Sums of complemented subspaces in locally convex spaces

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1. Introduction

Let $P_1, \ldots, P_m$ be continuous linear projections onto the subspaces $N_1, \ldots, N_m$ of a topological vector space X. Two natural questions arise:

(a) Is $N_1+\ldots+N_m$ closed?
(b) Is $N_1+\ldots+N_m$ complemented?

In [3], H. Lang answers (a) affirmatively in case X is a Fréchet space and all products P_iP_j, $i\neq j$, are compact. This generalizes a similar result by L. Svensson [4] for reflexive Banach spaces.

The aim of this paper is to answer question (b). In fact we will prove that if X is a Hausdorff locally convex topological vector space and P_iP_j is compact for $i\neq j$, then $N_1+\ldots+N_m$ is complemented. Moreover a continuous linear projection onto this sum is given by $P_1+\ldots+P_m$, modulo compact operators.

If X is a Hilbert space, we will prove that $N_1+\ldots+N_m$ is closed if N_i+N_j is closed for all i, j and every product $P_iP_jP_k$ is compact for $i\neq j\neq k\neq i$.

2. Sums of complemented subspaces in locally convex spaces

Throughout this paper we will use the following definitions and notations.

A continuous linear map from one topological vector space into another is called a *homomorphism* if it is relatively open, *compact* if it maps some open set onto some relatively compact set and a *projection* if it is idempotent.

A map T is a *compact perturbation* of a mapping S, if $S-T$ is compact. A subspace of a topological vector space (TVS) is called *complemented* (topologically supplemented or a direct summand) if it is the image of some continuous linear projection.

Lemma 2.1. *A subspace L in a TVS X is complemented precisely if the canonical map $X \to X/L$ has a right inverse, which also is a homomorphism.*

Proof. Strightforward verification.

Lemma 2.2. *Let $S: X \to Y$ and $T: Y \to Z$ be homomorphisms such that* $\operatorname{im} S \supset \ker T$. *Then TS is a homomorphism.*

Proof. Easy verification.

Lemma 2.3. *Let L and M be subspaces of a TVS X. Suppose that $L \subset M$ and that L is complemented in X. If, in addition, M/L is complemented in X/L, then M is complemented in X.*

Proof. Consider the commutative diagram below, where all topologies and mappings are the canonical ones

$$\begin{array}{ccc} X & \xrightarrow{p} & X/L \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle r} \\ X/M & \xrightarrow{s} & \dfrac{X}{L}\Big/\dfrac{M}{L} \end{array}$$

The reader may recall that s is a topological isomorphism. By the assumption and Lemma 2.1 there exist homomorphisms

$$p': X/L \to X \quad \text{and} \quad r': \frac{X}{L}\Big/\frac{M}{L} \to \frac{X}{L}$$

such that $p \circ p'$ and $r \circ r'$ are the identity mappings.

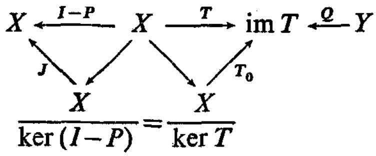
Thus both p' and r' are injective and we conclude from Lemma 2.2 that $p' \circ r'$ is an injective homomorphism. Hence $q' = p' \circ r' \circ s: X/M \to X$ is a homomorphism.

But $q \circ q' = s^{-1} \circ r \circ p \circ p' \circ r' \circ s$ is the identity on X/M.

This proves the lemma.

Lemma 2.4. *Let T be a homomorphism from one TVS X into another Y. Suppose that* $\ker T$ *and* $\operatorname{im} T$ *are complemented with projections P and Q respectively. Then T has a "left pseudo-inverse", i.e. there exists a homomorphism $T^{\#}: Y \to X$ such that $T^{\#} T = I - P$.*

Proof. Consider the diagram below, where T_0 is an isomorphism.



Now put $T^{\#}=JT_0^{-1}Q$.

By Lemma 2.2, $T^{\#}$ is a homomorphism.

Much of what remains, in this paper, is to refine the following.

Lemma 2.5. *Let P and Q be projections onto the subspaces L and M in a TVS X. Suppose that $I-PQ$ and $I-QP$ are homomorphisms with complemented kernels and images. Suppose, moreover, that their kernels are equal.*

Then $L+M$ is complemented in X.

Proof. By assumption $L\cap M=\ker(I-PQ)=\ker(I-QP)$. Thus, by Lemma 2.3, we may assume that $L\cap M=0$. Now it is straightforward to verify that $R=P(I-QP)^{\#}(I-Q)+Q(I-PQ)^{\#}(I-P)$ is a projection onto $L+M$.

Remark. The reader should have no difficulty in verifying that

$$R=I-S+SQS(I-PQ)^{\#}(I-P)S+SPS(I-QP)^{\#}(I-Q)S$$

is a projection onto $L+M$ (where $I-S$ is any projection onto $L\cap M$).

Lemma 2.6. *Let E be a finite dimensional and F a complemented subspace in some Hausdorff locally convex TVS (from now on abbreviated HLCTVS). Then $E+F$ is complemented. If moreover $E\cap F=0$, and Q is some projection onto F, there exists a projection P onto E satisfying $PQ=0$, with the property that $P+Q-QP$ is a projection onto $E+F$.*

Proof. It is no loss of generality to assume that

$$\dim E=1, \quad \text{and that} \quad E\cap F=0.$$

If $0\neq e\in E$, it follows from the Hahn—Banach theorem that some $e'\in X'$ annihilates F and satisfies $e'(e)=1$.

Put $Px=e'(x)e$.

The rest is plain verification.

Definition. A homomorphism from one *TVS* into another is called a *quasi-isomorphism* if its kernel has finite dimension and its image has finite codimension.

Lemma 2.7. *In a HLCTVS compact perturbations of isomorphisms are quasi-isomorphism.*

Proof. See Grothendieck [2].

We now have come to our main result.

Theorem 2.8. *Let L, M be complemented subspaces in a HLCTVS X with corresponding projections P, Q. Suppose that $I-PQ$ and $I-QP$ are compact perturbations of isomorphisms. Then $L+M$ is complemented. Furthermore, if PQ and QP are compact, then $P+Q$ is a compact perturbation of some projection onto $L+M$.*

Proof. We will reduce this theorem to Lemma 5. To do this we introduce

$$H = \ker(I-PQ) \subset L$$
$$K = \ker(I-QP) \subset M$$
$$\tilde{L} = L+K$$
$$\tilde{M} = M+H.$$

We observe that $L\cap M = H\cap K$ is finite dimensional. Thus, by passing to the quotient space $X/L\cap M$ we may, in view of Lemma 3, assume that $L\cap M=0$. Hence $H\cap M = K\cap L=0$.

Since H and K are finite dimensional we conclude, from Lemma 6, that there exist projections S and T onto H and K, respectively, such that $\tilde{P}=P+(I-P)T$ and $\tilde{Q}=Q+(I-Q)S$ are projections onto $\tilde{L}$ and $\tilde{M}$, respectively. Since S and T are compact, $I-\tilde{P}\tilde{Q}$ and $I-\tilde{Q}\tilde{P}$ are compact perturbations of isomorphisms.

Obviously $L+M=\tilde{L}+\tilde{M}$, so if we show that $\ker(I-\tilde{P}\tilde{Q})=\ker(I-\tilde{Q}\tilde{P})$, the proof will follow from Lemma 2.5. Since $H\subset L\cap\tilde{M}\subset\tilde{L}\cap\tilde{M}$ and $K\subset\tilde{L}\cap\tilde{M}$, we get $H+K\subset\tilde{L}\cap\tilde{M}\subset\tilde{H}\cap\tilde{K}$, where $\tilde{H}=\ker(I-\tilde{P}\tilde{Q})$ and $\tilde{K}=\ker(I-\tilde{Q}\tilde{P})$. Now we claim that $\tilde{H}\subset H+K$. Indeed if $x\in\tilde{H}$, then

$$x = \tilde{P}\tilde{Q}x = PQx+P(I-Q)Sx+(I-P)T\big(Q+(I-Q)S\big)x.$$

So $Px=PQx+P(I-Q)Sx$. But clearly $PQS=S=PS$. Hence $Px=PQx$ for all x in $\tilde{H}$. Since $\tilde{H}\subset\tilde{L}=L+K$, every x in $\tilde{H}$ can be written as $y+z$, where $y\in L$ and $z\in K$.

Therefore $Px=P(y+z)=PQ(y+z)$ yielding $y=PQy\in H$. Hence $\tilde{H}\subset H+K$, proving our claim.

Finally $\tilde{L}\cap\tilde{M}\subset\tilde{H}\subset H+K\subset\tilde{L}\cap\tilde{M}$, from which we conclude that $\tilde{L}\cap\tilde{M}=\tilde{H}=\tilde{K}=H+K$, and consequently that $L+M$ is complemented. The rest of the theorem follows easily from the remark made after Lemma 2.5.

An induction argument yields.

Corollary 2.9. *Let $N_1, \dots, N_m$ be complemented subspaces in a HLCTVS with corresponding projections $P_1 \dots P_m$. Assume that P_iP_j is compact whenever $i\neq j$. Then $N_1+\dots+N_m$ is complemented. Moreover, $P_1+\dots+P_m$ is a compact perturbation of a corresponding projection.*

3. Sums of closed subspaces in Hilbert spaces

Our aim in this section is to prove.

Theorem 3.10. *Let $P_1 \dots P_m$ be orthogonal projections onto the subspaces $N_1 \dots N_m$ of a Hilbert space H such that*

(i) *$N_i + N_j$ is closed for all i, j;*
(ii) *$P_i P_j P_k$ is compact for all $i \neq j \neq k \neq i$.*

Then $N_1 + \dots + N_m$ is closed in H.

Before we prove this, we need a couple of Lemmas.

Lemma 3.11. *Let P and Q be orthogonal projections onto L and M, subspaces of a Hilbert space H.*

Then $L+M$ is closed precisely if $I-PQ$ has closed image.

Proof. We may assume that $L \cap M = 0$, otherwise we just pass to the quotient space $H/L \cap M$. By duality, $\operatorname{im}(I-PQ)$ is dense. Thus, by the open mapping theorem, $I-PQ$ is invertible if and only if $\operatorname{im}(I-PQ)$ is closed.

Also, as is easily seen, $I-PQ$ is invertible precisely if $|PQ|<1$.

Finally, as is well-known, $L+M$ is closed precisely if $|PQ|<1$, proving our lemma.

Lemma 3.12. *Let P, Q, R be orthogonal projections onto the subspaces L, M, N of a Hilbert space H, such that*

(i) *$L+M$, $L+N$, $M+N$ are closed.*
(ii) *$L \cap N = M \cap N = 0$.*
(iii) *PQR, RPQ and QRP are compact.*

Then $L+M+N$ is closed.

Proof. Letting $P \wedge Q$ denote the orthogonal projection onto $L \cap M$, it is not too hard to verify that

$$(*) \qquad S = P \wedge Q + (I-Q)(I-PQ+P \wedge Q)^{-1}(P-P \wedge Q) + (I-P)(I-QP+P \wedge Q)^{-1}(Q-P \wedge Q)$$

is an orthogonal projection onto $L+M$. A simple calculation shows that

$$I-RS = (I-RP)(I-RQ)+K$$

for some compact operator K.

By Lemma 3.11, $I-RP$ and $I-RQ$ are invertible. Hence, $I-RS$, being a Fredholm mapping, has a closed image. Thus, by Lemma 3.11, $L+M+N$ is closed.

Lemma 3.13. *Let L, M, N be closed subspaces in a Hilbert space H, with orthogonal projections P, Q, R respectively. Suppose that*

(i) *$L+M$, $L+N$, $M+N$ are closed.*
(ii) *PQR, RPQ, QRP are compact.*

Then $L+M+N$ is closed.

Proof. Since $(R\wedge P)(R\wedge Q)$ is compact, it follows from Theorem 8 that $E=N\cap L+N\cap M$ is closed, and that $R\wedge P+R\wedge Q$ is a compact perturbation of an orthogonal projection T onto E. (The orthogonality follows easily from the fact that $R\wedge P+R\wedge Q$ is self adjoint).

$\tilde{R}=R-T$ is an orthogonal projection onto $\tilde{N}=N\cap E^{\perp}$ where $\perp$ denote orthogonal complement. We observe that $\tilde{N}\cap L=\tilde{N}\cap M=0$, and that $L+M+N=L+M+\tilde{N}$. Now we want to apply Lemma 3.12 to L, M and $\tilde{N}$. That $PQ\tilde{R}$, $\tilde{R}PQ$ and $Q\tilde{R}P$ are compact is easily checked. Hence it only remains to show that $L+\tilde{N}$ and $M+\tilde{N}$ are closed.

A straightforward calculation shows that

$$P\tilde{R}=PR(I-R\wedge P)+K$$

for some compact operator K.

But, since $L+N$ is closed, the norm of $PR(I-R\wedge P)$ is less than 1. Hence $I-P\tilde{R}$ is a compact perturbation of an isomorphism.

From Theorem 2.8 we therefore conclude that $L+\tilde{N}$ is closed. Similarly we get that $M+\tilde{N}$ is closed, completing the proof.

Proof of Theorem 3.10. Induction on m.

Remark. A generalization of Theorem 3.10 in terms of products of four projections is not valid, as the following counter example shows. Put for $n=1, 2, \ldots$

$$K_n = \text{span}\,(1, 0, 0, 0) \subset \mathbf{R}^4$$
$$L_n = \text{span}\,(0, 1, 0, 0)$$
$$M_n = \text{span}\,(0, 0, 1, 0)$$
$$N_n = \text{span}\,(1, 1, 1, n^{-1})$$
$$H_n = \mathbf{R}^4$$

and let H be the direct product of the H_n, with K, L, M, N similarly defined as subsets of H.

If S, P, Q and R denote the orthogonal projections onto K, L, M and N respectively, one easily verifies that the product, in any order, of P, Q, R and S is 0.

Moreover, every sum of three or less of the subspaces K, L, M and N is closed. However, $K+L+M+N$ is not closed. If we put $\tilde{L}=K+L$ we get an example showing that we really need the assumption that *every* permutation of the product in condition (ii) of Theorem 3.10, is compact.

Remark. In [4] Theorem 2.8 is used to study questions arising in theoretical tomography concerning the closure of a finite sum of subspaces of L^p consisting of functions constant on certain sets. In a future paper we will use Theorem 3.10 to give a functional analytic proof of a theorem in three-dimensional theoretical tomography, due to J. Boman [1]. Theorem 2.8 can also be used to prove theorems about extensions of functions and existence theorems for certain partial differential equations, see [4].

References

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