Uniqueness theorems for operator-valued functions with positive imaginary part, and the singular spectrum in the selfadjoint Friedrichs model

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Introduction

Let L be a selfadjoint weakly-perturbed operator of multiplication in $L_2(\mathbf{R})$, i.e. the so - called Friedrichs model [4, 5] in theory of smooth perturbations:

$$(Lu)(x) = xu(x)+(Vu)(x) \equiv xu(x)+\int_{\mathbf{R}} v(x, y)u(y)\,dy.$$

If the kernel $v(x, y)$ satisfies conditions of smoothness, symmetry and decreases at infinity in a sense then the absolutely continuous spectrum of the selfadjoint operator L coincides with $\mathbf{R}$ [4, 5]. It is enough to require that V belongs to the trace class S^1 [16]. It has been proved in [4] that under the condition $v \in \operatorname{Lip} a$, $\alpha > 1/2$, the spectrum of L besides the absolutely continuous part can contain only a finite number of eigenvalues of finite multiplicity. But in the case $v \in \operatorname{Lip}\alpha$ with $\alpha \leqq 1/2$ the singular continuous spectrum as well as the infinite point spectrum can occur. In [11] an example of such an operator with rank $V=1$ was constructed. The following result describing the structure of the singular continuous spectrum $\mathfrak{S}_s$ and of the point spectrum $\mathfrak{S}_p$ was also obtained there. Just for rank $V=1$ there proved the following estimate.

Let F be the support of the singular part of the spectral measure of L and F^δ be the δ-neighbourhood of F on $\mathbf{R}$. Then

$$\operatorname{mes} F^\delta = O(\delta^{2\alpha}), \quad \delta \to 0, \quad 0 < \alpha < 1/2,$$

mes E being Lebesgue measure on $\mathbf{R}$.

It has been shown in [11] that this estimate is precise in the power scale. In this paper the problem under consideration is reduced to the investigation of a zero-set of a scalar analytic function determining the spectrum of L. The solution of this problem is based on a uniqueness theorem for analytic functions with positive imaginary part [12].

Unfortunately, attempts to extend this result even to the case rank $V=2$ meet essential difficulties. This is connected with the fact that the functions under consideration are not scalar but matrix functions. The description of the spectral structure for rank $V<\infty$ given in [14] is related to the case when all eigenfunctions of V do not vanish simultaneousely. The problem of description of the singular spectrum in Friedrichs model in general setting was posed by Pavlov, B. S. and Faddeev, L. D. [13]. Below we give a solution of this problem. It is based upon the investigation of an operator-valued analytic function $M(\lambda)$ (see (1.2)) with the positive imaginary part in a neighbourhood of its roots. It seems plausible that in case rank $V>1$ (even for rank $V=2$) it is impossible, to reduce the problem to the investigation of only scalar characteristics (type det $M(\lambda)$). This leads to an essential complication of the problem for rank $V>1$.

Let us describe the contents of the paper.

Section 1 is devoted to the spectral analysis of the Friedrichs model. Here we reduce the investigation of the spectrum to that of roots of an analytic operator-function with the positive imaginary part. Section 2 deals with the uniqueness theorem for analytic operator-functions with positive imaginary part on the circle or half-plane. Classes of such operator-valued functions which naturally appear in the selfadjoint perturbation theory are studied. In the next section on the contrary we make use of operator theory to prove some facts for analytic operator functions with positive imaginary part, and, in particular, uniqueness theorems.

The main Section 4 concerns the detailed study of the roots of the operator-function $M(\lambda)$. Here we prove the main theorem on the "doubling of the root order" 2α of the operator-function $M(\lambda)$ when the kernel v satisfies the condition of type Lip α.

Finally, in Section 5, basing on the results of the preceeding sections, we prove the main theorem on the spectral structure of operators in the Friedrichs model. Different variants of conditions on v are considered. This allows us to compare the latter theorem with the other results on the spectral analysis of Friedrichs model obtained by Friedrichs and Faddeev.

Note that the main results of this paper were announced in [10].

I. *Friedrichs model in perturbation theory and the operator-function* $M(\lambda)$.

Recall that for the operator L in the Friedrichs model we have

$$(Lu)(x) = xu(x)+\int_{\mathbf{R}} v(x, y)\,u(y)\,dy, \quad u\in L_2(\mathbf{R})$$

so that $L=x\cdot\ +V$, where V is an integral operator with kernel $v(x, y)$ satisfying ($\alpha<1/2$).

$$v(x+h, x+h)+v(x, x)-v(x+h, x)-v(x, x+h) \leqq C_L|h|^{2\alpha}. \tag{1.1}$$

Let $V\geqq 0$, $V\in S^1$. We shall consider the operator-valued function $M(\lambda)$, $\operatorname{Im}\lambda\neq 0$, taking values in the space of bounded operators on E, where $E=\overline{R(V)}$ is the closure of the range $R(V)$ of V:

$$M(\lambda)\equiv I+(V)^{1/2}(x-\lambda)^{-1}(V)^{1/2},\quad \operatorname{Im}\lambda\neq 0, \tag{1.2}$$

where $(x-\lambda)^{-1}$ means the operator of multiplication by the function $(x-\lambda)^{-1}$ on $L_2(\mathbf{R})$.

Obviously $\operatorname{Im} M(\lambda)\geqq 0$, $\operatorname{Im}\lambda>0$, and we have $(M(\lambda)-I)\in S^1$ if $V\in S^1$. The next identities are the direct consequences of the Hilbert identity for resolvents:

$$\begin{aligned} M(\lambda)(V)^{1/2}(L-\lambda)^{-1} &= (V)^{1/2}(x-\lambda)^{-1},\\ M^{-1}(\lambda) &= I-(V)^{1/2}(L-\lambda)^{-1}(V)^{1/2},\quad \operatorname{Im}\lambda\neq 0.\end{aligned} \tag{1.3}$$

Since $V\in S^1$, the absolutely continuous spectrum of L coincides with the real axis $\mathbf{R}$ [16]. The following elementary proposition describes the connection between the singular spectrum of L and the "roots" of M.

Proposition 1.1. *Let $M(\lambda)$ be defined by* (1.2). *If*

$$\sup_{\substack{|\lambda-\lambda_0|\leqq\varepsilon\\ \operatorname{Im}\lambda>0}}\|M^{-1}(\lambda)\|<\infty,\quad \operatorname{Im}\lambda_0=0,\quad \varepsilon>0, \tag{1.4}$$

then the spectrum of L in an ε-neighbourhood of $\lambda_0\in\mathbf{R}$ is absolutely continuous.

Proof. Let $\Delta\equiv(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\subset\mathbf{R}$. It is enough to check that $((L-k-i\varepsilon)^{-1}\varphi,\varphi)\in L_p(\Delta)$ uniformly on ε for a dense set of vectors $\varphi\in H\overset{\text{def}}{=\!=}L_2(\mathbf{R})$ and for some $p>1$.

If $\varphi=(V)^{1/2}h$, $h\in H$, then by (1.3) we have

$$\begin{aligned}|((L-\lambda)^{-1}\varphi,\varphi)| &= |((V)^{1/2}(L-\lambda)^{-1}(V)^{1/2}h,h)|\\ &= |(M^{-1}(\lambda)(V)^{1/2}(x-\lambda)^{-1}(V)^{1/2}h,h)|\\ &= |(M^{-1}(\lambda)(M(\lambda)-I)h,h)|\\ &= |((I-M^{-1}(\lambda))h,h)|\leqq(1+\|M^{-1}(\lambda)\|)\|h\|^2.\end{aligned}$$

So under (1.4) $((L-\lambda)^{-1}\varphi,\varphi)$ belongs to $L_\infty(\Delta)$. The dense set of vectors $\varphi\in H$ can be determined as follows. Let $\operatorname{Im}\mu<0$, $\varphi=(L-\mu)^{-1}(V)^{1/2}h$, then the Hilbert identily yields

$$\begin{aligned} &(V)^{1/2}(L-\lambda)^{-1}(L-\bar\mu)^{-1}(L-\mu)^{-1}(V)^{1/2}h\\ &=(\mu-\lambda)^{-1}(V)^{1/2}(L-\bar\mu)^{-1}((L-\lambda)^{-1}-(L-\mu)^{-1})(V)^{1/2}h\\ &=(\mu-\lambda)^{-1}(V)^{1/2}(L-\bar\mu)^{-1}(-1)(L-\mu)^{-1}(V)^{1/2}h\\ &+(\mu-\lambda)^{-1}(\bar\mu-\lambda)^{-1}(V)^{1/2}((L-\lambda)^{-1}-(L-\bar\mu)^{-1})(V)^{1/2}h.\end{aligned}$$

This fact together with the obvious inclusion $(\mu-\lambda)^{-1}\|(L-\mu)^{-1}(V)^{1/2}h\|^2\in L_\infty$ (as a function of the variable λ, $\operatorname{Im}\lambda>0$) proves the proposition.

Note in conclusion that the operator L on the subspace

$$H \ominus \bigvee_{\mathrm{Im}\,\mu<0} (L-\mu)^{-1} R(V^{1/2})$$

coincides with multiplication by x and therefore its spectrum in the orthogonal complement is also absolutely continuous.

Suppose that $V \in S^1$ and satisfies (1.1) and that (1.4) does not hold for any $\varepsilon > 0$. Then there exists $f \in E$, $f \neq 0$, such that $M(\lambda_0) f = 0$.

Theorem 1.1. *Under condition* (1.1) *and* $V \in S^1$ *the operator-valued function* $M(\lambda)$ *belongs to the Lipschitz class* Lip α *with index* α, $\alpha < 1/2$ *in* S^1 *norm, i.e.*

$$\|M(\lambda) - M(\lambda')\|_{S^1} \leqq \mathrm{const}\, |\lambda - \lambda'|^{\alpha}, \quad \mathrm{Im}\, \lambda, \lambda' \geqq 0. \tag{1.5}$$

Proof. Suppose first $\mathrm{Im}\,\lambda, \lambda' > 0$ and that $G \in B(H)$. Then

$$\begin{aligned} \mathrm{tr}\left((M(\lambda)-I)\, G\right) &= \mathrm{tr}\left(V^{1/2}(x-\lambda)^{-1} V^{1/2} G\right) \\ &= \textstyle\sum_i \left((V)^{1/2}(x-\lambda)^{-1}(V)^{1/2} G\varphi_i, \varphi_i\right) \\ &= \textstyle\sum_{i,j} \sqrt{\lambda_i \lambda_j} \left((x-\lambda)^{-1}\varphi_j, \varphi_i\right)(G\varphi_i, \varphi_j) \\ &= \int_{\mathbf{R}} (x-\lambda)^{-1} \left\{\textstyle\sum_{i,j} \sqrt{\lambda_i\lambda_j}(G\varphi_i, \varphi_j)\, \varphi_j(x)\, \overline{\varphi_i(x)}\right\} dx. \end{aligned}$$

Where $V = \sum_i \lambda_i \cdot (\cdot, \varphi_i)\varphi_i$ is the spectral decomposition of the selfadjoint operator V, $(\varphi_i, \varphi_k) = \delta_{ik}$, $\lambda_i > 0$, $\|V\|_{S^1} = \sum_i \lambda_i < \infty$.

We obtain by Privalov's theorem [15] for a scalar analytic function

$$\begin{aligned} \left|\mathrm{tr}\,(M(\lambda)-M(\lambda'))\,G\right| \leqq \mathrm{const}\, |\lambda-\lambda'|^{\alpha} \Big\{ &\Big\| \textstyle\sum_{i,j} \sqrt{\lambda_i\lambda_j}(G\varphi_i, \varphi_j)\, \varphi_j(x)\, \overline{\varphi_i(x)} \Big\|_{\mathrm{Lip}\,\alpha} \\ &+ \Big\| \textstyle\sum_{i,j} \sqrt{\lambda_i\lambda_j}(G\varphi_i, \varphi_j)\, \varphi_j(x)\, \overline{\varphi_i(x)} \Big\|_{L_1(\mathbf{R})} \Big\}. \end{aligned}$$

Use the convenient notations

$$\left\{\sqrt{\lambda_i}\, \bar{\varphi}_i(x)\right\} \overset{\mathrm{def}}{=\!=} \Phi(x) \in L_2(\mathbf{R}, l_2), \quad \|\Phi(x)\|^2_{l^2} \overset{\mathrm{a.e.}}{=\!=} \textstyle\sum_i \lambda_i |\varphi_i(x)|^2 < \infty$$

then

$$\textstyle\sum_{i,j} \sqrt{\lambda_i\lambda_j}(G\varphi_i, \varphi_j)\, \varphi_j(x)\, \overline{\varphi_i(x)} \equiv \langle \Phi(x), G^* \Phi(x)\rangle_{l_2}$$

where for the realization of G in l_2 we keep the same notations. Since

$$\begin{aligned} \|\langle \Phi(x), G^*\Phi(x)\rangle_{l_2}\|_{L_1(\mathbf{R})} &\leqq \int_{\mathbf{R}} \|\Phi(x)\|^2 \|G^*\|\, dx \\ &= \int_{\mathbf{R}} \|\Phi(x)\|^2 dx\, \|G\| = \left(\textstyle\sum_i \lambda_i\right)\|G\| = \|V\|_{S^1}\|G\|, \end{aligned}$$

we have

$$\begin{aligned} \left|\mathrm{tr}\,(M(\lambda)-M(\lambda'))\,G\right| &\leqq \mathrm{const}\, |\lambda-\lambda'|^{\alpha} \left\{2\,\|G\| \left(\sup_x \|\Phi(x)\|\right) \|\Phi(x)\|_{\mathrm{Lip}\alpha} + \|G\|\, \|V\|_{S^1}\right\} \\ &= \mathrm{const}\, |\lambda-\lambda'|^{\alpha} \|G\| \left\{2\left(\sup_x \|\Phi(x)\|\right) \|\Phi(x)\|_{\mathrm{Lip}\alpha} + \|V\|_{S^1}\right\}. \end{aligned}$$

Lemma 1.1. *Let v satisfies* (1.1) *then*

$$\|\Phi(x+h)-\Phi(x)\|_{l_2} \leqq C_L^{1/2}|h|^\alpha, \quad x\in\mathbf{R}.$$

Proof. If $v(x,y)=\sum_i \lambda_i\varphi_i(x)\cdot\overline{\varphi_i(y)}$ is an eigenfunction expansion of the kernel of the selfadjoint operator V then the simple calculation shows

$$\|\Phi(x+h)-\Phi(x)\|^2 = \|\Phi(x+h)\|^2+\|\Phi(x)\|^2-(\Phi(x+h),\Phi(x))-(\Phi(x),\Phi(x+h))$$
$$=\sum_i\lambda_i|\varphi_i(x+h)|^2+\sum_i\lambda_i|\varphi_i(x)|^2-\sum_i\lambda_i\varphi_i(x+h)\overline{\varphi_i(x)}-\sum_i\lambda_i\varphi_i(x)\overline{\varphi_i(x+h)}$$
$$= v(x+h,x+h)+v(x,x)-v(x+h,x)-v(x,x+h) \leqq C_L|h|^{2\alpha}.$$

The uniform boundedness of $\|\Phi(x)\|$ is equivalent to the condition $\sup_x v(x,x)<\infty$. Indeed

$$\|\Phi(x)\|^2=\sum_i\lambda_i|\varphi_i(x)|^2 = v(x,x)\in L_1(\mathbf{R})$$

since $V\geqq 0$, $V\in S^1$. The local estimate $\|\Phi(x+h)-\Phi(x)\|\leqq C_L^{1/2}|h|^\alpha$ (see Lemma 1.1) together with the condition $\|\Phi(x)\|\in L_2(\mathbf{R})$ of course leads to the uniform boundedness of $\|\Phi(x)\|$. Finally,

$$\left|\operatorname{tr}\left\{(M(\lambda)-M(\lambda'))G\right\}\right| \leqq \text{const}\,|\lambda-\lambda'|^\alpha\,\|G\|$$

which is equivalent to (1.3) by the equality

$$\sup_{\substack{G\in B(H)\\ \|G\|=1}} |\operatorname{tr}(AG)| = \|A\|_{S^1}$$

[6]. Here $B(H)$ denotes the space of all bounded operators on H.

It is not difficult now to prove that the boundary values

$$M(k) \stackrel{\text{def}}{=} \text{s}-\lim_{\varepsilon\to+0} M(k+i\varepsilon)$$

belong to S^1 and that (1.3) holds for $\operatorname{Im}\lambda,\lambda'\geqq 0$.

The corollary of the theorem is that $M(\lambda)$ is continuous (in S^1-norm) in the closed upper half-plane. Let there exist the operator $M^{-1}(k)$ for some $k\in\mathbf{R}$ then by the continuity the estimate (1.4) holds in a neighbourhood of k on $\mathbf{C}$. Hence the failure of (1.4) leads to that $M^{-1}(k)$ does not exist. Since $(M(k)-I)\in S^1\subset S^\infty$, this means that there exists an eigenvector of $M(k)$ corresponding to the eigenvalue 0.

Theorem 1.1 reduces the problem description of the singular spectrum in selfadjoint Friedrichs model to the description of the rootsets Λ of the operator-function $M(\lambda)$ with positive imaginary part.

$$\Lambda \stackrel{\text{def}}{=} \{k\in\mathbf{R}\colon \exists e\neq 0,\, e\in E,\, M(k)e=0\},$$
$$\Lambda \supset \mathfrak{S}_S(L)\cup\mathfrak{S}_p(L).$$

Note that the converse statement is not true (if $\alpha<1/2$) even for $\operatorname{rank} V=1$. This can easily be confirmed by an example.

II. *The uniqueness theorems for the operator-functions with positive imaginary part.*

First we examine the operator-valued function on the unit circle. Let the "modulus of continuity" $\omega(t)$ be a monotone continuous function, $t \geqq 0$, $\omega(0)=0$, and S^p, $p \geqq 1$, be the Schatten classes [6].

Theorem 2.1. *Let E be a Hilbert space, $N(z)$, $\operatorname{Im} N(z) \geqq 0$ be an E-valued analytic operator-function on the unit disc $|z|<1$ satisfying $(N(z)-I) \in S^1$, $|z|<1$. If $N(z)$ is invertible at some point of the circle (and hence at every point of the circle [3]) then the Lebesgue measure of the δ-neighbourhood on the unit circle T of the set*

$$F \overset{\text{def}}{=} \{w \in T\colon \|N^{-1}(z)-I\|_{S^p}^{-1} \leqq \omega(|z-w|),\ |z|<1\}$$

satisfies the condition

$$\operatorname{mes} F^\delta \leqq C\omega(\delta) \tag{2.1}$$

with a constant C depending only on p, ω and on $\|N^{-1}(0)-I\|_{S^1}$ $p>1$.

This theorem and its proof are the generalizations to the operator case of the uniqueness theorem from [7, 12]. Note that the condition (2.1) for $\omega(t)=t$ leads to the finiteness of the set F.

Proof. Use the fact that Calderon—Zygmund theory of the singular integral operators (and of the Hilbert transform in particular) can be generalized to the case of Banach space S^p-valued functions [2, 17, 18]. Note that the statement is not a consequence of the corresponding "scalar" theorem. The operator-functions $\operatorname{Im} N^{-1}(re^{i\theta})$ and $(\operatorname{Re} N^{-1}(re^{i\theta}) - \operatorname{Re} N^{-1}(0))$ connected by the Hilbert transform take values in $S^1 \subset S^p$, $p>1$ for every $r<1$. Since this transformation of the S^p-valued function on T is of week type (1,1) in $L_1(T, S^p)$ [2, 18], we get for any $y>0$

$$\operatorname{mes}\{\theta\colon \|\operatorname{Re} N^{-1}(re^{i\theta}) - \operatorname{Re} N^{-1}(0)\|_{S^p} > y\}$$
$$\leqq (C_p/y)\int_T \|\operatorname{Im} N^{-1}(re^{i\theta})\|_{S^p}\, d\theta.$$

Hence

$$y \operatorname{mes}\{\theta\colon \|\operatorname{Re} N^{-1}(re^{i\theta}) - \operatorname{Re} N^{-1}(0)\|_{S^p} > y\}$$
$$\leqq C_p \int_T \|\operatorname{Im} N^{-1}(re^{i\theta})\|_{S^1}\, d\theta = -C_p \operatorname{tr} \int_T \operatorname{Im} N^{-1}(re^{i\theta})\, d\theta = -C_p \operatorname{tr}(\operatorname{Im} 2\pi N^{-1}(0))$$
$$= 2\pi C_p \|\operatorname{Im} N^{-1}(0)\|_{S^1} \leqq 2\pi C_p \|N^{-1}(0)-I\|_{S^1}$$

and therefore

$$y \operatorname{mes}\{\theta;\ \|N^{-1}(re^{i\theta}) - \operatorname{Re} N^{-1}(0)\|_{S^p} > y\} \leqq (C_p+1)4\pi \|N^{-1}(0)-I\|_{S^1}.$$

Fix $r, 0<r<1$, then

$$F^\delta \subset \{w \in T: \|N^{-1}(rw)-I\|_{S^p}^{-1} \leqq \omega(\delta+(1-r))\}$$

and

$$\operatorname{mes} F^\delta \leqq \operatorname{mes}\{\theta: \|N^{-1}(re^{i\theta})-I\|_{S^p}^{-1} \leqq \omega(\delta+(1-r))\}$$

$$\leqq \operatorname{mes}\{\theta: \|N^{-1}(re^{i\theta})-\operatorname{Re} N^{-1}(0)\|_{S^p} \geqq \omega^{-1}(\delta+(1-r))-\|\operatorname{Re} N^{-1}(0)-I\|_{S^p}\}$$

$$\leqq (C_p+1)\,4\pi\,\|N^{-1}(0)-I\|_{S^1}(\omega^{-1}(\delta+(1-r))-\|\operatorname{Re} N^{-1}(0)-I\|_{S^p})^{-1}$$

$$= 4\pi(C_p+1)\,\|N^{-1}(0)-I\|_{S^1}\,\omega(\delta+(1-r))(1-\omega(\delta+(1-r))\,\|\operatorname{Re} N^{-1}(0)-I\|_{S^p})^{-1}.$$

Choose δ to be small enough (for example let $\omega(\delta)\cdot\|\operatorname{Re} N^{-1}(0)-I\|_{S^p}<1/2$ that is however not essential).

If $r\to 1$ we get the inequality

$$\operatorname{mes} F^\delta \leqq (C_p+1)\,4\pi\,\|N^{-1}(0)-I\|_{S^1}\,\omega(\delta)\,2.$$

In case $\omega(t)=t^\alpha$, $0<\alpha\leqq 1$ we can describe in particular the structure of the set of "roots of order α" for operator-functions with positive imaginary part.

Now we consider analogous results for operator-functions on the upper half-plane $\mathbf{C}_+$.

Let us introduce a class $R_0(X)$ of operator-valued functions $T(\lambda)$ analytic on $\mathbf{C}_+$ which take values in some Banach space $X\subseteqq B(E)$ of operators on Hilbert space E (interesting examples are $X=S^p$, $1\leqq p\leqq\infty$, $X=B(E)$) and which satisfy the conditions:

(2.2)

(1) $$\operatorname{Im} T(\lambda) \geqq 0, \quad \operatorname{Im}\lambda > 0;$$

(2) $$T(i\tau) \xrightarrow[\tau\to+\infty]{w} \mathbf{0}.$$

(3) $$\int_{\mathbf{R}} \operatorname{Im} T(k+i\varepsilon)\,dk \in X, \quad \varepsilon > 0.$$

It is easy to show that the last integral does not depend on $\varepsilon>0$. Moreover, it turns out that condition (2) can be substituted by $\|T(i\tau)\|_X=O(1/\tau)$, $\tau\to+\infty$.

The above definition is a natural generalization of the class R_0 of scalar-valued analytic functions f on $\mathbf{C}_+$ such that: (1) $\operatorname{Im} f(\lambda)\geqq 0$; (2) $\sup_{\tau>0}\tau\,|f(i\tau)|<\infty$ (see Appendix II in the Russian translation of the book [1]).

Assume now that X satisfies the following "monotonicity" property: if $0\leqq A\leqq B$, $A=A^*$, $B=B^*$, $B\in X$, $A\in B(E)$ then $A\in X$ and $\|A\|_X\leqq\|B\|_X$, and in addition X-norm $\|\cdot\|_X$ is a majorant of the usual operator norm in $B(E)$ ($\|T\|\leqq\|T\|_X$, $T\in X$). The next theorem is a simple generalization of the corresponding "scalar" ascertion from [1] (for $X=B(E)$ the similar theorem was proved in [3]).

Theorem 2.2. *Assume that $T(\lambda)$ satisfies* (1), (2). *Then the following are equivalent;*

(3.1) $$\int_{\mathbf{R}} \operatorname{Im} T(k+i\varepsilon)\, dk \in X, \quad \varepsilon > 0;$$

(3.2) $$\sup_{\tau>0} \tau \|\operatorname{Im} T(i\tau)\|_X < \infty;$$

(3.3) $$\lim_{\tau\to+\infty} \tau \|T(i\tau)\|_X < \infty;$$

(3.2), (3.3) in case X is a cross-ideal [6], $X \neq S_\infty$

(3.4) $$\operatorname{tr} T(\lambda) \in R_0 \quad \textit{in case} \quad X = S^1;$$

(3.5) *There exist a selfadjoint operator $\mathscr{L}$ on a Hilbert space $\mathscr{H}$ and a bounded operator $R\colon E \to \mathscr{H}$ such that $R^*R \in X$ and*

$$T(\lambda) = R^*(\mathscr{L}-\lambda)^{-1}R, \quad \operatorname{Im}\lambda > 0.$$

Proof. The equivalence of (3.1)—(3.4) can be checked by the Riesz—Herglotz theorem in the same way as in the "scalar" case. To this end one should consider the functions $(T(\lambda)\varphi, \varphi)$, $\varphi \in E$, and take some care for the estimates obtained be uniformly dependent on φ. Here we only prove the equivalence of conditions (3.1), (3.5). This permits us to find a connection between $R_0(X)$ and a class of operator-functions arising in the perturbation theory of selfadjoint operators. The connection can be used for studying of some properties of $R_0(X)$ (see for example § 3) with the help of operator theory.

(3.1)⇒(3-5). Let $\varphi \in E$, $T_\varphi(\lambda) \stackrel{\text{def}}{=} (T(\lambda)\varphi, \varphi)$. By the Riesz—Herglotz theorem and $T_\varphi(i\tau) \xrightarrow[\tau\to\infty]{} 0$ we get

(2.3) $$T_\varphi(\lambda) = \int_{\mathbf{R}} (\mu-\lambda)^{-1}\, d\tau_\varphi(\mu)$$

for a monotone function $\tau_\varphi(\mu)$ of bounded variation:

$$\int_{\mathbf{R}} \operatorname{Im} T_\varphi(k+i\varepsilon)\, dk = \int_{\mathbf{R}} dk \int_{\mathbf{R}} \frac{\varepsilon}{(\mu-k)^2+\varepsilon^2}\, d\tau_\varphi(\mu) = \int_{\mathbf{R}} \pi\, d\tau_\varphi(\mu).$$

The uniqueness of the representation (2.3) implies that $\tau_\varphi(\mu)$ is a bounded quadratic form in φ, $\tau_\varphi(\mu) = (B_\mu\varphi, \varphi)$ for some selfadjoint operators B_μ on E, $\mu \in \mathbf{R}$. Obviously, $B_\mu \geqq 0$,

$$\|B_\mu\| \leqq \left\| \int_{\mathbf{R}} \operatorname{Im} T(k+i\varepsilon)\, dk \right\|$$

and B_μ is a monotone function of μ, $\mu \in \mathbf{R}$. As in [10], by Naimark's theorem [1], there exist a Hilbert space $\mathscr{H}$, a selfadjoint spectral resolution E_μ and a bounded operator $R\colon E \to \mathscr{H}$ such that $B_\mu = R^*E_\mu R$, $\mu \in \mathbf{R}$. Hence

$$T_\varphi(\lambda) = \int_{\mathbf{R}} (\mu-\lambda)^{-1}\, d(E_\mu R\varphi, R\varphi) = \left(R^*(\mathscr{L}-\lambda)^{-1}R\varphi, \varphi\right)$$

where $\mathscr{L}$ is the selfadjoint operator on $\mathscr{H}$ with the spectral resolution E_μ. Then

$$\int_{\mathbf{R}} \operatorname{Im} T_\varphi(\lambda)\,dk = \int_{\mathbf{R}} dk \int_{\mathbf{R}} \frac{\varepsilon}{(\mu-k)^2+\varepsilon^2}\, d(E_\mu R\varphi, R\varphi)$$

$$= \pi \int_{\mathbf{R}} d(E_\mu R\varphi, R\varphi) = \pi(R^*R\varphi, \varphi),$$

i.e.

$$\int_{\mathbf{R}} \operatorname{Im} T(k+i\varepsilon)\,dk = \pi R^*R\in X.$$

(3.5)⇒(3.1). Note that under condition (3.5) we can obtain (3) as well as (1), (2) from the identity

$$\int_{\mathbf{R}} \operatorname{Im} T(k+i\varepsilon)\,dk = \pi R^*R.$$

Therefore only condition (3.5) is a criterion for $T(\lambda)\in R_0(X)$. Making use of this criterion one can easily prove the rest of the theorem.

Remark. (1) By the same technique we obtain for a general operator-function $M(\lambda)$, $\operatorname{Im} M(\lambda)\geqq 0$ that

$$M(\lambda) = A+B\lambda+R^*(I+\lambda\mathscr{L})(\mathscr{L}-\lambda)^{-1}R$$

where $A=A^*$, $B\geqq 0$, R is a bounded operator $R\colon E\to\mathscr{H}$, and $\mathscr{L}=\mathscr{L}^*$ on $\mathscr{H}$.

(2) This description of the analytic operator-valued functions with positive imaginary part permits one to show that $M(\lambda)\in X$, $\lambda\in\mathbf{C}_+$ if $M(\lambda_0)\in X$ for some $\lambda_0\in\mathbf{C}_+$. It is assumed here X satisfies the condition of "monotonicity".

The following result is important in what follows.

Theorem 2.3. *If* $T(\lambda)\in R_0(X)$ *then* $(I-(I+T(\lambda))^{-1})\in R_0(X)$.

Proof. Although the theorem can easily be proved by the usual function-theoretic technique we give an "operator" proof. If

$$T(\lambda) = R^*(\mathscr{L}-\lambda)^{-1}R,\quad R\colon E\to\mathscr{H},\quad R^*R\in X$$

then

$$(I+T(\lambda))^{-1} = I-R^*(\mathscr{L}+RR^*-\lambda)^{-1}R.$$

$RR^*\colon \mathscr{H}\to\mathscr{H}$. Indeed by the Hilbert identity

$$\big(I-R^*(\mathscr{L}+RR^*-\lambda)^{-1}R\big)\big(I+R^*(\mathscr{L}-\lambda)^{-1}R\big)$$

$$= I-R^*(\mathscr{L}+RR^*-\lambda)^{-1}R+R^*(\mathscr{L}-\lambda)^{-1}R-R^*(\mathscr{L}+RR^*-\lambda)$$

$$\cdot[(\mathscr{L}+RR^*-\lambda)-(\mathscr{L}-\lambda)](\mathscr{L}-\lambda)^{-1}R = I.$$

Since RR^* is selfadjoint and bounded on $\mathscr{H}$, $(\mathscr{L}+RR^*)$ is selfadjoint too. This completes the proof of the theorem.

Fix $\varepsilon_0>0$ and a modulus of continuity ω.

Theorem 2.4. *Let E be a Hilbert space and let $M(\lambda)$ be an operator-function, such that $(M(\lambda)-I)\in R_0(S^1)$. Then the Lebesgue measure on* $\mathbf{R}$ *of the δ-neighbourhood of*

$$F=\{k\in\mathbf{R}\colon \|M^{-1}(\lambda)-I\|_{S^p}^{-1}\leqq\omega(|k-\lambda|),\ \operatorname{Im}\lambda>0,\ |k-\lambda|\leqq\varepsilon_0\}$$

satisfies the condition

$$\text{mes}\, F^\delta\leqq C\omega(\delta),\quad \delta\leqq\varepsilon_0, \tag{2.4}$$

with constant C depending only on p, and

$$\lim_{\tau\to\infty}\tau\,\|M(i\tau)-I\|_{S^1},\quad p>1.$$

The "local" version of the theorem follows from Theorem 2.1 by the simple transformation. Actually we only need to check that F is bounded.

Proof. As in Theorem 2.1, we have for the Hilbert transform on $L_1(\mathbf{R}, S^p)$

$$y\,\text{mes}\,\{k\in\mathbf{R}\colon \|M^{-1}(k+i\varepsilon)-I\|>y\}\leqq C_p\left\|\int_{\mathbf{R}}dk\,\operatorname{Im}M^{-1}(k+i\varepsilon)\right\|_{S^1}$$
$$=C_p\pi\,\|R^*R\|_{S^1}=C_p\pi\lim_{\tau\to\infty}\tau\,\|M(i\tau)-I\|_{S^1}.$$

Here Theorems 2.2, 2.3 were used. Therefore for any $\varepsilon>0$, $\delta>0$ such that $\delta^2+\varepsilon^2\leqq\varepsilon_0^2$ we get

$$\text{mes}\,F^\delta\leqq\text{mes}\,\{k\in\mathbf{R}\colon \|M^{-1}(k+i\varepsilon)-I\|_{S^p}^{-1}\leqq\omega(\delta+\varepsilon)\}$$
$$\leqq C_p\pi\,\|R^*R\|_{S^1}\,\omega(\delta+\varepsilon).$$

If $\varepsilon\to+0$ then necessary inequality holds

$$\text{mes}\,F^\delta\leqq C_p\pi\big(\lim_{\tau\to\infty}\tau\,\|M(i\tau)-I\|_{S^1}\big)\,\omega(\delta),\quad \delta\leqq\varepsilon_0$$

which clearly implies the desired inequality.

III. *An operator proof of the Uniqueness theorem.*

In this section, making use of the spectral theorem for selfadjoint operators, we give the proof of the uniqueness theorem for the operator-valued functions with positive imaginary part.

Let $M\big(M(\lambda)\colon H\to H,\ \lambda\in\mathbf{C}\setminus\mathbf{R}\big)$ be the operator-function

$$M(\lambda)=I+\sqrt{V}(A-\lambda)^{-1}\sqrt{V}, \tag{3.1}$$

where $L\overset{\text{def}}{=}A+V$, A, V are selfadjoint operators on a Hilbert space H, $V\geqq0$ and V is bounded (or is an (A)-bounded operator [8] whose relative bound is smaller than 1). Evidently, $\operatorname{Im}M(\lambda)\geqq0$, $\operatorname{Im}\lambda>0$. It is easy to check that the "general" case of operator-functions M, $(M-I)\in R_0(S^1)$, $M(\lambda)=I+R^*(\mathscr{L}-\lambda)^{-1}R$ can be reduced to (3.1). For this purpose we can consider the polar decomposition $R=J\cdot T$ where $T\geqq 0$, $T\colon E\to E$ and J is an isometric embedding of E into $\mathcal{H}$. Here $T=\sqrt{V}$ and $R^*R=T^2$ which implies that $T\in S^2$.

Theorem 3.1. *Let $V\in S^1$ then the set F of "roots of M of the order α", $1/2<\alpha\leqq 1$,*

$$F \overset{\text{def}}{=} \{k\in\mathbf{R}\colon \|I-M^{-1}(\lambda)\|_{S^1}^{-1} \leqq C|\lambda-k|^{\alpha},\ 0<|k-\lambda|<\varepsilon_0,\ \operatorname{Im}\lambda>0\}$$

satisfies the estimate

$$\text{mes}\, F^{\delta} \leqq 4\cdot\pi C^2\|V\|_{S^1}^2\delta^{2\alpha-1},\quad \delta\leqq\varepsilon_0. \tag{3.2}$$

Note that for $\alpha=1$ the theorem is close to Theorem 2.4 (about finiteness of the set of roots) but in the present case the set F is wider than that in Theorem 2.4 since in its definition the S^1-norm is involved. Unfortunately for $\alpha<1$ the estimate (3.2) is essentially weaker than (2.4).

Proof. It is easy to check that

$$M^{-1}(\lambda)-I=-\sqrt{V}(L-\lambda)^{-1}\sqrt{V}.$$

Consider the orthonormal system of eigenvectors $\varphi_k\in H$ of V, $V=\sum_k\lambda_k(\cdot,\varphi_k)\varphi_k$. Then we have

$$\|M^{-1}(\lambda)-I\|_{S^1}^2=\left\|\sqrt{V}(L-\lambda)^{-1}\sqrt{V}\right\|_{S^1}^2\leqq\left\|\sqrt{V}\right\|_{S^2}^2$$

$$\cdot\left\|(L-\lambda)^{-1}\sqrt{V}\right\|_{S^2}^2=\|V\|_{S^1}\sum_k\left\|(L-\lambda)^{-1}\sqrt{V}\varphi_k\right\|^2.$$

Therefore ($\varepsilon>0$)

$$\int_{\mathbf{R}}dk\,\|M^{-1}(k+i\varepsilon)-I\|_{S^1}^2\leqq\|V\|_{S^1}\sum_k\lambda_k\int_{\mathbf{R}}\|(L-k-i\varepsilon)^{-1}\varphi_k\|^2dk$$

$$=\|V\|_{S^1}\sum_k\lambda_k(2\pi/\varepsilon)\,\|\varphi_k\|^2=\|V\|_{S^1}^2\,(2\pi/\varepsilon)$$

where we have made use of the following identity

$$\varepsilon\int_{\mathbf{R}}\|(L-k-i\varepsilon)^{-1}\varphi\|^2\,dk=2\pi\,\|\varphi\|^2,\quad\varphi\in H. \tag{3.3}$$

It is valid for any selfadjoint operator L on H. This follows from the spectral theorem by the direct computation [16]. Finally

$$\varepsilon\int_{\mathbf{R}}\|M^{-1}(k+i\varepsilon)-I\|_{S^1}^2\,dk\leqq 2\pi\,\|V\|_{S^1}^2,\quad\varepsilon>0. \tag{3.4}$$

Let us consider any finite subset F' of F together with δ-neighbourhoods of its points. We can suppose of course that any point on $\mathbf{R}$ belongs to at most two of such neighbourhoods. We denote them by Δ_i, $i=1,2,\dots,n$ and their centres by x_i.

Then $(\lambda \equiv k+i\delta)$

$$\int_{\Delta_i} \|M^{-1}(k+i\delta)-I\|_{S^1}^2 dk \geqq \int_{\Delta_i} C^{-2}[\text{dist}(\lambda, F)]^{-2\alpha} dk$$

$$\geqq C^{-2}\int_{\Delta_i} ((k-x_i)^2+\delta^2)^{-\alpha} dk = C^{-2}2\delta^{1-2\alpha}\int_0^1 (u^2+1)^{-\alpha} du$$

$$\geqq C^{-2}\delta^{1-2\alpha}.$$

By the inequality (3.4) we get

$$(4\pi/\delta)\|V\|_{S^1}^2 \geqq 2\int_{\bigcup_{i=1}^n \Delta_i} \|M^{-1}(k+i\delta)-I\|_{S^1}^2 dk \geqq (2/C^2)\delta^{1-2\alpha}n,$$

hence $n\delta \leqq c^2 \cdot 2\pi \cdot \|V\|_{S^1}^2 \cdot \delta^{2\alpha-1}$. This estimate $(n\delta)$ of the measure of δ-neighbourhood of F' leads to (3.2).

Note that the main point of the proof was the identity (3.3) i.e. the spectral theorem for selfadjoint operators. In view of this it will be interesting to find an operator proof for the more exact estimate with δ^α in the right-hand side of (3.2).

Remark. (1) Similar theorems (and proofs) are also valid for operator-valued function with J-positive imaginary part $(J=J^*,\ J^2=I)$. Indeed, let

$$L = A+J\sqrt{|V|}\sqrt{|V|},\ V=J|V|,\ J\overset{\text{def}}{=}\operatorname{sign} V$$

then

$$M(\lambda) = I+J\sqrt{|V|}(A-\lambda)^{-1}\sqrt{|V|},\quad \operatorname{Im} JM(\lambda) \geqq 0$$

if $\operatorname{Im}\lambda>0$. Again $M^{-1}(\lambda)=I-J\sqrt{|V|}\cdot(L-\lambda)^{-1}\sqrt{|V|}$ and we can proceed analogously

(2) The inequality (3.2) has an equivalent form:

$$\text{mes } F^\delta \leqq 4\cdot\pi C^2\delta^{2\alpha-1}(\lim_{\tau\to\infty}\tau\|\operatorname{Im} M(i\tau)\|_{S^1})^2.$$

It holds for any $(M-I)\in R_0(S^1)$.

(3) Clearly inequality (3.4) is equivalent to the following one $(T\in R_0(S^1))$

$$\varepsilon\int_{\mathbf{R}} \|T(k+i\varepsilon)\|_{S^1}^2 dk \leqq 2\pi(\lim_{\tau\to\infty}\tau\|T(i\tau)\|_{S^1})^2$$

or what is the same, to

$$\varepsilon\int_{\mathbf{R}} \|(I+T(k+i\varepsilon))^{-1}-I\|_{S^1}^2 dk \leqq 2\pi(\lim_{\tau\to\infty}\tau\|T(i\tau)\|_{S^1})^2.$$

(4) Let us note without proof that methods of operator theory permit us to obtain the following inequality as well $(T\in R_0(S^1))$

$$\int_{\mathbf{R}} dk \frac{\|T(k+i\varepsilon)\|_{S^2}^2}{(|\det(I-iT(k+i\varepsilon))|)^2} \leqq \pi \lim_{\tau\to+\infty}\tau\|T(i\tau)\|_{S^1},\quad \varepsilon>0.$$

IV. *Investigation of roots of the operator-function M.*

Let $f\in E$ and $V=\sum_k \lambda_k(\cdot, \varphi_k)\varphi_k$ be the spectral resolution of the selfadjoint operator V. Then $\sqrt{V}f=\sum_k \sqrt{\lambda_k}(f, \varphi_k)\varphi_k$. Making use of the usual isomorphism between E and

$$l_2(\text{i.e. } f\mapsto \{(f, \varphi_k)\}_{k=1}^{\infty},\ \|f\|^2=\sum_k |(f, \varphi_k)|^2)$$

we get a new formula for $M(\lambda)$ in l_2-representation:

$$M(\lambda)=I+\int_{\mathbf{R}}(x-\lambda)^{-1}(\cdot, \Phi(x))\Phi(x)\,dx.$$

Here for any $x\in\mathbf{R}(\cdot, \Phi(x))\Phi(x)$ is a rank one operator on l_2 with the eigenvector

$$\Phi(x)\overset{\text{def}}{=}\{\sqrt{\lambda_k}\bar{\varphi}_k(x)\}_{k=1}^{\infty}\in l_2, \quad x\in\mathbf{R}. \tag{4.1}$$

Thus in the new representation $(M(\lambda)-I)$ is the Cauchy transform of the operator-function on $\mathbf{R}$ with values in the set of rank one operators. Recall that (see Theorem 1.1) under condition (1.1) we have proved that $\Phi(x)\in\operatorname{Lip}\alpha$.

The boundary values of M on $\mathbf{R}\,(\lambda=k+i0)$ are

$$M(k)=I+\text{v.p.}\int_{\mathbf{R}}(x-k)^{-1}(\cdot, \Phi(x))\Phi(x)\,dx+i\pi\Phi(k)(\cdot, \Phi(k)).$$

Thus

$$F(k)\overset{\text{def}}{=}\operatorname{Im}M(k)=\pi\Phi(k)(\cdot, \Phi(k))\geqq 0$$

and $M(u_0)f=0$, $u_0\in\mathbf{R}$, $f\in E$, $f\neq 0$ if and only if

(1) $(f, \Phi(u_0))=0$,

(2) $f+\text{v.p.}\int_{\mathbf{R}}(x-u_0)^{-1}(f, \Phi(x))\Phi(x)\,dx=0$

The following assertion is the main result of the paper.

Theorem 4.1. *Suppose that* $V\in S^1$, $V\geqq 0$ *and that the kernel of the perturbation satisfies* (1.1). *Then for* $\alpha<1/2$ *and for a root* $u_0\in\mathbf{R}$ of M *the estimate*

$$\|M^{-1}(u)\|^{-1}\leqq C|u-u_0|^{2\alpha}, \quad u\in\mathbf{R}, \tag{4.2}$$

holds on a neighbourhood of u_0 *with* C *only depending on* α, $\|V\|_{S^1}$ *and* C_L (*see* (1.1)).

The proof is divided into several lemmas. We prove first the following refinement of the Pavlov—Petras lemma [11] which gives the needed estimate for rank one perturbations.

Lemma 4.1. *Let* $u, u_0\in\mathbf{R}$, $|u-u_0|\overset{\text{def}}{=}\delta$ *and*

$$S\overset{\text{def}}{=}(u_0-2|u-u_0|,\ u_0+2|u-u_0|)\subset\mathbf{R}.$$

If a function ψ *on* $\mathbf{R}$ *satisfies*

(1) $|\psi(t)-\psi(u)| \leqq C_1|t-u|^\alpha, \quad t\in S;$

(2) $|\psi(t)| \leqq C_1|t-u_0|^\alpha, \quad t\in S;$

(3) $\int_{\mathbf{R}} |\psi(t)|\, dt \leqq C_2;$

(4) $|\psi(t)| \leqq |t-u_0|^\beta (C_0+C_3(\operatorname{dist}(t, s))^\beta), \quad t\notin S$

then

$$|\varphi(u)-\varphi(u_0)| \leqq C_1 C_\alpha \delta^\alpha+2C_2\delta+C_0 C'_\beta \delta^\beta+C_3 C''_\beta \delta^{2\beta},$$

where C_α, C'_β, C''_β *are constants only depending on* $\alpha, \beta<1/2$ *and* φ *is the Hilbert transform of* ψ:

$$\varphi(u) = \text{v.p.}\int_{\mathbf{R}} \frac{\psi(t)\,dt}{t-u} = \text{v.p.}\int_{\mathbf{R}} \frac{\psi(t)-\psi(u)}{t-u}\,dt. \tag{4.3}$$

The proof is rather close to that of the paper [11] but for our purposes (even in case $\operatorname{rank} V=2$) an essential complication of the conditions on ψ is necessary. Since $\psi(u_0)=0$, we have

$$\varphi(u)-\varphi(u_0) = -\int_S \frac{\psi(t)-\psi(u_0)}{t-u_0}\,dt+\int_S \frac{\psi(t)-\psi(u)}{t-u}\,dt$$

$$-\lim_{N\to\infty}\int_{(-N,N)\setminus S} \frac{\psi(u)}{t-u}\,dt+\int_{\mathbf{R}\setminus S}(u-u_0)\frac{\psi(t)-\psi(u_0)}{(t-u)(t-u_0)}\,dt.$$

For the first two integrals above conditions (2) and (1) respectively imply

$$\left|\int_S \frac{\psi(t)-\psi(u_0)}{t-u_0}\,dt\right| \leqq \int_S \frac{C_1|t-u_0|^\alpha}{|t-u_0|}\,dt \leqq C_1\delta^\alpha 2^{\alpha+1}/\alpha,$$

$$\left|\int_S dt\frac{\psi(t)-\psi(u)}{t-u}\right| \leqq C_1\delta^\alpha 2^{\alpha+1}/\alpha.$$

The third integral does not exeed $\ln 3\cdot C_1\cdot\delta^\alpha$ and

$$\left|\int_{\mathbf{R}\setminus S}(u-u_0)\frac{\psi(t)-\psi(u_0)}{(t-u)(t-u_0)}\,dt\right| \leqq \delta\int_{|t-u_0|>1}\frac{|\psi(t)|}{|t-u|\,|t-u_0|}\,dt$$

$$+\delta\int_\Sigma \frac{|\psi(t)|\,dt}{|t-u_0|\,|t-u|} \leqq 2\delta C_2+2\delta\int_\Sigma\frac{|\psi(t)|\,dt}{|t-u_0|^2},$$

where

$$\Sigma \overset{\text{def}}{=} \{t\in\mathbf{R}\colon\ t\notin S,\ |t-u_0|<1\}.$$

We have by the property (4)

$$2\delta\int_{\Sigma}\frac{|\psi(t)|}{|t-u_0|^2}\,dt \leqq 4\delta\int_{(u_0+2\delta)}^{1} dt\,\frac{|t-u_0|^\beta}{|t-u_0|^2}\left(C_0+C_3|t-u_0-2\delta|^\beta\right)$$

$$\leqq 4\delta\left(C_0(\delta^{\beta-1}/(1-\beta))+C_3\int_0^1\frac{\tau^\beta\,d\tau}{(\tau+2\delta)^{2-\beta}}\right) \leqq C_0 4\delta^\beta/(1-\beta)+4C_3\delta^{2\beta}/(1-2\beta).$$

Finally,

$$|\varphi(u)-\varphi(u_0)| \leqq 4C_1 2^\alpha\delta^\alpha/\alpha+\ln 3C_1\delta^\alpha+2\delta C_3+C_0 4\delta^\beta/(1-\beta)+4C_3\delta^{2\beta}/(1-2\beta).$$

This completes the proof of the lemma with $C_\alpha=4\cdot 2^\alpha/\alpha+\ln 3$, $C''_\beta=4/(1-2\beta)$, $C'_\beta=4/(1-\beta)$.

Note that in the case of $C_3=0$ the lemma is valid for $\alpha<1/2$, $\beta<1$.

We consider first the simple case which is analogous to the scalar case when $\Phi(u_0)=0$. Theorem 4.1 is a direct consequence of the following lemma.

Lemma 4.2. *If* $\Phi(u_0)=0$, $u_0\in\mathbf{R}$ *then*

$$\|M(u)f\| \leqq C_L\,\mathrm{const}\,|u-u_0|^{2\alpha},\quad u\in\mathbf{R},\ \alpha<1/2,$$

where f satisfies $M(u_0)f=0$, $\|f\|=1$ *and* C_L *is the constant from* (1.1).

Proof. Let $|u-u_0|\equiv\delta$. Then

$$\|M(u)f\| = \left\|(M(u)-M(u_0))f\right\|$$

$$\leqq \pi\|F(u)f\|+\left\|\text{v.p.}\int_{\mathbf{R}}(x-u)^{-1}F(x)f\,dx-\text{v.p.}\int_{\mathbf{R}}(x-u_0)^{-1}F(x)f\,dx\right\|,$$

$$\|F(u)f\| \leqq \|\Phi(u)\|^2\|f\| = \|\Phi(u)-\Phi(u_0)\|^2 \leqq C_L|u-u_0|^{2\alpha}.$$

Therefore

$$\|M(u)f\| \leqq \pi C_L\delta^{2\alpha}+\sup_{\|g\|=1}\Big|\text{v.p.}\int_{\mathbf{R}}(x-u)^{-1}(F(x)f,g)\,dx-$$

$$-\text{v.p.}\int_{\mathbf{R}}(x-u_0)^{-1}(F(x)f,g)\,dx\Big|.$$

Consider the function

$$\psi(t)\overset{\text{def}}{=}(F(t)f,g)=(f,\Phi(t))(\Phi(t),g)$$

and check conditions (1)—(4) of Lemma 4.1. We have $\psi(0)=0$ and

$$|\psi(u_1)-\psi(u_2)| \leqq \left|(f,\Phi(u_1))(\Phi(u_1),g)-(f,\Phi(u_1))(\Phi(u_2),g)\right|$$

$$+\left|(f,\Phi(u_1))(\Phi(u_2),g)-(f,\Phi(u_2))(\Phi(u_2),g)\right|$$

$$\leqq \|f\|\,\|\Phi(u_1)\|\,\|\Phi(u_1)-\Phi(u_2)\|\,\|g\|+\|f\|\,\|\Phi(u_2)\|\,\|\Phi(u_1)-\Phi(u_2)\|\,\|g\| \leqq$$

$$\leqq C_L|u_1-u_2|^\alpha\left(|u_1-u_0|^\alpha+|u_2-u_0|^\alpha\right)$$

which proves the properties (1), (2) with $C_1=C_L(2^\alpha+1)\cdot\delta^\alpha$. By the definition of Φ (see (4.1))

$$\|\psi(t)\|_{L_1(\mathbf{R})} \leqq \int_{\mathbf{R}} dt\,\|f\|\,\|g\|\,\|\Phi(t)\|^2$$

$$= \int_{\mathbf{R}} \sum_i \lambda_i |\varphi_i(t)|^2\,dt = \|V\|_{S^1} \equiv C_2 < \infty.$$

The last condition of Lemma 4.1 with $\beta=2\alpha$ (it is possible for $\alpha<1/2$ since $C_3=0$), $C_3=0$, $C_0=C_L$ obviously follows from the inequality $|\psi(t)|\leqq\|f\|\,\|g\|\,\|\Phi(t)\|^2\leqq C_L\cdot|t-u_0|^{2\alpha}$. As a result we obtain

$$\|M(u)f\| \leqq \pi C_L\delta^{2\alpha}+C_L(2^\alpha+1)\,\delta^\alpha\,C_\alpha\,\delta^\alpha+2\delta\,\|V\|_{S^1}$$

$$+C''_\alpha\,C_L\delta^{2\alpha} = \delta^{2\alpha}\big[C_L\big(\pi+(2^\alpha+1)\,C_\alpha+C''_\alpha\big)+2\delta^{1-2\alpha}\,\|V\|_{S^1}\big].$$

This implies (4.2) with

$$C = C_L(\pi+3C_\alpha+C''_\alpha)+2\,\|V\|_{S^1} \quad \text{for} \quad \delta<1.$$

Let us turn to the general case $\Phi(u_0)\neq 0$ when the "scalar" proof fails. Define the normalized vector

$$e_u \overset{\text{def}}{=} \Phi(u)/\|\Phi(u)\|\in l_2,$$

which is correctly determined for u in some neighbourhood of u_0 on $\mathbf{R}$. Let $E'\subset E$ be the orthogonal complement in E of the vector f, $M(u_0)f=0$, $\|f\|=1$. Hence $e_{u_0}\in E'$. For the matrix $M(u)$ in "basis" $E=\{w\cdot f,\ w\in\mathbf{C}\}\oplus E'$ we have the formula

$$\text{(4.4)} \qquad M(u) = \begin{Vmatrix} a & (\cdot\,,b) \\ c & \mathscr{D} \end{Vmatrix}$$

with $a\overset{\text{def}}{=}(M(u)f,f)$, $c\overset{\text{def}}{=}P_{E'}M(u)f\in E'$, $b\overset{\text{def}}{=}P_{E'}M^*(u)f\in E'$ and $\mathscr{D}\equiv\mathscr{D}(u)\colon E'\to E'$, $\mathscr{D}(u)\overset{\text{def}}{=}P_{E'}M(u)P_{E'}$, where $P_{E'}$ is the orthogonal projection of E onto E'. The formula (4.4) is equivalent to the following calculation ($w\in\mathbf{C}$, $e\in E'$):

$$M(u)(wf+e) = wM(u)f+M(u)P_{E'}e = w(I-P_{E'})$$

$$\cdot M(u)f+wP_{E'}M(u)f+P_{E'}M(u)P_{E'}e+(I-P_{E'})M(u)$$

$$\cdot P_{E'}e = w\big(M(u)f,f\big)f+wP_{E'}M(u)f+\mathscr{D}e+\big(M(u)P_{E'}e,f\big)f$$

$$= waf+wc+\mathscr{D}e+\big(e,P_{E'}M^*(u)f\big)f = awf+wc+\mathscr{D}e+(e,b)f.$$

It follows straightforwardly that

$$\text{(4.5)} \qquad M^{-1}(u) = \frac{1}{\Delta(u)}\begin{Vmatrix} 1 & -(\cdot\,,(\mathscr{D}^{-1})^*b) \\ -\mathscr{D}^{-1}c & \Delta(u)\mathscr{D}^{-1}+\mathscr{D}^{-1}c\big(\cdot\,,(\mathscr{D}^{-1})^*b\big) \end{Vmatrix}$$

where $\Delta(u)\overset{\text{def}}{=}a-(\mathscr{D}^{-1}(u)c,b)$ plays the role of determinant of $M(u)$. The existence of the inverse operator $D^{-1}(u)$ a.e. u on $\mathbf{R}$ follows directly from Theorem 1.1. Otherwise there exists $h\in E'$ such that $h\neq 0$, $\mathscr{D}(u)h=0$ and we can obtain (4.2) as in Lemma 4.7 (see below).

The following estimates of the matrix elements of the operator $M(\lambda)$ are needed in the sequel.

Lemma 4.3. *If* $|u-u_0|\leqq 1$ *then*

$$|a(u)| \leqq C|u-u_0|^{2\alpha},$$

$$\|b(u)\| \leqq C|u-u_0|^{\alpha}(\sup_{t\in S}\|\Phi(t)\|+|u-u_0|^{\alpha}),$$

$$\|c(u)\| \leqq C|u-u_0|^{\alpha}(\sup_{t\in S}\|\Phi(t)\|+|u-u_0|^{\alpha}), \tag{4.6}$$

$$\|P_{E''}b(u)\| \leqq C|u-u_0|^{2\alpha},$$

$$\|P_{E''}c(u)\| \leqq C|u-u_0|^{2\alpha},$$

where $P_{E''}$ *is the orthogonal projection of E onto subspace* $E''\overset{\text{def}}{=}E\ominus[f\vee\Phi(u_0)]$, *and the interval* $S=(u_0-2|u-u_0|,\ u_0+2|u-u_0|)$ *is the same as in Lemma* 4.1. *The constant C in estimates* (4.6) *depends only on* C_L *(see* (1.1)), $\|V\|_{S^1}$ and $\alpha<1/2$.

Proof. (1) For the proof of the first estimate we define a scalar function $\psi(t)\equiv|(\Phi(t),f)|^2$. Then

$$a=(M(u)f,f)=((M(u)-M(u_0))f,f)$$

$$=i\pi|(\Phi(u),f)|^2+\text{v.p.}\int_{\mathbf{R}}\psi(t)(t-u)^{-1}\,dt-\text{v.p.}\int_{\mathbf{R}}\psi(t)(t-u_0)^{-1}\,dt.$$

It is easy to check that Lemma 4.1 can be applied to ψ:

$$|\psi(u_1)-\psi(u_2)| \leqq |(\Phi(u_1),f)|\,\|f\|\,\|\Phi(u_1)-\Phi(u_2)\|$$

$$+|(\Phi(u_2),f)|\,\|f\|\,\|\Phi(u_1)-\Phi(u_2)\|$$

$$\leqq C_L^{1/2}|u_1-u_2|^{\alpha}(|((\Phi(u_1)-\Phi(u_0)),f)|+|((\Phi(u_2)-\Phi(u_0)),f)|)$$

$$\leqq C_L|u_1-u_2|^{\alpha}(|u_1-u_0|^{\alpha}+|u_2-u_0|^{\alpha}),$$

$$\|\psi(t)\|_{L_1(\mathbf{R})} \leqq \|\Phi(t)\|^2_{L_2(\mathbf{R})}=\|V\|_{S^1},$$

$$|\psi(t)|=|(\Phi(u)-\Phi(u_0),f)|^2 \leqq C_L|u-u_0|^{2\alpha}.$$

To prove the estimate we use Lemma 4.1 and argue exactly as in the preceding lemma. Finally

$$|a| \leqq |u-u_0|^{2\alpha}[C_L(\pi+C_\alpha+C'_{2\alpha})+2\|V\|_{S^1}].$$

(2) As for the next pair of estimates (4.6) we need to check only the estimate for $\|c\|$ since $M^*(u_0)f=0$. We have

$$\|c\| \leqq \|M(u)f\| = \sup_{\|g\|=1,\,g\in E}|(M(u)f,g)|.$$

Let

$$\psi(t) \overset{\text{def}}{=} (f, \Phi(t))(\Phi(t), g).$$

We have to check the conditions of Lemma 4.1 for ψ. If $u_1, u_2 \in S$ then

$$|\psi(u_1)-\psi(u_2)| \leqq |(f, \Phi(u_1))| \, \|g\| \, \|\Phi(u_1)-\Phi(u_2)\|$$

$$+|(\Phi(u_2), g)| \, \|f\| \, \|\Phi(u_1)-\Phi(u_2)\| \leqq C_L^{1/2}|u_1-u_2|^\alpha 2 \sup_{t\in S} \|\Phi(t)\|,$$

$$\|\psi(t)\|_{L_1(\mathbf{R})} \leqq \|\Phi(t)\|^2_{L_2(\mathbf{R})} = \|V\|_{S^1}.$$

Let $\bar{u}$ be the nearest number to t from the interval $\bar{S}$:

$$|\psi(t)| \leqq |(f, \Phi(t))(\Phi(t)-\Phi(\bar{u}), g)| + |(f, \Phi(t))| \, |(\Phi(\bar{u}), g)|$$

$$\leqq |(f, \Phi(t)-\Phi(u_0))| \, \|\Phi(t)-\Phi(\bar{u})\| \, \|g\| + |(f, \Phi(t)-\Phi(u_0))| \, \|\Phi(\bar{u})\| \, \|g\|$$

$$\leqq C_L^{1/2}|t-u_0|^\alpha \cdot [C_L^{1/2}|t-\bar{u}|^\alpha + \|\Phi(\bar{u})\|]$$

$$\leqq C_L^{1/2}|t-u_0|^\alpha \big[\sup_{\tau\in S} \|\Phi(\tau)\| + C_L^{1/2}(\operatorname{dist}(t, S))^\alpha\big].$$

By Lemma 4.1 with

$$C_1 = 2C_L^{1/2}\sup_{\tau\in S}\|\Phi(\tau)\|, \quad C_2 = \|V\|_{S^1}, \quad C_3 = C_L, \quad C_0 = C_L^{1/2}\sup_{\tau\in S}\|\Phi(\tau)\|, \quad \beta = \alpha$$

we get

$$\|c\| \leqq \pi C_L^{1/2}|u-u_0|^\alpha\big(C_L^{1/2}|u-u_0|^\alpha + \sup_{t\in S}\|\Phi(t)\|\big)$$

$$+2C_L^{1/2}\sup_{t\in S}\|\Phi(t)\| \, C_\alpha|u-u_0|^\alpha + 2|u-u_0| \, \|V\|_{S^1}$$

$$+C_L^{1/2}\sup_{t\in S}\|\Phi(t)\| \, C'_\alpha|u-u_0|^\alpha + C''_\alpha C_L|u-u_0|^{2\alpha}$$

$$= |u-u_0|^{2\alpha}\{\pi C_L + 2\|V\|_{S^1}|u-u_0|^{1-2\alpha} + C''_\alpha C_L\}$$

$$+|u-u_0|^\alpha\sup_{t\in S}\|\Phi(t)\| \, \{\pi C_L^{1/2} + 2C_L^{1/2}C_\alpha + C_L^{1/2}C'_\alpha\}.$$

(3) Now we prove the last pair of estimates (4.6). Since

$$\|P_{E''}b\| = \sup_{\|g\|\leqq 1, g\in E} |(P_{E''}b, g)| = \sup_{\|g\|=1, g\in E} |(P_{E''}M^*(u)f, g)|$$

we can apply Lemma 4.1 with the function $\psi(t)=(f, \Phi(t))\cdot(\Phi(t), P_{E''}g)$. The following estimates are valid

$$\|\psi(t)\|_{L_1(\mathbf{R})} \leqq \|V\|_{S^1},$$

$$|\psi(t)| = |(f, \Phi(t)-\Phi(u_0))| \cdot |(\Phi(t)-\Phi(u_0), P_{E''}g)|$$

$$\leqq \|\Phi(t)-\Phi(u_0)\|^2 \leqq C_L|t-u_0|^{2\alpha}.$$

By the orthogonality $\Phi(u_0)$ to E'' we have for $u_1, u_2 \in S$

$$|\psi(u_1)-\psi(u_2)| \leqq |(f, \Phi(u_1))(\Phi(u_1)-\Phi(u_2), P_{E''}g)| + |(\Phi(u_2), P_{E''}g)|\,|(f, \Phi(u_1)-\Phi(u_2))|$$

$$\leqq \|\Phi(u_1)-\Phi(u_2)\|\,[\|P_{E''}g\|\,|(f, \Phi(u_1)-\Phi(u_0))| + |(\Phi(u_2)-\Phi(u_0), P_{E''}g)|\,\|f\|]$$

$$\leqq C_L|u_1-u_2|^\alpha(|u_1-u_0|^\alpha+|u_2-u_0|^\alpha) \leqq C_L|u_1-u_2|^\alpha 2^{1+\alpha}|u-u_0|^\alpha.$$

By Lemma 4.1 with $C_1=C_L 2^{\alpha+1}|u-u_0|^\alpha$, $C_2=\|V\|_{S^1}$, $C_3=0$, $C_0=C_L$, $\beta=2\alpha$ we get the inequality

$$\|P_{E''}b\| \leqq \pi C_L|u-u_0|^{2\alpha}+C_L 2^{\alpha+1}|u-u_0|^{2\alpha}C_\alpha+2|u-u_0|\,\|V\|_{S^1}+C'_{2\alpha}|u-u_0|^{2\alpha}C_L$$

$$= |u-u_0|^{2\alpha}[C_L(\pi+2^{\alpha+1}C_\alpha+C'_{2\alpha})+2\|V\|_{S^1}|u-u_0|^{1-2\alpha}].$$

Since

$$\|P_{E''}c\| = \|P_{E''}M(u)f\| = \sup_{\|g\|=1, g\in E} |(P_{E''}M(u)f, g)|$$

the proof of the last estimate is quite analogous to the preceding one. This completes the proof of the lemma.

Theorem 4.1 for $\Phi(u_0)\neq 0$ can be obtained by the careful analysis of the matrix elements in the representation (4.5). The following consideration is devoted to this purpose. By the formula (4.5) we have

$$\|M^{-1}(u)\|^{-1} \leqq 4|\Delta|/(1+\|(\mathscr{D}^{-1})^* b\| + \|\mathscr{D}^{-1}\bar{c}\| + \|\Delta\mathscr{D}^{-1}+\mathscr{D}^{-1}\bar{c}(\cdot, (\mathscr{D}^{-1})^* b)\|). \tag{4.7}$$

In particular, there are 3 variants of estimates:

$$\|M^{-1}(u)\|^{-1} \leqq 4\min\{|\Delta|, |a|+\|b\|, |a|+\|c\|\}.$$

Further estimates essentially depend on the value of $\|\Phi(u_0)\|$.

I. We suppose first that the following conditions are valid:

(C)

(1) $$\sup_{t\in S}\|\Phi(t)-\Phi(u_0)\| < 1/2\,\|\Phi(u_0)\|,$$

(2) $$\|\Phi(u_0)\| \geqq |u-u_0|^\alpha \equiv \delta^\alpha.$$

Only the second condition is essential here. We can replace (C) by the one condition $\|\Phi(u_0)\| \geqq C_L^{1/2}\cdot\delta^\alpha/2$ but the preceding form is more convenient. By conditions (C) we get

$$1/2\,\|\Phi(u_0)\| \leqq \|\Phi(u)\| \leqq 3/2\,\|\Phi(u_0)\|. \tag{4.8}$$

It should be mentioned that conditions (C) are valid for fixed values u, u_0. In what follows we shall use the following estimates for the matrix elements of the operator $\mathscr{D}^{-1}(u)$.

Lemma 4.4. *If* $e_t \overset{\text{def}}{=} \Phi(t)/\|\Phi(t)\|$, $t \in \mathbf{R}$ *then*

$$\left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, e_{u_0})\right| \geqq \pi\,\|\Phi(u)\|^2 \left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, e_u)\right|^2,$$

$$\left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, e_{u_0})\right| \geqq \pi\,\|\Phi(u)\|^2 \left|(\mathscr{D}^{-1}(u)\,P_{E'}\,e_u,\, e_{u_0})\right|^2.$$

Proof. Let $h \equiv \mathscr{D}^{-1}(u)e_{u_0} \in E'$, $e_{u_0} \in E'$ then

$$(h, e_{u_0}) = (h, \mathscr{D}(u)\,h) = (h, P_{E'}M(u)\,P_{E'}h) = (h, M(u)\,h),$$

$$|(h, e_{u_0})| \geqq \left|\operatorname{Im}(h, M(u)\,h)\right| = \pi\left|(h, \Phi(u)\,h)\right|^2.$$

This implies the first inequality (4.9). The second one can be obtained similarly:

$$\left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, e_{u_0})\right| = \left|(e_{u_0},\, (\mathscr{D}^{-1}(u))^* e_{u_0})\right| \geqq \pi\left|((\mathscr{D}^{-1}(u))^* e_{u_0},\, \Phi(u))\right|^2$$
$$= \pi\,\|\Phi(u)\|^2 \left|(e_{u_0},\, \mathscr{D}^{-1}(u)\,P_{E'}e_u)\right|^2 = \pi\,\|\Phi(u)\|^2 \left|(\mathscr{D}^{-1}(u)\,P_{E'}e_u,\, e_{u_0})\right|^2.$$

Lemma 4.5. $\left|(\mathscr{D}^{-1}(u)P_{E'}e_u,\, e_u)\right| \leqq 1/\pi \cdot \|\Phi(u)\|^{-2}$.

Proof. Let $h \equiv \mathscr{D}^{-1}(u)P_{E'}e_u \in E'$ then

$$\left|(\mathscr{D}^{-1}(u)P_{E'}\,e_u,\, e_u)\right| = |(h, P_{E'}\,e_u)| = \left|(h, \mathscr{D}(u)\,h)\right|$$
$$= \left|(h, P_{E'}M(u)\,P_{E'}h)\right| = \left|(h, M(u)\,h)\right| \geqq \left|\operatorname{Im}(h, M(u)\,h)\right|$$
$$= \pi\left|(h, \Phi(u))\right|^2 = \pi\,\|\Phi(u)\|^2\,|(h, e_u)|^2 = \pi\,\|\Phi(u)\|^2 \left|(\mathscr{D}^{-1}(u)P_{E'}e_u,\, e_u)\right|^2.$$

This proves the lemma.

Let us introduce the notation:

$$d \equiv (\mathscr{D}^{-1}(u)\,e_{u_0},\, e_{u_0}), \quad F \equiv (\mathscr{D}^{-1}(u)\,e_{u_0},\, \Phi(u) - \Phi(u_0)),$$

$$\tilde{F} \equiv (\mathscr{D}^{-1}(u)\,P_{E'}\,e_u,\, \Phi(u) - \Phi(u_0)).$$

By Lemmas 4.4, 4.5 we have the following

Corollary.

$$\sqrt{|d|} \geqq \sqrt{\pi}\,(|d|\,\|\Phi(u_0)\| - |F|),$$

$$\sqrt{|d|} \geqq \sqrt{\pi/4}\,(|\tilde{F}| - 1/(\pi\,\|\Phi(u)\|)).$$

Indeed by Lemma 4.4

$$|d| \geqq \pi\left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, \Phi(u))\right|^2 = \pi\left|(\mathscr{D}^{-1}(u)\,e_{u_0},\, \Phi(u_0)) + F\right|^2$$

hence

$$\sqrt{|d|} \geqq \sqrt{\pi}\,(|d|\,\|\Phi(u_0)\| - |F|).$$

In view of Lemmas 4.4, 4.5 we get

$$|d| \geqq \pi\left|(\mathscr{D}^{-1}(u)P_{E'}\,e_u,\, \Phi(u)) - (\mathscr{D}^{-1}(u)P_{E'}\,e_u,\, \Phi(u) - \Phi(u_0))\right|^2 (\|\Phi(u)\|/\|\Phi(u_0)\|)^2$$
$$\geqq \pi/4\left|(\mathscr{D}^{-1}(u)P_{E'}\,e_u,\, \Phi(u)) - \tilde{F}\right|^2 \geqq \pi/4\,(|\tilde{F}| - \|\Phi(u)\|\left|(\mathscr{D}^{-1}(u)P_{E'}\,e_u,\, e_u)\right|^2.$$

This shows that $\sqrt{|d|} \geqq \sqrt{\pi/4} \cdot (|\tilde{F}| - 1/(\pi \cdot \|\Phi(u)\|))$. It is easy to estimate the value $|F|$ by means of $|\tilde{F}|$:

$$\begin{aligned} |F| &= (\|\Phi(u_0)\|)^{-1} |(\mathcal{D}^{-1}(u) P_{E'} \Phi(u_0), \Phi(u) - \Phi(u_0))| \\ &\leqq (\|\Phi(u_0)\|)^{-1} [|(\mathcal{D}^{-1}(u) P_{E'} \Phi(u), \Phi(u) - \Phi(u_0))| \\ &\quad + |(\mathcal{D}^{-1}(u) P_{E'} (\Phi(u) - \Phi(u_0)), \Phi(u) - \Phi(u_0))|] \\ &\leqq (\|\Phi(u)\| / \|\Phi(u_0)\|) |\tilde{F}| + \|\mathcal{D}^{-1}(u)\| \, \|\Phi(u) - \Phi(u_0)\|^2 (\|\Phi(u_0)\|)^{-1} \\ &\leqq 3/2 |\tilde{F}| + \|\mathcal{D}^{-1}(u)\| C_L |u - u_0|^{2\alpha} (\|\Phi(u_0)\|)^{-1}. \end{aligned}$$

Suppose first that $\|\mathcal{D}^{-1}(u)\|$ is not too big

$$\|\mathcal{D}^{-1}(u)\| \, C_L \delta^{2\alpha} \leqq 1. \tag{4.10}$$

Then $|F| \leqq 3/2 |\tilde{F}| + 1/\|\Phi(u_0)\|$. This permits us to estimate $\tau \overset{\text{def}}{=} \sqrt{|d|} \ \|\Phi(u_0)\|$. By Lemmas 4.4, 4.5, by conditions (C), (4.10) and by the last Corollary we have

$$\begin{aligned} |d| \, \|\Phi(u_0\| &\leqq \pi^{-1/2} \sqrt{|d|} + |F| \leqq \pi^{-1/2} \sqrt{|d|} + 3/2 |\tilde{F}| + 1/\|\Phi(u_0)\| \\ &\leqq \pi^{-1/2} \sqrt{|d|} + 3/2 ((4|d|/\pi)^{1/2} + 1/(\pi \|\Phi(u)\|)) + 1/\|\Phi(u_0)\|. \end{aligned}$$

Therefore

$$\begin{aligned} |d| \, \|\Phi(u_0)\|^2 &\leqq \sqrt{|d|} (1/\sqrt{\pi} + 3/\sqrt{\pi}) \|\Phi(u_0)\| + 3/(2\pi)(\|\Phi(u_0)\| / \|\Phi(u)\|) + 1 \\ &\leqq \sqrt{|d|} \, \|\Phi(u_0)\| \, 4/\sqrt{\pi} + (3/\pi + 1), \end{aligned}$$

i.e. $\tau > 0$, $\tau^2 \leqq \tau \cdot 4/\sqrt{\pi} + (3/\sqrt{\pi} + 1)$. Hence $\tau = \sqrt{|d|} \cdot \|\Phi(u_0)\| \leqq 10$, which will be used in what follows.

Lemma 4.6. *Suppose that conditions* (C) *and* (4.10) *hold then the estimate* (4.2) *is valid.*

Proof. By the formula (4.5)

$$\|M^{-1}(u)\|^{-1} \leqq |a| + |(b, \mathcal{D}^{-1}c)| (1 + \|\mathcal{D}^{-1}c\| + \|(\mathcal{D}^{-1})^* b\|)^{-1} \tag{4.11}$$

where $|a| \leqq c \cdot \delta^{2\alpha}$, $\delta \equiv |u - u_0|$ (see Lemma 4.3). Let us denote the orthogonal projection of E' onto vector e_{u_0} by P_0, then

$$\begin{aligned} (b, \mathcal{D}^{-1}c) &= (P_{E''} b, \mathcal{D}^{-1}c) + (P_0 b, \mathcal{D}^{-1}c) \\ &= (P_{E''} b, \mathcal{D}^{-1}c) + (P_0 b, \mathcal{D}^{-1} P_0 c) + (P_0 b, \mathcal{D}^{-1} P_{E''} c) \\ &= (P_{E''} b, \mathcal{D}^{-1}c) + (P_0 b, \mathcal{D}^{-1} P_0 c) + ((\mathcal{D}^{-1})^* b, P_{E''} c) - (P_{E''} b, \mathcal{D}^{-1} P_{E''} c). \end{aligned}$$

In view of the estimates from Lemma 4.3 and the inequality $d \leqq 100 \cdot \|\Phi(u_0)\|^{-2}$ we have

$$|(b, \mathscr{D}^{-1}c)| \leqq \|P_{E''}b\| \, \|\mathscr{D}^{-1}c\| + |(\mathscr{D}^{-1}e_{u_0}, e_{u_0})| \, \|P_0 b\| \, \|P_0 c\| + \|(\mathscr{D}^{-1})^* b\| \, \|P_{E''}c\|$$
$$+ \|P_{E''}b\| \, \|\mathscr{D}^{-1}(u)\| \, \|P_{E''}c\| \leqq C\delta^{2\alpha} \|\mathscr{D}^{-1}c\| + 100 \, \|\Phi(u_0)\|^{-2} (C\delta^{\alpha}(\delta^{\alpha} + 3/2 \, \|\Phi(u_0)\|))^2$$
$$+ \|(\mathscr{D}^{-1})^* b\| \, C\delta^{2\alpha} + \|\mathscr{D}^{-1}(u)\| \, C^2 \delta^{4\alpha}.$$

If we make use of the formula (4.11), condition (C) and the last estimate, then we get

$$\|M^{-1}(u)\|^{-1} \leqq C\delta^{2\alpha} + C\delta^{2\alpha} + 100C^2\delta^{2\alpha}(5/2)^2 + C\delta^{2\alpha} + (\|\mathscr{D}^{-1}(u)\| \, \delta^{2\alpha}) C^2 \delta^{2\alpha}.$$

Finally, by the condition (4.10) we have

$$\|M^{-1}(u)\|^{-1} \leqq \delta^{2\alpha}(3C + (25C)^2 + C^2/C_L)$$

which completes the proof of the estimate (4.2).

Assume now that (4.10) fails i.e.

$$(4.12) \qquad \|\mathscr{D}^{-1}(u)\| \, C_L \delta^{2\alpha} > 1.$$

Lemma 4.7. *If conditions* (C) *and* (4.12) *are satisfied, then the estimate* (4.2) *is valid,*

Proof. By the condition (4.12) there is a vector $g \in E'$, $\|g\| = 1$, such that $\|\mathscr{D}(u)g\| < C_L \cdot \delta^{2\alpha}$. It is sufficient to prove that $\|M(u)g\| \leqq \text{const.} \cdot \delta^{2\alpha}$ with the constant only depending on C_L and $\|V\|_{S^1}$.

$$\|M(u)g\| \leqq \|P_{E'}M(u)g\| + \|(I - P_{E'})M(u)g\|$$
$$= \|\mathscr{D}(u)g\| + |(M(u)g, f)| < C_L\delta^{2\alpha} + |(g, b)| \leqq C_L\delta^{2\alpha} + |(g, P_{E''}b)| + |(g, P_0 b)|$$
$$\leqq C_L\delta^{2\alpha} + C\delta^{2\alpha} + C\delta^{\alpha}(\delta^{\alpha} + 3/2 \, \|\Phi(u_0)\|) \, \|P_0 g\|.$$

It is necessary here to evaluate the norm $\|P_0 g\| = |(g, e_{u_0})|$. We use the condition (4.12) again:

$$C_L\delta^{2\alpha} \geqq |(\mathscr{D}(u)g, g)| = |(M(u)g, g)| \geqq \operatorname{Im}(M(u)g, g) = \pi \, |(\Phi(u), g)|^2.$$

Since

$$|(g, \Phi(u_0))| \leqq |(g, \Phi(u))| + |(g, \Phi(u) - \Phi(u_0))|$$
$$\leqq C_L^{1/2}\delta^{\alpha}/\pi^{1/2} + \|\Phi(u) - \Phi(u_0)\| \leqq 2C_L^{1/2}\delta^{\alpha},$$

we have

$$\|M(u)g\| \leqq C_L\delta^{2\alpha} + C\delta^{2\alpha} + C\delta^{\alpha}(\delta^{\alpha} + 3/2 \, \|\Phi(u_0)\|) \, \|\Phi(u_0)\|^{-1} 2C_L^{1/2}\delta^{\alpha}$$
$$\leqq (C_L + C + 3CC_L^{1/2})\delta^{2\alpha} + 2CC_L^{1/2}\delta^{3\alpha} \|\Phi(u_0)\|^{-1} \leqq (C_L + C + 5CC_L^{1/2})\delta^{2\alpha},$$

where in the last inequality the condition (C) has been used. This finishes the proof of the lemma as well as the estimate (4.2) (under the additional condition (C)).

II. Suppose that the condition (C) fails. Then it is easy to see that

$$\sup_{t\in S}\|\Phi(t)\| \leqq \bar{c}\delta^{\alpha} \tag{4.13}$$

with

$$\bar{c} \overset{\text{def}}{=} \max\{1+2^{\alpha}C_L^{1/2},\ 3\cdot 2^{\alpha}C_L^{1/2}\}.$$

Lemma 4.8. *Suppose that Φ satisfies* (4.13) *then the inequality* (4.2) *is valid.*

Proof. In view of (4.13) and Lemma 4.3

$$\|M^{-1}(u)\| \leqq |a|+\|c\| \leqq C\delta^{2\alpha}+C\delta^{\alpha}\big(\sup_{t\in S}\|\Phi(t)\|+\delta^{\alpha}\big)$$
$$\leqq 2C\delta^{2\alpha}+C\delta^{\alpha}\bar{c}\delta^{\alpha} = C(2+\bar{c})\delta^{2\alpha}$$

with the constant $C(2+\bar{c})$ depending only on C_L and $\|V\|_{S^1}$. This completes the proof of Theorem 4.1.

V. *The main theorem. The discussion of different smooth conditions on the kernel v.*

Theorem 4.1 together with the uniqueness theorem from § 3 imply the following result.

Theorem 5.1. *Let $V\in S^1$, $V\geqq 0$ and suppose that the kernel of perturbation $v(x, y)$ satisfies the condition $(\alpha<1/2)$*

$$v(x+h, x+h)+v(x, x)-v(x+h, x)-v(x, x+h) \leqq C_L|h|^{2\alpha}, \quad x, h\in\mathbf{R}. \tag{5.1}$$

Then the singular continuous and the point spectra of L (see §1) are contained in Λ (the set of roots of the operator-function $M(\lambda)$) satisfying the condition

$$\operatorname{mes}\Lambda^{\delta} \leqq \operatorname{const}\delta^{2\alpha}, \quad \delta<1, \tag{5.2}$$

where Λ^{δ} is the δ-neighbourhood of the set Λ on $\mathbf{R}$.

Proof. The inclusion $\mathfrak{S}_s\cup\mathfrak{S}_p\subset\Lambda$ was established earlier. The Theorem 4.1 implies that in a neighbourhood of a root $u_0\in\mathbf{R}$ on $\mathbf{R}$ the estimate

$$\|M^{-1}(u)\|^{-1} \leqq C(\alpha, C_L, \|V\|_{S^1})|u-u_0|^{2\alpha}$$

is valid. Since $M(\lambda)$ belongs to Lip α (see Theorem 1.1), it is easy to extend this inequality to the line $(\mathbf{R}+i\delta^2)$ of the complex plane

$$\|M^{-1}(u+i\delta^2)\|^{-1} \leqq \big(C(\alpha, C_L, \|V\|_{S^1})+C_1\big)\delta^{2\alpha},$$

$|u-u_0|\leqq\delta$, $u_0\in\Lambda$, where C_1 is a constant from the Lipshitz condition $\|M(\lambda)-M(\lambda')\|\leqq C_1|\lambda-\lambda'|^{\alpha}$ only depending on C_L, $\|V\|_{S^1}$. By the last inequality we get

$$\|M^{-1}(u+i\delta^2)-I\|_{S^2}^{-1} \leqq \|M^{-1}(u+i\delta^2)-I\|^{-1}$$
$$\leqq \big(\|M^{-1}(u+i\delta^2)\|-1\big)^{-1} \leqq \|M^{-1}(u+i\delta^2)\|^{-1}\big(1-\|M^{-1}(u+i\delta^2)\|^{-1}\big)^{-1}$$
$$\leqq (C+C_1)\delta^{2\alpha}\big(1-(C+C_1)\delta^{2\alpha}\big)^{-1} \leqq 2(C+C_1)\delta^{2\alpha}.$$

Without loss of generality we can assume here that $(C+C_1)\cdot\delta^{2\alpha}<1/2$. Then we can use the same arguments as in the proof of Theorem 2.4

$$\begin{aligned}\operatorname{mes} \Lambda^{\delta} &\leqq \operatorname{mes}\{k\in\mathbf{R}\colon \|M^{-1}(k+i\delta^2)-I\|_{S^2}^{-1}\leqq 2(C+C_1)\delta^{2\alpha}\}\\ &\leqq \operatorname{const}(C+C_1)\delta^{2\alpha}\Big\|\int_{\mathbf{R}}\operatorname{Im} M^{-1}(k+i\delta^2)\,dk\Big\|_{S^1}\\ &= \operatorname{const}(C+C_1)\delta^{2\alpha}\pi\|V\|_{S^1}.\end{aligned}$$

The proof is finished.

Corollary. *It is well-known* [12] *that the condition* (5.2) *implies the inequality*

$$\sum_n |l_n|^{1-2\alpha}<\infty,$$

$|l_n|$ being the lengths of bounded complementary intervals l_n of Λ.

Let us note that the part of the theorem concerning the relationship between the "smoothness" of v and the "smallness" of Λ is exact even for rank one perturbations V [11].

Compare the condition (5.1) with other conditions on the kernel of perturbation v which have been appeared in the spectral analysis of Friedrichs model. Note that the left-hand side of the inequality (5.1) is positive in view of the condition $V\geqq 0$ (see (5.3)). Actually this condition means that the function of two variables $v(x, y)$ is smooth (in terms of the second difference) only on the diagonal. It is useful to compare the condition (5.1) with the norm for v introduced by Friedrichs [5] for the investigation of the spectral structure of the operator L:

$$\begin{aligned}\sup_{h_1,h_2,x,y\in\mathbf{R}} &(1+|x|^{\alpha})(1+|y|^{\alpha})|h_1|^{-\alpha}|h_2|^{-\alpha}|v(x+h_1, y+h_2)+v(x, y)\\ &-v(x+h_1, y)-v(x, y+h_2)|<\infty.\end{aligned}$$

The condition (5.1) can be obtained from the expression if we put $x=y$, $h_1=h_2$ there and drop the multipliers $(1+|x|^{\alpha})$, $(1+|y|^{\alpha})$ increasing at infinity. Note that in Theorem 5.1 the decreasing of $v(x, y)$ at infinity only appears in condition $V\in S^1$.

The other condition on v from the paper [4]:

$$|v(x+h_1, y+h_2)-v(x, y)|\leqq C(1+|x|+|y|)^{-\theta}(|h_1|^{\alpha}+|h_2|^{\alpha}),$$

$\theta>1/2$, $\alpha>1/2$, leads to the absence of singular continuous spectrum of L as well as to the "triviality" of the point spectrum. To compare it with (5.1) we define a kernel $v_{1/2}(x, y)$ of the integral operator corresponding to the non-negative square root $(V)^{1/2}\in S^2$. The connection with (5.1) follows from the following identity.

Proposition 5.1. *Let $V\geqq 0$, $V\in S^1$ then for kernels of integral operators V, $V^{1/2}$ the identity*

(5.3) $$\begin{aligned}&v(x+h, x+h)+v(x, x)-v(x+h, x)-v(x, x+h)\\ &\equiv\int_{\mathbf{R}}|v_{1/2}(x+h, y)-v_{1/2}(x, y)|^2\,dy\end{aligned}$$

holds.

Proof. Let $\{\varphi_i(x)\}$ be an orthonomal system of eigenfunctions of the selfadjoint operator $(V)^{1/2}$ then

$$\sqrt{\lambda_i}\,\varphi_i(x) \equiv \int_{\mathbf{R}} v_{1/2}(x, y)\,\varphi_i(y)\,dy.$$

Therefore

$$\sqrt{\lambda_i}(\varphi_i(x+h)-\varphi_i(x)) = \int_{\mathbf{R}} (v_{1/2}(x+h, y)-v_{1/2}(x, y))\,\varphi_i(y)\,dy.$$

By the properties of the orthogonal expansion in $L_2(\mathbf{R})$ we get

$$\sum_i \lambda_i\,|\varphi_i(x+h)-\varphi_i(x)|^2 = \int_{\mathbf{R}} |v_{1/2}(x+h, y)-v_{1/2}(x, y)|^2\,dy.$$

The left-hand side of the equality can be transformed into the needed expression by the same method as in Lemma 1.1. Of course we can obtain the analogous result by the direct substitution

$$v_{1/2}(x, y) = \sum_i \sqrt{\lambda_i}\,\varphi_i(x)\,\overline{\varphi_i(y)}.$$

By the last proposition the condition (5.1) is valid if

$$\Big(\int_{\mathbf{R}} |v_{1/2}(x+h, y)-v_{1/2}(x, y)|^2\,dy\Big)^{1/2} \leqq C_L^{1/2}\,|h|^{\alpha}. \tag{5.4}$$

It is sufficient to require that

$$\sup_{x, y, h\in\mathbf{R}} (1+|y|)^{\theta}\,|v_{1/2}(x+h, y)-v_{1/2}(x, y)|\,|h|^{-\alpha} < \infty$$

with $\theta > 1/2$, i.e. the kernel of operator $(V)^{1/2}$ belongs to $\operatorname{Lip}\alpha$. Note that in view of the formula (1.2) for M we obtain a restriction on the operator $(V)^{1/2}$. The investigation of M is the cornerstone of the proof of Theorem 5.1. In case $\operatorname{rank} V < \infty$ the conditions of type (5.4) for v and $v_{1/2}$ are equivalent.

Remark. It is easy to check that the condition $V \geqq 0$ can be dropped. Consider the operator-valued function

$$M(\lambda) = J + (|V|)^{1/2}(x-\lambda)^{-1}(|V|)^{1/2}, \quad \operatorname{Im} M(\lambda) \geqq 0,$$

where $V = J\cdot|V|$, $J = \operatorname{sign} V$ is a polar decomposition of the selfadjoint operator V. The Theorem 5.1 is valid if in condition (5.1) we substitute v by the kernel of "modulus" of the operator $|V|$. In case $\operatorname{rank} V < \infty$ it is equivalent to the condition of type (5.4) but for the kernel v itself. For example, in terms of eigenfunctions we get

$$\max_{1\leqq i\leqq \operatorname{rank} V} \Big(\int_{\mathbf{R}} |\varphi_i(x+h)-\varphi_i(x)|^2\,dx\Big)^{1/2} \leqq C\,|h|^{\alpha}.$$

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Added in proof. The more general uniqueness theorems for S^p-valued $(p \geqq 1)$ functions with positive imaginary part were proved by the "operator" methods. See the author's paper: "On the root structure for the operator-valued functions with positive imaginary part in S^p-classes", to be published in Dokl. Akad. Nauk SSSR, 1987.

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