F(x,u(x),Du(x))\,dx$, *F satisfying the General assumptions. Then there exists an open set* $\Omega_0\subset\Omega$ *with* $\operatorname{meas}(\Omega\setminus\Omega_0)=0$, *such that u has Hölder continuous first-order derivatives in* Ω_0. *For the singular set* $\Omega\setminus\Omega_0$ *we have* $\Omega\setminus\Omega_0=\Sigma_1\cup\Sigma_2$, *where*

$$\Sigma_1=\Big\{x_0\in\Omega;\ \sup_{\varrho>0}\Big|\fint_{B_\varrho(x_0)}Du(x)\,dx\Big|=+\infty\Big\},$$

$$\Sigma_2=\Big\{x_0\in\Omega;\ \liminf_{\varrho\to 0}\fint_{B_\varrho(x_0)}|Du(x)-(Du)_{x_0,\varrho}|^m\,dx>0\Big\}.$$

Proof. We again fix a point $x_0\in\Omega$ and an $R<\operatorname{dist}(x_0,\partial\Omega)$. Put $u_0=\fint_{B_R(x_0)}u(x)\,dx$ and define the function $F^0\colon \mathbf{R}^{Nn}\to\mathbf{R}$ by $F^0(p)=F(x_0,u_0,p)$.

Let v be the unique function (for existence and uniqueness, see e.g. [8]) satisfying

$$\text{(2.1)}\qquad \int_{B_{1/2R}(x_0)}F^0(Dv(x))\,dx\le\int_{B_{1/2R}(x_0)}F^0(Dv(x)+D\varphi(x))\,dx$$

$$\text{for all}\quad \varphi\in H_0^{1,m}(B_{1/2R}(x_0);\ \mathbf{R}^N)$$

$$v-u|_{B_{1/2R}(x_0)}\in H_0^{1,m}(B_{1/2R}(x_0);\ \mathbf{R}^N).$$

As in the previous section, we get the estimate

$$\fint_{B_{1/2R}}(1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\,dx\le c\fint_{B_{1/2R}}[F^0(Du)-F^0(Dv)]\,dx$$

$$\le c\fint_{B_{1/2R}}[F^0(Du)-F(x,u,Du)+F(x,v,Dv)-F^0(Dv)]\,dx.$$

Now $|F^0(p)-F(x,u,p)|\le c_0(1+|p|^2)^{m/2}(|x-x_0|^\sigma+|u-u_0|^\sigma)$, and thus

$$\fint_{B_{1/2R}}(1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\,dx$$

$$\le c\fint_{B_{1/2R}}(1+|Du|^2+|Dv|^2)^{m/2}(R^\sigma+|u-u_0|^\sigma+|v-u|^\sigma)\,dx.$$

Now we have

Lemma 2.1. *Under the assumptions of Theorem* 2.1, *and with v defined by* (2.1), *there is a positive number K and a number* $r>1$, *both depending only on n, N, m and* c_0, *such that*

$$\Big\{\fint_{B_{1/2R}}(1+|Du|^2+|Dv|^2)^{mr/2}\,dx\Big\}^{1/r}\le K\fint_{B_R}(1+|Du|^2)^{m/2}\,dx.$$

For the proof, see [4].

Since the number r as well as the exponent σ of General assumptions can be decreased if necessary, it means no loss of generality to assume that $r\sigma/(r-1)=$

$2n/(n-2)$, so from Lemma 2.1 and the preceding estimate we get

$$(2.2)\qquad \fint_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\,dx$$
$$\leqq c\left\{\fint_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{mr/2}\,dx\right\}^{1/r}$$
$$\times\left\{\fint_{B_{1/2R}} (R^{2n/(n-2)}+|u-u_0|^{2n/(n-2)}+|u-v|^{2n/(n-2)})\,dx\right\}^{1-(1/r)}$$
$$\leqq cR^\sigma \fint_{B_R} (1+|Du|^2)^{m/2}\,dx\left\{\fint_{B_R} (1+|Du|^2+|Dv|^2)\,dx\right\}^{\sigma/2}$$
$$\leqq cR^\sigma\left\{\fint_{B_R} (1+|Du|^2)^{m/2}\,dx\right\}^{1+(\sigma/m)}.$$

The theorem will follow from

Proposition 2.1. *Under the assumptions of Theorem* 2.1, *there is for each positive number L a positive number $\varepsilon_0(L)$ such that if $|(Du)_{x_0,R}|<L$ and $U(x_0,R)<\varepsilon_0$ for some $R<\operatorname{dist}(x_0,\partial\Omega)$, then*

$$(2.3)\qquad U(x_0,\varrho)\leqq D\left[\left(\frac{\varrho}{R}\right)^\mu U(x_0,R)+R^\sigma\left(\frac{R}{\varrho}\right)^n\left\{\fint_{B_R}(1+|Du|^2)^{m/2}\,dx\right\}^{1+(\sigma/m)}\right]$$

for all $\varrho<R$, where D depends only on n, N, m, c_0 and ω, and μ is the exponent from Proposition 1.1.

Proof. We want to study the quantity

$$U(x_0,r)=\fint_{B_r(x_0)}|Du-(Du)_{x_0,r}|^2\,dx+\fint_{B_r(x_0)}|Du-(Du)_{x_0,r}|^m\,dx,$$

comparing it with the corresponding quantity $V(x_0,r)$, defined in the same way with Du and $(Du)_{x_0,r}$ replaced by Dv and $(Dv)_{x_0,r}$.

If we first prescribe that ε_0 be less than 1, we see that $|(Du)_R|<L$ and $U(x_0,R)<\varepsilon_0$ imply that

$$\fint_{B_R}(1+|Du|^2)^{m/2}\,dx\leqq L_1$$

with L_1 depending only on L and m, and hence, from the minimizing property of v,

$$\fint_{B_{1/2R}}(1+|Dv|^2)^{m/2}\,dx\leqq L_2(L,m,c_0).$$

Furthermore, since v satisfies the Euler equation for the functional $\int_{B_{1/2R}}F^0(Dv(x))\,dx$, written in the form

$$\int_{B_{1/2R}}[F^0_{p^j_\beta}(Dv(x))-F^0_{p^j_\beta}((Du)_R)]D_\beta\varphi^j(x)\,dx=0,\quad \varphi\in H_0^{1,m}(B_{1/2R};\,\mathbf{R}^N),$$

it is easily seen, choosing as a test function

$$\varphi(x)=v(x)-u(x)=[v(x)-(Du)_R\cdot x]-[u(x)-(Du)_R\cdot x]$$

that $V(x_0,\frac{1}{2}R)\leqq c(L)\,U(x_0,R)$, and thus, if ε_0 is small enough, depending on L, the hypothesis of Proposition 1.1 holds for v, and therefore (1.9) holds for $V(x_0,\varrho)$.

Thus

$$U(x_0, \varrho) \leqq c \int_{B_\varrho} (|Du-(Dv)_\varrho|^2+|Du-(Dv)_\varrho|^m)\, dx$$

$$\leqq cV(x_0, \varrho)+c\left(\frac{R}{\varrho}\right)^n \int_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\, dx$$

$$\leqq cB\left(\frac{\varrho}{R}\right)^\mu V\left(x_0, \frac{1}{2}R\right)+c\left(\frac{R}{\varrho}\right)^n \int_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\, dx$$

$$\leqq cB\left(\frac{\varrho}{R}\right)^\mu U(x_0, R)+c(1+B)\left(\frac{R}{\varrho}\right)^n \int_{B_{1/2R}} (1+|Du|^2+|Dv|^2)^{(m-2)/2}|Du-Dv|^2\, dx.$$

Using (2.2) we now get (2.3), and the proposition is proved.

By an easy modification of the iteration argument used in the proof of Proposition 1.1 (see e.g. [5]) one now deduces from Proposition 2.1 that for each positive number L there are positive numbers $\varepsilon_0(L)$ and $R_0(L)$ such that if $|(Du)_{x_0,R}|<L$ and $U(x_0, R)<\varepsilon_0$ for some $R<\min(R_0, \operatorname{dist}(x_0, \partial\Omega))$, then for $\varrho<R$ we have $U(x_0, \varrho)\leqq c(\varrho/R)^\tau$ for some positive number τ, and the conclusion of Theorem 2.1 follows exactly as in the proof of Theorem 1.1.

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