Entropy and Lorentz—Marcinkiewicz operator ideals

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0. Introduction

In this paper we continue our research on the Lorentz—Marcinkiewicz operator ideals that we introduced in [7]. Here our attention will mainly be focussed on the entropy ideals $\mathscr{L}^{(e)}_{\varphi,q}$ generated by the entropy numbers e and the Lorentz—Marcinkiewicz sequence space $\lambda^q(\varphi)$.

During the last few years entropy numbers and interpolation theory have turned out to be powerful tools for the investigation of eigenvalue problems (see e.g. [6], [2], [3], [10]). In this article we establish an interpolation formula between $\mathscr{L}^{(e)}_{\varphi,q}$-ideals for the real method with function parameter developed by J. Peetre [16], T. F. Kalugina [9], J. Gustavsson [8] (in the normed case) and C. Merucci [12], [13], [14], [15] (in the quasi-normed case). As a consequence we extend results of B. Carl [3] and T. Kühn [10] on the characterization, in terms of entropy numbers, of operators from l_v into a Banach space of type p factorizing through l_1, and of operators in the "dual" situation, i.e. operators acting from a Banach space whose dual is of type p into l_v, admitting a factorization through l_∞. Some information on distributions of eigenvalues is also obtained. We estimate the asymptotic behaviour of eigenvalues of certain classes of factorable operators, complementing earlier results of B. Carl [3].

I should like to express my gratitude to the editors for suggesting several improvements of the first version of this paper.

1. Preliminaries

For the standard notions of the theory of operator ideals we refer to the book by A. Pietsch [17]. For interpolation theory our general references are the books by J. Bergh and J. Löfström [1] and by H. Triebel [19]. The definition of Banach space of (Rademacher) type p can be found in [11].

The class of all functions $\varphi\colon (0, +\infty)\to(0, +\infty)$ continuous, with $\varphi(1)=1$ and such that

$$\bar{\varphi}(t) = \sup_{s>0} \frac{\varphi(ts)}{\varphi(s)} < \infty \quad \text{for every} \quad t > 0$$

is denoted by $\mathscr{B}$.

The Boyd indices $\alpha_{\bar{\varphi}}$ and $\beta_{\bar{\varphi}}$ of the function $\bar{\varphi}$ are defined by

$$\alpha_{\bar{\varphi}} = \inf_{1<t<+\infty} \frac{\log \bar{\varphi}(t)}{\log t} = \lim_{t\to+\infty} \frac{\log \bar{\varphi}(t)}{\log t}$$

$$\beta_{\bar{\varphi}} = \sup_{0<t<1} \frac{\log \bar{\varphi}(t)}{\log t} = \lim_{t\to 0} \frac{\log \bar{\varphi}(t)}{\log t}.$$

The indices $\alpha_{\bar{\varphi}}$ and $\beta_{\bar{\varphi}}$ satisfy $-\infty<\beta_{\bar{\varphi}}\leqq\alpha_{\bar{\varphi}}<+\infty$ and indicate when $\bar{\varphi}$ belongs to $L_1\big((1, +\infty), dt/t\big)$ and $L_1\big((0, 1), dt/t\big)$ (see [12], [13]).

For $\varphi\in\mathscr{B}$ and $0<q\leqq\infty$, we denote by $\lambda^q(\varphi)$ the Lorentz—Marcinkiewicz sequence space [12], formed by all bounded sequences of scalars $\zeta=(\zeta_n)$ with a finite quasi-norm

$$\|\zeta\|_{\varphi,q} = \begin{cases} \big(\sum_{n=1}^{\infty} (\varphi(n)\, s_n(\zeta))^q n^{-1}\big)^{1/q} & \text{if} \quad 0 < q < \infty \\ \sup\limits_{n\geqq 1} (\varphi(n) s_n(\zeta)) & \text{if} \quad q = \infty \end{cases}$$

where $(s_n(\zeta))$ is the non-increasing rearrangement of ζ, defined by

$$s_n(\zeta) = \inf\{\delta > 0\colon \quad \operatorname{card}(k\colon |\zeta_k| \geqq \delta) < n\}.$$

For properties of spaces $\lambda^q(\varphi)$ see [12] and [7]. We only remind the reader of the following generalization of a classical inequality of Hardy [7], Lemma 2.4:

Lemma H. *Let* $\varphi\in\mathscr{B}$ *and* $0<r<\infty$ *with* $0<\beta_{\bar{\varphi}}\leqq\alpha_{\bar{\varphi}}<1/r$, *and let* $0<q\leqq\infty$. *Then there is a constant* $C=C(\varphi, r, q)$ *such that for all monotone non-increasing sequence* (δ_n) *of non-negative numbers*

$$\|(\delta_n)\|_{\varphi,q} \leqq \Big\|\big(n^{-1/r} \sup_{1\leqq k\leqq n} k^{1/r}\delta_k\big)\Big\|_{\varphi,q} \leqq C\|(\delta_n)\|_{\varphi,q}.$$

The class of all bounded linear operators between arbitrary Banach spaces is denoted by $\mathscr{L}$, while $\mathscr{L}(E, F)$ stands for the set of those operators acting from E into F.

If $\varphi\in\mathscr{B}$, $0<q\leqq\infty$ and s is an additive s-function in the sense of A. Pietsch [17], then the Lorentz—Marcinkiewicz operator ideal $[\mathfrak{L}^{(s)}_{\varphi,q}, \sigma^{(s)}_{\varphi,q}]$ consists of all $T\in\mathscr{L}$ which have a finite quasi-norm

$$\sigma^{(s)}_{\varphi,q}(T) = \big\|\big(s_n(T)\big)\big\|_{\varphi,q} \quad \text{(see [7])}.$$

Examples of additive s-functions are the Gelfand numbers $\big(c_n(T)\big)$, the Kolmogorov numbers $\big(d_n(T)\big)$ or the approximation numbers $\big(a_n(T)\big)$, see [17] and [18].

The class of all compact operators is denoted by $\mathfrak{K}$.

We conclude these preliminaries by recalling some simple facts about the real interpolation space with function parameter (see [16], [9], [8], [13]).

Let (A_0, A_1) be a compatible couple of quasi-normed spaces, let $0<q\leqq\infty$ and $\varphi\in\mathscr{B}$. The space $(A_0, A_1)_{\varphi,q;K}$ consists of all $x\in A_0+A_1$ which have a finite quasi-norm

$$\|x\|_{\varphi,q;K} = \begin{cases} \left(\int_0^\infty (\varphi(t)^{-1}K(t,x))^q\, dt/t\right)^{1/q} & \text{if } 0<q<\infty \\ \sup\limits_{t>0} (\varphi(t)^{-1}K(t,x)) & \text{if } q=\infty, \end{cases}$$

where $K(t,x)$ is the functional of J. Peetre, defined by

$$K(t,x) = \inf\{\|x_0\|_{A_0}+t\|x_1\|_{A_1}\colon x = x_0+x_1,\ x_0\in A_0,\ x_1\in A_1\}.$$

For $\varphi(t)=t^\theta$ $(0<\theta<1)$ we get the classical real interpolation space $((A_0, A_1)_{\theta,q}, \|\,\|_{\theta,q})$ (see [1], [19]).

2. Entropy ideals $\mathfrak{L}^{(e)}_{\varphi,q}$

The n-th entropy number $e_n(T)$ of an operator $T\in\mathscr{L}(E,F)$ is defined as the infimum of all $\varepsilon\geqq 0$ such that there are $y_1, y_2, \dots, y_q\in F$ with $q\leqq 2^{n-1}$ for which

$$T(U_E) \subseteqq \bigcup_{j=1}^q \{y_j+\varepsilon U_F\}$$

holds, where U_E, U_F are the closed unit balls of E and F respectively.

The theory of entropy numbers was developed by A. Pietsch for the first time in [17], §12, where the properties of entropy numbers were described in detail.

Definition 2.1. For $\varphi\in\mathscr{B}$ and $0<q\leqq\infty$ we put

$$\mathfrak{L}^{(e)}_{\varphi,q} = \{T\in\mathfrak{L}\colon (e_n(T))\in\lambda^q(\varphi)\}$$

and

$$\sigma^{(e)}_{\varphi,q}(T) = \varepsilon_{\varphi,q}\|(e_n(T))\|_{\varphi,q} \quad \text{for} \quad T\in\mathfrak{L}^{(e)}_{\varphi,q}.$$

Here the norming constant $\varepsilon_{\varphi,q}$ is chosen such that $\sigma^{(e)}_{\varphi,q}(I_{\mathbf{K}})=1$, where $I_{\mathbf{K}}$ is the identity map of the scalar field $\mathbf{K}$.

Since $(\lambda^q(\varphi), \|\,\|_{\varphi,q})$ is a maximal quasi-normed sequence ideal (in the sense of A. Pietsch [17], §13) and entropy numbers are additive, it follows from [17], Thm. 14.1.8, that $[\mathfrak{L}^{(e)}_{\varphi,q}, \sigma^{(e)}_{\varphi,q}]$ is a quasi-normed operator ideal.

The special case $\varphi(t)=t^{1/p}$ $(0<p\leqq\infty)$ gives the entropy classes $\mathfrak{L}^{(e)}_{p,q}$, which have been extensively studied (see e.g., [17], [2], [4], [5]).

Next we state an interpolation formula between $\mathfrak{L}^{(e)}_{\varphi,q}$-ideals for the $(\varphi, q; K)$-method. The proof is based on an idea previously used by A. Pietsch in the case of the (θ, q)-method [18], Thm. 14.

Theorem 2.2. *Let E, F be Banach spaces, let $0<q_0, q_1, q\leqq\infty$, $\chi, \varphi_0, \varphi_1\in\mathscr{B}$, and put $\varphi(t)=\varphi_0(t)/\varphi_1(t)$ and $\varrho(t)=\varphi_0(t)/\chi(\varphi(t))$. If $0<\beta_{\bar\chi}\leqq\alpha_{\bar\chi}<1$, $\beta_{\bar\varphi_i}>0$ $(i=0, 1)$ and $\beta_{\bar\varphi}>0$ or $\alpha_{\bar\varphi}<0$, then $\varrho\in\mathscr{B}$ and*

$$\left(\mathfrak{L}^{(e)}_{\varphi_0, q_0}(E, F), \mathfrak{L}^{(e)}_{\varphi_1, q_1}(E, F)\right)_{\chi, q; K} \subseteqq \mathfrak{L}^{(e)}_{\varrho, q}(E, F).$$

Proof. We first give the proof when $0<q<\infty$ and $\beta_{\bar\varphi}>0$. The fact that $\varrho\in\mathscr{B}$ was proved in [7], Thm. 5.3.

Since $\beta_{\bar\varphi_i}>0$ $(i=0, 1)$, $\beta_{\bar\chi}>0$ and $\beta_{\bar\varphi}>0$, without loss of generality we may assume that φ_i $(i=0, 1)$ and χ are increasing and that φ is an increasing bijection belonging to $\mathscr{C}^1((0, +\infty))$ with

$$(1) \qquad 0 < C = \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)}$$

(see [14], Prop. 1 or [15], Prop. 4.1.1). Furthermore, as

$$\mathfrak{L}^{(e)}_{\varphi_i, q_i}(E, F) \subseteqq \mathfrak{L}^{(e)}_{\varphi_i, \infty}(E, F)$$

we may also assume that $q_0=q_1=\infty$.

Let $T\in\left(\mathfrak{L}^{(e)}_{\varphi_0, \infty}(E, F), \mathfrak{L}^{(e)}_{\varphi_1, \infty}(E, F)\right)_{\chi, q; K}$ and let $T=T_0+T_1$ be any decomposition with $T_0\in\mathfrak{L}^{(e)}_{\varphi_0, \infty}(E, F)$ and $T_1\in\mathfrak{L}^{(e)}_{\varphi_1, \infty}(E, F)$. Given any $n\in\mathbf{N}$, let m be the greatest integer not exceeding $(n+1)/2$, then $2m-1\leqq n\leqq 2m$ and

$$\begin{aligned} e_n(T) &\leqq e_{2m-1}(T_0+T_1) \leqq e_m(T_0)+e_m(T_1) \\ &\leqq \varphi_0(m)^{-1}\sigma^{(e)}_{\varphi_0, \infty}(T_0)+\varphi_1(m)^{-1}\sigma^{(e)}_{\varphi_1, \infty}(T_1) \\ &\leqq \varphi_0(n/2)^{-1}\sigma^{(e)}_{\varphi_0, \infty}(T_0)+\varphi_1(n/2)^{-1}\sigma^{(e)}_{\varphi_1, \infty}(T_1) \\ &\leqq \bar\varphi_0(2)\varphi_0(n)^{-1}\sigma^{(e)}_{\varphi_0, \infty}(T_0)+\bar\varphi_1(2)\varphi_1(n)^{-1}\sigma^{(e)}_{\varphi_1, \infty}(T_1) \\ &\leqq C_1\varphi_0(n)^{-1}[\sigma^{(e)}_{\varphi_0, \infty}(T_0)+\varphi(n)\sigma^{(e)}_{\varphi_1, \infty}(T_1)] \end{aligned}$$

where $C_1=\max\{\bar\varphi_0(2), \bar\varphi_1(2)\}$. Thus we get

$$\varphi_0(n)e_n(T) \leqq C_1K(\varphi(n), T).$$

Consequently, taking into account that $K(\cdot, T)$, $\bar\varphi$ and $\bar\chi$ are non-decreasing and making the substitution $u=\varphi(t)$, we obtain with $C_2=2(\bar\chi(\bar\varphi(2)))^qC_1^q$ and C the constant of (1)

$$\begin{aligned} \sum_{n=1}^{\infty}(\varrho(n)e_n(T))^q\frac{1}{n} &= \sum_{n=1}^{\infty}\left(\frac{1}{\chi(\varphi(n))}\varphi_0(n)e_n(T)\right)^q\frac{1}{n} \\ &\leqq C_1^q\sum_{n=1}^{\infty}\left(\frac{1}{\chi(\varphi(n))}K(\varphi(n), T)\right)^q\frac{1}{n} \\ &\leqq C_2\int_0^{\infty}\left(\frac{1}{\chi(\varphi(t))}K(\varphi(t), T)\right)^q\frac{dt}{t} \\ &\leqq C^{-1}C_2\int_0^{\infty}(\chi(u)^{-1}K(u, T))^q\frac{du}{u} < \infty. \end{aligned}$$

Therefore $T\in\mathfrak{L}^{(e)}_{\varrho, q}(E, F)$.

Suppose now that $\alpha_{\bar{\varphi}}<0$. Put $\chi^*(t)=t\chi(1/t)$. Then we have with equal quasi-norms

$$\big(\mathfrak{L}^{(e)}_{\varphi_0,\infty}(E, F), \mathfrak{L}^{(e)}_{\varphi_1,\infty}(E, F)\big)_{\chi,q;K} = \big(\mathfrak{L}^{(e)}_{\varphi_1,\infty}(E, F), \mathfrak{L}^{(e)}_{\varphi_0,\infty}(E, F)\big)_{\chi^*,q;K}.$$

Furthermore

$$0 < \beta_{\bar{\chi}^*} \leqq \alpha_{\bar{\chi}^*} < 1 \quad \text{and} \quad \varphi^*(t) = \varphi_1(t)/\varphi_0(t) = \varphi(t)^{-1},$$

whence

$$\beta_{\bar{\varphi}^*} = -\alpha_{\bar{\varphi}} > 0 \quad \text{and} \quad \varrho^*(t) = \varphi_1(t)/\chi^*\big(\varphi^*(t)\big) = \varrho(t).$$

Hence the result follows from the case just proved.

The proof of the remaining case $q=\infty$ can be carried out in the same way. □

3. Relationships between $\mathfrak{L}^{(e)}_{\varphi,q}$ and $\mathfrak{L}^{(s)}_{\varphi,q}$

In the following we compare the entropy ideals $\mathfrak{L}^{(e)}_{\varphi,q}$ and the ideals $\mathfrak{L}^{(s)}_{\varphi,q}$ generated by either the approximation (a) or the Gelfand (c) or the Kolmogorov numbers (d).

Theorem 3.1. *Let* $\varphi\in\mathscr{B}$ *with* $\beta_{\bar{\varphi}}>0$ *and let* $0<q\leqq\infty$. *If* $s\in\{a, c, d\}$, *then* $\mathfrak{L}^{(s)}_{\varphi,q}(E, F)\subseteqq\mathfrak{L}^{(e)}_{\varphi,q}(E, F)$ *for all Banach spaces E and F.*

Proof. Choose $r>0$ with $0<\beta_{\bar{\varphi}}\leqq\alpha_{\bar{\varphi}}<1/r$. According to [2], Thm. 1, there exists a constant $M=M(r)<\infty$ such that for every $T\in\mathscr{L}(E, F)$

$$\sup_{1\leqq k\leqq n} k^{1/r}e_k(T) \leqq M \sup_{1\leqq k\leqq n} k^{1/r}s_k(T) \quad n = 1, 2, \ldots$$

Therefore, using the generalized Hardy inequality, we have for every $T\in\mathfrak{L}^{(s)}_{\varphi,q}(E, F)$ with $C=C(\varphi, r, q)$ the constant of Lemma H

$$\begin{aligned}\sigma^{(e)}_{\varphi,q}(T) &= \varepsilon_{\varphi,q}\big\|\big(e_n(T)\big)\big\|_{\varphi,q}\\ &\leqq \varepsilon_{\varphi,q}\Big\|\Big(n^{-1/r}\sup_{1\leqq k\leqq n} k^{1/r}e_k(T)\Big)\Big\|_{\varphi,q}\\ &\leqq M\varepsilon_{\varphi,q}\Big\|\Big(n^{-1/r}\sup_{1\leqq k\leqq n} k^{1/r}s_k(T)\Big)\Big\|_{\varphi,q}\\ &\leqq MC\varepsilon_{\varphi,q}\big\|\big(s_n(T)\big)\big\|_{\varphi,q} = MC\varepsilon_{\varphi,q}\sigma^{(s)}_{\varphi,q}(T). \quad \square\end{aligned}$$

Remark 3.2. This inclusion can be improved if $q<\infty$: It can be easily checked using [7], Thm. 5.1 and [17], Lemma 14.2.8/1 that finite rank operators from E into F are dense in $\mathfrak{L}^{(a)}_{\varphi,q}(E, F)$. Therefore it follows from Theorem 3.1, from [5], Thm. 1.2 and [4], Thm. 2.1, that

$$\mathfrak{L}^{(s)}_{\varphi,q} \subseteqq \mathfrak{L}^{(e)}_{\varphi,q}\circ\mathfrak{K} \quad \text{for} \quad s = a, d. \tag{2}$$

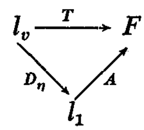
Furthermore, since the operator ideal $\mathfrak{L}^{(e)}_{\varphi,q}\circ\mathfrak{K}$ is injective, we also get from (2)

$$\mathfrak{L}^{(c)}_{\varphi,q} \subseteqq \mathfrak{L}^{(e)}_{\varphi,q}\circ\mathfrak{K}.$$

4. Entropy numbers of factorable operators

We shall now extend results of B. Carl [3] and T. Kühn [10] by means of the interpolation formula of Section 2.

Theorem 4.1. *Let F be a Banach space of type p and let $T\in\mathscr{L}(l_v, F)$ admit a factorization*



with a diagonal operator $D_\eta\in\mathscr{L}(l_v, l_1)$ and $A\in\mathscr{L}(l_1, F)$. If $\eta\in\lambda^q(\varphi)$, then $T\in\mathfrak{L}^{(e)}_{\varrho,q}(l_v, F)$ provided that $1\leqq p\leqq 2$, $1\leqq v\leqq\infty$, $0<q\leqq\infty$, $\beta_{\bar\varphi}>1-1/v$ and

$$\varrho(t) = t^{1/v-1/p}\varphi(t).$$

Proof. Choose $0<p_0<p_1<\infty$ such that

$$1-1/v < 1/p_1 < \beta_{\bar\varphi} \leqq \alpha_{\bar\varphi} < 1/p_0.$$

Thus $1/v+1/p_i>1$ $(i=0, 1)$. Put $1/s_i=1/p_i+1/v-1/p$ $(i=0, 1)$ and let $\mathscr{D}$ be the operator assigning to every sequence η the composite operator AD_η. By [3], Thm. 2

$$\mathscr{D}(l_{p_i}) \subseteqq \mathfrak{L}^{(e)}_{s_i,\infty}(l_v, F)$$

then the Closed Graph Theorem guarantees that

$$\mathscr{D}\in\mathfrak{L}\big(l_{p_i}, \mathfrak{L}^{(e)}_{s_i,\infty}(l_v, F)\big) \quad (i = 0, 1). \tag{3}$$

Let us now consider the function $\chi\in\mathscr{B}$ defined by

$$\chi(t) = t^{p_1/(p_1-p_0)}\big(\varphi\big(t^{p_0p_1/(p_1-p_0)}\big)\big)^{-1}.$$

It follows from the interpolation property ([12], Thm. 1) and (3) that

$$\mathscr{D}\in\mathfrak{L}\big((l_{p_0}, l_{p_1})_{\chi,q;K}, (\mathfrak{L}^{(e)}_{s_0,\infty}(l_v, F), \mathfrak{L}^{(e)}_{s_1,\infty}(l_v, F))_{\chi,q;K}\big).$$

By [12], Thm. 5 and Prop. 8

$$(l_{p_0}, l_{p_1})_{\chi,q;K} = \lambda^q(\varphi)$$

and by Theorem 2.2

$$\big(\mathfrak{L}^{(e)}_{s_0,\infty}(l_v, F), \mathfrak{L}^{(e)}_{s_1,\infty}(l_v, F)\big)_{\chi,q;K} \subseteqq \mathfrak{L}^{(e)}_{\varrho,q}(l_v, F).$$

Consequently, if $\eta\in\lambda^q(\varphi)$ we obtain that $T=AD_\eta\in\mathfrak{L}^{(e)}_{\varrho,q}(l_v, F)$. □

For the "dual" situation, a similar reasoning and [10], Thm. 4, allow us to derive:

Theorem 4.2. *Let E be a Banach space whose dual is of type p and let $S \in \mathfrak{L}(E, l_v)$ admit a factorization*

$$\begin{array}{ccc} E & \xrightarrow{\ S\ } & l_v \\ & {\scriptstyle B}\searrow \quad \nearrow{\scriptstyle D_\eta} & \\ & l_\infty & \end{array}$$

with a diagonal operator $D_\eta \in \mathfrak{L}(l_\infty, l_v)$ and $B \in \mathfrak{L}(E, l_\infty)$. If $\eta \in \lambda^q(\varphi)$, then $S \in \mathfrak{L}^{(e)}_{\varrho, q}(E, l_v)$ where $1 \leqq p \leqq 2$, $1 \leqq v \leqq \infty$, $0 < q \leqq \infty$, $\beta_{\bar{\varphi}} > \max(1/p, 1/v)$ and

$$\varrho(t) = t^{1-1/p-1/v}\varphi(t).$$

5. Eigenvalues

We shall now estimate the asymptotic behaviour of eigenvalues of certain classes of factorable operators, so all Banach spaces under consideration are assumed to be complex.

Let $\varphi \in \mathscr{B}$ with $\beta_{\bar{\varphi}} > 0$ and let $0 < q \leqq \infty$. If $T \in \mathfrak{L}^{(e)}_{\varphi, q}(E, E)$ it follows from [7], Lemma 2.2, that $\lim_{n\to\infty} e_n(T) = 0$. Therefore the operator T is compact. Let $(\lambda_n(T))$ denote the sequence of all eigenvalues of T counted according to their algebraic multiplicities and ordered such that $|\lambda_1(T)| \geqq |\lambda_2(T)| \geqq \ldots \geqq 0$. If T has less than n eigenvalues, we set $\lambda_n(T) = \lambda_{n+1}(T) = \ldots = 0$.

The following results extend earlier ones of B. Carl [3], Thm. 3 and Thm. 4. We shall prove them by using his techniques and our entropy results.

Theorem 5.1. *Let E be a Banach space of type p $(1 \leqq p \leqq 2)$ and let $T \in \mathfrak{L}(E, E)$ an operator which admits the factorization*

$$\begin{array}{ccc} E & \xrightarrow{\ T\ } & E \\ {\scriptstyle A}\big\downarrow & & \big\uparrow{\scriptstyle B} \\ l_v & \xrightarrow[\ D_\eta\ ]{} & l_1, \quad 1 \leqq v < \infty \end{array}$$

where $A \in \mathfrak{L}(E, l_v)$, $B \in \mathfrak{L}(l_1, E)$ are arbitrary operators and $D_\eta \in \mathfrak{L}(l_v, l_1)$ is a diagonal operator. If $\eta \in \lambda^q(\varphi)$, then $(\lambda_n(T)) \in \lambda^q(\varrho)$ whenever $0 < q \leqq \infty$, $\beta_{\bar{\varphi}} > 1 - 1/v$ and

$$\varrho(t) = t^{1/v - \min(1/p, \max(1/v, 1/2))}\varphi(t).$$

Proof. Theorem 4.1 implies $T \in \mathfrak{L}^{(e)}_{\varrho_0, q}(E, E)$ with $\varrho_0(t) = t^{1/v-1/p}\varphi(t)$. Therefore, according to [6], we have

$$(4) \qquad (\lambda_n(T)) \in \lambda^q(\varrho_0).$$

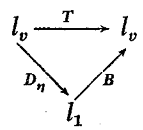
Let us now consider the operator $S=ABD_\eta\in\mathfrak{L}(l_v, l_v)$. Applying again Theorem 4.1 we get $S\in\mathfrak{L}^{(e)}_{\varrho_1,q}(l_v, l_v)$ where $\varrho_1(t)=t^{1/v-\max(1/v,1/2)}\varphi(t)$. So [6] yields

$$(\lambda_n(S))\in\lambda^q(\varrho_1). \tag{5}$$

But the eigenvalues of T and S coincide because the operators T and S are related (in the sense of A. Pietsch [17], 27.3). Consequently, we obtain from (4) and (5) that $(\lambda_n(T))\in\lambda^q(\varrho)$. □

In order to show our last result, which is an application of Theorem 5.1 to a special case, let us recall that $\lambda^q(\varphi)$ is equal to the Lorentz—Zygmund sequence space $l_{p,q}(\log l)^\gamma$ if $\varphi(t)=t^{1/p}(1+|\log t|)^\gamma$.

Example 5.2. Let $T\in\mathfrak{L}(l_v, l_v)$ be an operator such that $(\|T(x_n)\|_v)\in l_{r,r}(\log l)^\gamma$ where $1\leq v<\infty$, $0<r<\infty$, $1/v+1/r>1$, $-\infty<\gamma<+\infty$ and (x_n) is the unit vector basis of l_v. Then it is not hard to verify that the operator T admits the factorization



where $\eta=(\|T(x_n)\|_v)$ and $B((\zeta_n))=\sum_{n=1}^\infty \zeta_n(T(x_n)/\|T(x_n)\|_v)$. Whence Theorem 5.1 gives that

$$(\lambda_n(T))\in l_{s,r}(\log l)^\gamma \quad \text{for} \quad \frac{1}{s}=\frac{1}{r}+\frac{1}{v}-\max\left(\frac{1}{v},\frac{1}{2}\right).$$

References

1. Bergh, J. and Löfström, J., *Interpolation spaces, an introduction,* Springer, Berlin—Heidelberg—New York, 1976.
2. Carl, B., Entropy numbers, *s*-numbers, and eigenvalue problems, *J. Funct. Anal.* **41** (1981), 290—306.
3. Carl, B., On a characterization of operators from l_q into a Banach space of type q with some applications to eigenvalue problems, *J. Funct. Anal.* **48** (1982), 394—407.
4. Carl, B. and Stephani, I., On *A*-compact operators, generalized entropy numbers and entropy ideals, *Math. Nachr.* **119** (1984), 77—95.
5. Carl, B. and Stephani, I., Estimating compactness properties of operators by the aid of generalized entropy numbers, *J. Reine Angew. Math.* **350** (1984), 117—136.
6. Carl B. and Triebel, H., Inequalities between eigenvalues, entropy numbers and related quantities of compact operators in Banach spaces, *Math. Ann.* **251** (1980), 129—133.
7. Cobos, F., On the Lorentz—Marcinkiewicz operator ideal, *Math. Nachr.* **126** (1986), 281—300.
8. Gustavsson, J., A function parameter in connection with interpolation of Banach spaces, *Math. Scand.* **42** (1978), 289—305.

9. KALUGINA, T. F., Interpolation of Banach spaces with a functional parameter. The reiteration theorem, *Mos. Univ. Math. Bull.* **30** (6) (1975), 108—116.
10. KÜHN, T., Entropy numbers of r-nuclear operators in Banach spaces of type q, *Studia Math.* **80** (1984), 53—61.
11. MAUREY, B. and PISIER, G., Séries de variables aléatoires vectorielles indépendantes et propiétés géométriques des espaces de Banach, *Studia Math.* **58** (1976), 45—90.
12. MERUCCI, C., Interpolation réelle avec paramètre fonctionnel des espaces $L^{p,q}$, Université de Nantes, Sem. d'Anal. exposé n° 17 (1980/81), 350—373.
13. MERUCCI, C., Interpolation réelle avec paramètre fonctionnel des espaces $L^{p,q}$, *C. R. Acad. Sci. Paris,* Sér. I, **294** (1982), 653—656.
14. MERUCCI, C., Interpolation réelle avec fonction paramètre: réitération et applications aux espaces $A^p(\varphi)(0<p\leqq+\infty)$, *C. R. Acad. Sci. Paris, Sér.* I, **295** (1982), 427—430.
15. MERUCCI, C., *Interpolation réelle avec fonction paramètre. Dualité, reitération et applications,* thèse, Institut de Mathématiques et d'Informatique, Université de Nantes, 1983.
16. PEETRE, J., *A theory of interpolation of normed spaces,* lecture notes, Brasilia 1963 [Notas Mat. 39 (1968), 1—86].
17. PIETSCH, A., *Operator ideals,* North-Holland, Amsterdam—New York—Oxford, 1980.
18. PIETSCH, A., Weyl numbers and eigenvalues of operators in Banach spaces, *Math. Ann.* **247** (1980), 149—168.
19. TRIEBEL, H., *Interpolation theory, function spaces, differential operators,* North-Holland, Amsterdam—New York—Oxford 1978.

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