Sharp estimates of uniform harmonic majorants in the plane

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1. Introduction

If f is an analytic function in the unit disk U, the Dirichlet integral $D(f)$ is defined by

$$D(f) = \left(\int_U |f'(z)|^2 \, dx \, dy/\pi\right)^{1/2}, \quad z = x+iy.$$

The following result is due to A. Chang and D. Marshall (cf. [4], [7]). It is inspired by work of A. Beurling and J. Moser (cf. [3], [8]).

Theorem A. *There is a constant $C<\infty$ such that if f is analytic in U, $f(0)=0$ and $D(f)\leqq 1$, then*

$$\int_0^{2\pi} \exp\left(|f(e^{i\theta})|^2\right) d\theta \leqq C.$$

If f is univalent, $\pi D(f)^2$ is the area $|f(U)|$ of the range $f(U)$ of f. What can be said about functions f which are not necessarily univalent if the assumption on f is replaced by $|f(U)|\leqq\pi$? Can this condition on the area of $f(U)$ be generalized?

Let D be an open, connected subset in the plane and let $\theta(r)=\left|\{\theta\colon re^{i\theta}\in D\}\right|$. Let F be the class of locally bounded functions $\Psi\colon (0, \infty)\to(0, \infty)$ which have the following properties:

i) near the origin, we have $\Psi(r)=cr^2$, c constant,
ii) for each $a>0$, we have $\inf_{r\geqq a} \Psi(r)>0$,
iii) there exists $R>0$ such that Ψ is increasing on (R, ∞); in the interval $(0, R)$, Ψ is continuous except possibly at finitely many points.

Let $p(r)=\int_0^r (\Psi(t))^{1/2}dt/t$ and let $\Phi(r)=\exp\left(p(r)^2\right)$. The function $\Phi(|z|)$ is subharmonic in $\{|z|>R\}$: this is clear since $r\Phi'(r)$ is increasing for $r>R$ and $\Delta\Phi=r^{-1}(d/dr)\left(r\Phi'(r)\right)$. Natural examples of functions satisfying these conditions are given in Corollaries 2 and 3.

We shall assume that the domain D is such that

$$\int_0^\infty \Psi(r)\theta(r)\,dr/r = \iint_D \Psi(|z|)\,|z|^{-2}\,dx\,dy = \pi. \tag{1.1}$$

Theorem 1. *Let* $0\in D$ *and let* $\Psi\in F$. *If* (1.1) *holds,* $\Phi(|z|)$ *has a harmonic majorant* h *in* D, *and* $h(0)$ *has an upper bound* $c(\Psi)$ *only depending on* Ψ *and not on the special form of* D.

Corollary 1. *Let* D *and* Ψ *be as in Theorem 1, and let* $f\colon U\to D$ *be analytic in* U *with* $f(0)=0$. *Then* $\Phi(|f(e^{i\theta})|)\in L^1(\partial U)$ *and*

$$\int_0^{2\pi} \Phi(|f(e^{i\theta})|)\,d\theta \leqq c(\Psi). \tag{1.2}$$

Proof of Corollary 1.

$$\int_0^{2\pi} \Phi(|f(re^{i\theta})|)\,d\theta \leqq \int_0^{2\pi} h(f(re^{i\theta}))\,d\theta = 2\pi h(f(0)) = 2\pi h(0) \leqq c(\Psi).$$

Letting $r\uparrow 1$, we obtain Corollary 1.

Remark. Absolute constants are denoted by $C, C_0, C_1, \dots$ and constants determined by Ψ by $c(\Psi), c_0(\Psi), \dots$. They are not necessarily the same at each occurrence.

Corollary 2. *Let* $\lambda>0$ *be given. If* D *is a domain such that* $0\in D$ *and*

$$\int_0^1 r\theta(r)\,dr + \lambda^2\int_1^\infty t^{2\lambda-1}\theta(t)\,dt = \pi,$$

then $\exp(|w|^{2\lambda})$ *has a harmonic majorant* h *in* D *and there is a constant* $c(\lambda)$ *such that for all such domains, we have* $h(0)\leqq c(\lambda)$.

Proof. In Theorem 1, we choose $\Psi(r)=\lambda^2 r^{2\lambda}$, $r>1$, and $\Psi(r)=r^2$, $0<r<1$.

Corollary 3. *If* D *is a domain such that* $0\in D$ *and*

$$\int_0^1 r\theta(r)\,dr + \int_1^\infty \theta(r)/r\,dr = \pi,$$

then $\exp((1+\log^+|w|)^2)$ *has a harmonic majorant* h *in* D *and there is a constant* C *such that* $h(0)\leqq C$ *for all such domains* D.

Proof. In Theorem 1, we choose $\Psi(r)=\min(1, r^2)$, $r>0$.

It is easy to write down the corresponding results for boundary values of analytic functions $f\colon U\to D$ with $f(0)=0$ (cf. (1.2)).

The case $\lambda=1$ in Corollary 2 generalizes Theorem A of Chang and Marshall: also in cases when $D(f)>1$, conclusions of type (1.2) hold provided that $|f(U)|\leqq\pi$.

As a weak consequence of Theorem 1, we have a result of Phragmén—Lindelöf type.

Corollary 4. *Let D and Ψ be as in Theorem 1 and assume that* (1.1) *holds. If u is subharmonic in D with non-positive boundary values at all finite boundary points and if*

$$\liminf_{r\to\infty} M(r)/\Phi(r) < \infty, \tag{1.3}$$

where $M(r)=\sup u(re^{i\theta})$, $re^{i\theta}\in D$, then we have $u\leqq 0$ in D.

Remark. For $\zeta\in\partial D$, we define $u(\zeta)=\limsup u(z)$, $z\to\zeta$, $z\in D$.

The proof will be given in Section 4.

The heart of our proof of Theorem 1 is an estimate of harmonic measure in certain multiply-connected domains which follows from Lemmas 1 and 2 in Section 2. A second essential tool is an integral inequality due to J. Moser [8]:

Theorem B. *There is a constant $C<\infty$ such that if N is absolutely continuous on $[0, \infty)$, $N(0)=0$ and $\int_0^\infty N'(t)^2 dt\leqq 1$, then*

$$\int_0^\infty \exp\left(N(t)^2-t\right) dt \leqq C.$$

An alternative proof of Moser's theorem has been given by D. Marshall in connection with his proof of Theorem A (cf. [7]).

We note that if (1.1) is replaced by

$$\int_0^\infty \Psi(r)\theta(r)\,dr/r = \pi/A, \quad A>0 \quad \text{given}, \tag{1.1a}$$

our proof shows that we get a harmonic majorant with a uniform upper bound for $h(0)$ for the function $\exp\left(Ap(|z|)^2\right)$.

2. Proof of Theorem 1

If other domains than D appear, we use the notation $\theta_j(r)=\left|\{\theta\colon re^{i\theta}\in D_j\}\right|$, $j=1, 2, \ldots$. We write

$$I(a, b) = I\left(a, b, \theta(\cdot)\right) = \int_a^b \theta(t)^{-1}\,dt/t,$$

$$I_j(a, b) = I\left(a, b, \theta_j(\cdot)\right), \quad j=1, 2, \ldots.$$

Let R be a positive number and let $\omega_R(\cdot)=\omega_R(\cdot, D)$ be the harmonic measure of $\{|z|=R\}\cap\bar{D}$ in that component D_R of $D\cap\{|z|<R\}$ which contains the origin. It is easy to see that if

$$J(\Phi, D) = \int_0^\infty \Phi(t)\,d\left(-\omega_t(0, D)\right) = 1+\int_0^\infty \omega_t(0, D)\,\Phi'(t)\,dt < \infty, \tag{2.1}$$

then $\Phi(|z|)$ has a harmonic majorant h in D and $h(0)\leqq J(\Phi, D)$. Thus it will be sufficient to study the integral in (2.1).

Remark. Conversely, if $\Phi(|z|)$ has a harmonic majorant in D, then we have $J(\Phi, D)<\infty$ provided that Φ satisfies certain regularity conditions. A detailed discussion of these questions can be found in Essén—Haliste—Lewis—Shea [6].

We introduce

$$\theta^*(r) = \begin{cases} \theta(r), & \text{if } \{|z| = r\} \cap CD \neq \emptyset \\ \infty, & \text{if } \{|z| = r\} \subset D. \end{cases}$$

The circular symmetrization D_R^* of D_R is defined by

$$D_R^* = \{re^{i\theta}: |\theta| < \theta^*(r)/2, \ r < R\},$$

with the convention that $\theta^*(r)=\infty$ means that $\{|z|=r\}\subset D_R^*$. Let ω_R^* be the harmonic measure of $\{|z|=R\}\cap\partial D_R^*$ in D_R^*. It is known that $\omega_R(0)\leqq\omega_R^*(0)$ (cf. Theorem 7 in Baernstein [2]; also cf. Theorem 9.4 in Essén [5]). Thus, if the analogue of (2.1) holds for the symmetrized region D^*, (2.1) will hold for the original region D. Hence it suffices to study the symmetrized case. From now on, we assume that

$$D = \{re^{i\theta}: |\theta| < \theta^*(r)/2\}.$$

If $\theta^*(r)=\infty$, this means that $\{|z|=r\}\subset D$. To give a rough description of the proof, we introduce

$$D_0 = \{re^{i\theta}: |\theta| < \theta(r)/2\},$$

which is a simply connected domain, and

$$D_{0R} = D_0 \cap \{|z| < R\}.$$

A schematic sketch of the situation is given by Fig. 1. D_R is the connected set containing the origin, i.e., the centre of the figure, and D_{0R} is D_R cut along the negative real axis. Let $h^0=h_R^0$ be the harmonic measure of $D_{0R}\cap\{|z|=R\}$ with respect to D_{0R}.

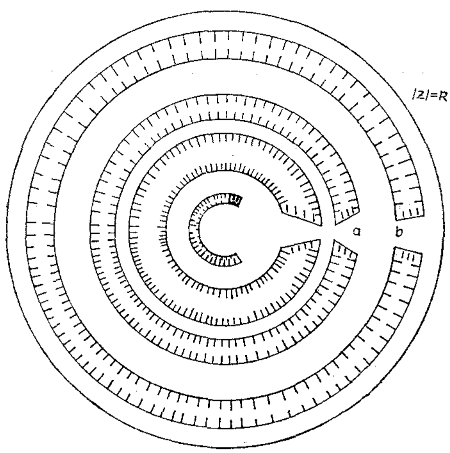

Figure 1

According to the Ahlfors distortion theorem (cf. [1], Corollary p. 78), we have

$$h^0(r) \leqq 32\exp\left(-\pi I(r, R)\right), \quad r < R, \tag{2.2}$$

provided that $I(r, R) \geqq 1/2$. The point of the following principal lemmas is to prove a similar estimate for ω_R. The difficulty is that D is not necessarily simply connected. For simplicity, we assume that $\Psi(r)=r^2$, $0<r<1$.

Lemma 1. *Assume that* (1.1) *holds and that* $0\in D$. *Then there is a constant* $c(\Psi)$ *such that if* $q\geqq e^{-2}$, *we have*

$$\omega_R(q, D) \leqq c(\Psi)\exp\left(-\pi I(q, R)\right), \quad q \leqq R. \tag{2.3}$$

Without loss of generality, we can assume that $\theta(\cdot)$ is continuous.

Lemma 2. *Let D be as in Lemma 1. Assume that*

$$\min\theta(t) \leqq 1/8, \quad e^{-3/2} \leqq t \leqq e^{-1}. \tag{2.4}$$

Then there exists $q\in[e^{-2}, 1]$ *and a domain* D_2 *with*

$$\left|D_2\cap\{|z| < q\}\right| \leqq \left|D\cap\{|z|<q\}\right| \quad \text{and} \quad D_2\cap\{|z| > q\} = D\cap\{|z| > q\}, \tag{2.5}$$

and a constant $c(\Psi)$ *such that*

$$\omega_R(0, D) \leqq c(\Psi)\exp\left(-\pi I_2(d_2, R)\right), \quad R \geqq q > d_2, \tag{2.6}$$

where d_2 is the radius of the largest disk centered at the origin contained in D_2.

If $D\cap\{|z|<q\}$ is simply connected, we can take $D_2=D$. If this is not the case, we shall choose D_2 almost as the union of the Steiner symmetrization of $D\cap\{|z|<q\}$ with respect to the real axis and $D\cap\{|z|\geqq q\}$. The point q must be chosen in a careful way (cf. Lemma 3.3 in Section 3).

The proofs of Lemmas 1 and 2 are given in Section 3.

Using (2.3) and (2.6), we shall find a uniform upper bound for the integrals in (2.1). First, we re-write Theorem B as

Lemma 3. *Let* $k\colon[0, \infty)\to[0, \infty)$ *be such that* $\int_0^\infty k(t)\,dt\leqq 1$. *Let* $K(r)=\int_0^r (k(t)/\lambda(t))^{1/2}\,dt$ *where λ is positive and* $L(r)=\int_0^r \lambda(t)^{-1}dt$ *is unbounded as* $t\to\infty$. *Then there is a constant C such that*

$$1+\int_0^\infty 2K'(r)K(r)\exp\left(K(r)^2-L(r)\right)dr = \int_0^\infty \lambda(r)^{-1}\exp\left(K(r)^2-L(r)\right)dr \leqq C. \tag{2.7}$$

Proof of Lemma 3. We put $L(r)=s$ and $K(L^{-1}(s))=N(s)$. Then $N'(s)=K'(r)(dr/ds)$ and we have

$$\int_0^\infty N'(s)^2\,ds = \int_0^\infty k(r)(\lambda(r))^{-1}(dr/ds)^2\,ds = \int_0^\infty k(r)\,dr \leqq 1,$$

$$\int_0^\infty \lambda(r)^{-1}\exp\left(K(r)^2-L(r)\right)dr = \int_0^\infty \exp\left(N(s)^2-s\right)ds \leqq C,$$

where the last inequality follows from Theorem B.

The basic idea in the estimate of $J(\Phi, D)$ defined in (2.1) is to apply Lemma 3 on an interval (α, ∞) with

$$\lambda(r) = r\theta(r)/\pi,$$

$$k(r) = r^{-1}\Psi(r)\theta(r)/(\pi J_\alpha),$$

$$K(r) = \big(p(r)-p(\alpha)\big)J_\alpha^{-1/2},$$

where $\pi J_\alpha = \int_\alpha^\infty \theta(r)\Psi(r)\,dr/r$. From Lemma 3, we see that there is an absolute constant C such that

$$\int_\alpha^\infty \Phi_\alpha'(t)\exp\big(-\pi I(\alpha, t)\big)\,dt \leqq C, \tag{2.8}$$

where $\Phi_\alpha(r) = \exp\big((p(r)-p(\alpha))^2/J_\alpha\big)$.

If (2.4) does not hold, there is a constant $C_1 = C_1(\Psi)$ such that $\int_0^1 \Psi(t)\,\theta(t)\,dt/t \geqq \pi C_1 > 0$. Let us choose $\alpha = 1$ in (2.8). It is easy to see that if $p(r) \geqq p(r_0) = \max\big(2p(1)/C_1, p(2)\big)$, we have

$$\Phi'(r) \leqq p(2)\big(p(2)-p(1)\big)^{-1}\Phi_1'(r).$$

Combining this inequality with (2.3), we see that it follows from (2.8) that

$$J(\Phi, D) \leqq \Phi(r_0) + c(\Psi)\int_{r_0}^\infty \Phi_1'(t)\exp\big(-\pi I(1, t)\big)\,dt \leqq c(\Psi).$$

Thus $J(\Phi, D)$ has an upper bound depending only on Ψ and the proof is complete in this case.

In the remaining case when (2.4) holds, we use (2.6) and see that it is sufficient to estimate

$$\int_{d_2}^\infty \Phi'(r)\exp\big(-\pi I_2(d_2, r)\big)\,dr,$$

where we know that (cf. (2.5))

$$\pi d_2^2 + \int_{d_2}^\infty \Psi(r)\,\theta_2(r)\,dr/r \leqq \pi. \tag{2.9}$$

In the rest of the argument, we drop the subscript 2. We shall use (2.8) with $\alpha = d \leqq e^{-1}$. In particular, we have $p(r) = r$, $r \leqq 2d$. We shall prove that

$$\Phi'(r) \leqq 2\exp\{(p(d)/d)^2\}\,\Phi_d'(r), \quad r \geqq 2d. \tag{2.10}$$

If (2.10) holds, the same argument as in the previous case shows that we have a good bound for $J(\Phi, D)$ and the proof of Theorem 1 is complete.

To prove (2.10), we first note that

$$p(r) \leqq p(2d)\big(p(2d)-p(d)\big)^{-1}\big(p(r)-p(d)\big), \quad r \geqq 2d.$$

From (2.9), we know that $J_d \leqq 1-d^2$. To handle the exponents, it suffices to prove that

$$p(r)^2 \leqq (p(r)-p(d))^2(1-d^2)^{-1}+(p(d)/d)^2.$$

It is easy to check that this inequality holds for all positive r. This finishes the proof of (2.10).

3. Proofs of Lemma 1 and Lemma 2

The arguments are different in the two intervals $(0, q)$ and (q, ∞), where $q \in [e^{-2}, 1]$ will be defined below after Lemma 3.3. In (q, ∞), we would like to prove that if ω_R and h^0 are as in Section 2, we have $\omega_R \leqq Ch^0$. However, it will not be possible to delete all intervals $(-b, -a)$ with the property that $\{a<|z|<b\} \subset D$ from D: we have to leave finitely many 'big' intervals in D.

Without loss of generality, we can assume that $\theta(\cdot)$ is continuous. For simplicity, we assume in the proof that $\Psi \in F$ is increasing on (e^{-2}, ∞) and that $\Psi(r)=r^2$, $0<r<1$. We start with the case (q, ∞).

We recall that the region D is of the form $\{re^{i\theta}: |\theta|<\theta^*(r)/2\}$. This means that the "annuli" in Fig. 1 are not necessarily bounded by circular arcs centered at the origin. To pick out those annuli which make D multiply connected, we introduce the set $\{r>0: \theta(r)>\pi\}=\bigcup I'_j$, where the open intervals $\{I'_j\}$ are disjoint. Let $\{I_j\}$ be those intervals in $\{I'_j\}$ which are such that $I'_j \cap \{r: \theta^*(r)=\infty\} \neq \emptyset$. If $I_j=(a_j, b_j)$, we have $\theta(a_j)=\theta(b_j)=\pi$ for all indices j. Using the mapping $w=\log z$, we map D_0 onto

$$\mathcal{D} = \{w=u+iv: |v|<\theta(e^u)/2\}.$$

Let $\alpha_j=\log a_j$ and $\beta_j=\log b_j$. We recall that D_0 is the region D cut along the negative real axis. In Fig. 2, we have sketched the graphs of $v=\pm\theta(e^u)/2$ in such an interval $(\alpha, \beta)=(\log a, \log b)$ which is the image of the interval (a, b) in Fig. 1 (there is of course much more fine structure in Fig. 2): (a, b) is one of the intervals in $\{I_j\}$. The next step is to find an estimate of the harmonic measure F of $\partial\mathcal{D} \cap \{w=u+iv: \alpha<u<\beta, |v|=\pi\}$ in $\mathcal{D}$. In Fig. 2 the boundary values of F in (α, β) are given. Let $f(z)=\mathrm{F}(\log z)$ be the associated harmonic measure in the z-plane. To prove the estimate in Lemma 3.2, we need a preliminary result.

Lemma 3.1. *Let* $I(a, b)=\int_a^b \theta(t)^{-1}\,dt/t$ *and let* (a, b) *be one of the intervals in* $\{I_j\}$ *with* $a \geqq e^{-2}$. *Then*

$$I(a, b) \leqq (\pi\Psi(e^{-2}))^{-1}. \tag{3.1}$$

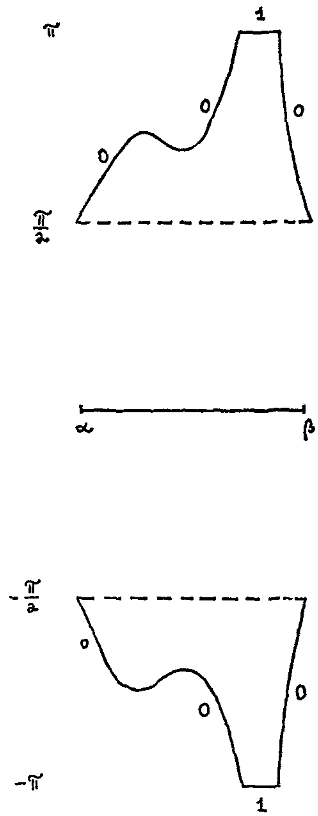

Figure 2

Proof. Since $\theta(t) \geqq \pi$ in $[a, b]$, it follows from (1.1) that

$$\pi \Psi(a) \log (b/a) \leqq \int_a^b \Psi(t)\theta(t)\, dt/t \leqq \pi,$$

$$I(a, b) \leqq \pi^{-1} \log (b/a) \leqq \big(\pi \Psi(a)\big)^{-1}.$$

The lemma is proved.

Lemma 3.2. *Let* $\delta = \int_a^b \Psi(t)\,\theta(t)\,dt/t$. *If* $\delta \leqq \pi^2 \Psi(a)$, *there is an absolute constant* C_1 *such that*

$$\max_{\alpha \leqq u \leqq \beta} \mathrm{F}(u \pm i\pi/2) \leqq C_1 \exp\big(-\pi^3 \Psi(a)(2\delta)^{-1}\big). \tag{3.2}$$

Remark. This is an estimate of F on the dotted lines in Fig. 2.

Proof. Since $\mathscr{D}$ is symmetric with respect to the real axis, it is sufficient to consider the case $v=\pi/2$. If $\gamma(v)=|\{u: u+iv\in\mathscr{D}, \alpha<u<\beta\}|$, we have

$$\pi^2/4 \leqq \int_{\pi/2}^{\pi}\gamma(v)\,dv\int_{\pi/2}^{\pi}\gamma(v)^{-1}\,dv \leqq (2\Psi(a))^{-1}\delta\int_{\pi/2}^{\pi}\gamma(v)^{-1}\,dv,$$

$$\int_{\pi/2}^{\pi}\gamma(v)^{-1}\,dv \geqq \pi^2\Psi(a)(2\delta)^{-1}.$$

According to the Ahlfors distortion theorem, (3.2) is true provided that $\pi^2\Psi(a)\geqq\delta$ (cf. p. 78 in [1]; also cf. §2.20 in [9]).

Returning to the z-plane, we have proved that if f_j^* is the harmonic measure of $\partial D_0\cap\{z=-r: a_j<r<b_j\}$ in D_0, we have

(3.3) $$\max\{f_j^*(re^{it}): a_j\leqq r\leqq b_j, |t|\leqq\pi/2\}\leqq C_1\exp(-\Psi_j/\delta_j),$$

where $\delta_j=\int_{a_j}^{b_j}\Psi(t)\,\theta(t)\,dt/t$, $\Psi_j=\pi^3\Psi(a_j)/2$ and $\delta_j\leqq\pi^2\Psi(a_j)$.

Let $c_1(\Psi)=\max(2C_2, 2C_2\exp\{(\pi\Psi(e^{-2}))^{-1}\})$, where C_2 is the absolute constant in (3.7) below produced by the Ahlfors distortion theorem and the absolute constant C_1 in (3.2).

This means that we have got an estimate of the harmonic measure f_j^* near the real axis in the interval (a_j, b_j). Once more applying the Ahlfors distorsion theorem in the simply connected domain D_0, we can deduce an estimate of f_j^* on the whole real axis (cf. (3.7) below).

To describe the main idea in remaining part of the proof, we let ω_R be as in Section 2 and let h_R^* be the harmonic measure of $D_0\cap\{|z|=R\}$ with respect to D_{0R}. Since D is circularly symmetric, $\max_\theta\omega_R(re^{i\theta})$ is assumed on the positive real axis and is increasing as a function of r (cf. Theorem 7 in Baernstein [2]). It follows that

$$\omega_R(z)\leqq h_R^*(z)+\sum_j\omega_R(b_j)f_j^*(z),\quad z\in D_R.$$

If we could prove that $\sum_j f_j^*(r)\leqq 1/2$, say, when $r<R$, then we would also be able to control $\omega_R(r)$ on the real axis. Unfortunately, this argument is too simple: we can only prove that $\sum f_j^*(r)\leqq 1/2$ if we restrict ourselves to summing over "small" intervals (a_j, b_j) in a sense to be made precise below. Also, we have to replace f_j^* and h_R^* by new harmonic measures f_j and h. The modified argument is as follows.

We shall divide the intervals in $\{I_j\}$ with $a_j\geqq e^{-2}$ into two classes: big and small intervals. Since $\sum_1^\infty\delta_j\leqq\pi$ (cf. (1.1)), it is easy to see that there exists a constant $c_0=c_0(\Psi)$ such that

(3.4) $$\sum{}'\exp(-\Psi(e^{-2})\pi^3(2\delta_j)^{-1})\leqq c_1(\Psi)^{-1},$$

where $\sum'$ means that we sum over all indices j with $\delta_j\leqq c_0(\Psi)$.

Let us say that those intervals in $\{I_j\}$ are big for which we have either

$$\delta_j > \pi^2 \Psi(a_j), \tag{3.5}$$

or

$$\delta_j > c_0(\Psi). \tag{3.6}$$

Intervals such that neither one of these alternatives hold are small. It follows from (1.1) that there are at most $(\pi\Psi(e^{-2}))^{-1}+\pi c_0^{-1}$ big intervals in $[e^{-2}, \infty)$.

Let us for a moment consider a block of small intervals $\{I_j\}_{j\in J}$ which are situated between two intervals (a', b') and (a, b) in $\{I_j\}$, $b'<a$. Let ω be the harmonic measure of $D\cap\{|z|=a\}$ with respect to $D\cup\{|z|<b'\}$ and let h be the harmonic measure of $D_0\cap\{|z|=a\}$ with respect to $D_0\cup\{|z|<b'\}$. Let f_j be the harmonic measure of $[-b_j, -a_j]$ with respect to $D_0\cup\{|z|<b'\}$. It is clear that (3.3) holds with f_j^* replaced by f_j. Once more using the distortion theorem, we find that there is an absolute constant C_2 such that

$$f_j(r) \leqq g_j(r) = C_2 \exp\left(-\Psi_j/\delta_j - \pi|I(r, a_j)|\right), \quad b' \leqq r \leqq a, \quad j\in J. \tag{3.7}$$

For $r>a_j$, we have used the estimate in Lemma 3.1.

Again referring to Theorem 7 in Baernstein [2] (cf. the preliminary discussion above), we know that $\max_\theta \omega(re^{i\theta})$ is assumed on the positive real axis and is increasing as a function of r. It follows that

$$\omega(z) \leqq h(z)+\textstyle\sum_{j\in J} \omega(b_j)f_j(z), \quad z\in D\cup\{|z| < b'\}. \tag{3.8}$$

We define

$$\Omega(r) = h(r)+2\textstyle\sum_{j\in J} h(b_j)\, g_j(r), \quad b' \leqq r \leqq a,$$

and claim that

$$\Omega(r) \geqq \omega(r), \quad b' \leqq r \leqq a. \tag{3.9}$$

To prove (3.9), we first use (3.4) and (3.7) to deduce that

$$\textstyle\sum_{j\in J} g_j(r) \leqq C_2 \sum' \exp\left(-\Psi_j/\delta_j\right) \leqq 1/2. \tag{3.10}$$

Assume that we can prove that

$$\Omega(r) - \textstyle\sum_{j\in J} \Omega(b_j)\, g_j(r) \geqq h(r), \quad b' \leqq r < a. \tag{3.11}$$

If (3.11) holds, it follows from (3.8) that

$$(\omega(r)-\Omega(r))^+ \leqq \textstyle\sum_{j\in J}(\omega(b_j)-\Omega(b_j))^+ g_j(r), \quad b' \leqq r < a,$$

which will imply (3.9) since we have (3.10).

A computation shows that (3.11) will be true if for all indices $k\in J$, we shall have

$$g_k(r) \geqq 2\textstyle\sum_{j\in J} g_k(b_j)\, g_j(r), \quad b' \leqq r < a,$$

which is equivalent to

$$(3.12)\qquad 1 \geqq 2C_2 \sum_{j\in J} \exp\Big(\pi\Big|\int_r^{a_k}\theta(t)^{-1}\,dt/t\Big| - \Psi_j/\delta_j - \pi\Big|\int_{b_j}^{a_k}\theta(t)^{-1}\,dt/t\Big| - \pi\Big|\int_r^{a_j}\theta(t)^{-1}\,dt/t\Big|\Big).$$

Since

$$\Big|\int_r^{a_k}\theta(t)^{-1}\,dt/t\Big| \leqq \Big|\int_r^{b_j}\theta(t)^{-1}\,dt/t\Big| + \Big|\int_{b_j}^{a_k}\theta(t)^{-1}\,dt/t\Big|,$$

the sum of the integrals in each exponent is at most (cf. Lemma 3.1)

$$\pi\Big\{\Big|\int_r^{b_j}\theta(t)^{-1}\,dt/t\Big| - \Big|\int_r^{a_j}\theta(t)^{-1}\,dt/t\Big|\Big\} \leqq \pi\int_{a_j}^{b_j}\theta(t)^{-1}\,dt/t \leqq (\pi\Psi(e^{-2}))^{-1}.$$

From (3.4), it follows that

$$1 \geqq c_1(\Psi)\sum_j{}' \exp(-\Psi_j/\delta_j) \geqq 2C_2 \exp\{(\pi\Psi(e^{-2}))^{-1}\} \sum_{j\in J}\exp\{-\Psi_j/\delta_j\}.$$

This inequality implies that (3.12) holds. Consequently, (3.11) and thus also (3.9) are true.

Combining the estimate of h given by (2.2) and the definition of g_j, we deduce that

$$h(b_j)\,g_j(r) \leqq 32\,C_2 \exp\{\pi I(a_j, b_j) - \Psi_j/\delta_j - \pi I(r, a)\}.$$

Once more using (3.4) and Lemma 3.1, we find our final estimate

$$(3.13)\qquad \omega(r) \leqq \Omega(r) \leqq 64\exp(-\pi I(r, a)), \quad b' \leqq r < a.$$

Let us now discuss how we can combine these estimates for blocks of small intervals and obtain an estimate of the harmonic measure ω_R in D. Let $\{(A_n, B_n)\}_1^N$ be the big intervals in (e^{-2}, ∞), where $\{A_n\}_1^N$ is an increasing sequence. We know that $N \leqq N(\Psi)$. If $R \geqq B_{n+1}$, it follows from (3.13) and Lemma 3.1 that

$$\omega_R(B_n) \leqq \omega_R(B_{n+1})\,64\exp(-\pi I(B_n, A_{n+1})) \leqq c(\Psi)\,\omega_R(B_{n+1})\exp(-\pi I(B_n, B_{n+1})),$$

and consequently that there is an absolute constant C such that if $q \geqq e^{-2}$,

$$(3.14)\qquad \omega_R(q) \leqq Cc(\Psi)^{N(\Psi)}\exp(-\pi I(q, R)), \quad R \geqq q.$$

This finishes the proof of Lemma 1.

The conclusion of Lemma 2 is trivial if $D \cap \{|z| < e^{-2}\}$ is simply connected: we take $D_2 = D$ and combine the Ahlfors distortion theorem with (3.14).

We recall that D was assumed to be of the form $\{re^{i\theta}\colon |\theta| < \theta^*(r)\}$. The main idea of the proof of Lemma 2 when $D \cap \{|z| < e^{-2}\}$ is not simply connected is to replace $D \cap \{|z| < q\}$ by its Steiner symmetrization $(D \cap \{|z| < q\})^S = \{z = x + iy\colon |y| < L(x)/2\}$, where $L(x) = |\{y\colon x + iy \in D \cap \{|z| < q\}\}|$. It is well-known that this operation increases harmonic measure on the real axis. The new domain is simply connected and we can handle it.

There is a discrepancy due to the fact that we compare harmonic measure for circular arcs to harmonic measure for segments contained in lines orthogonal to the real axis. The point of the next lemma is to define q in such a way that the discrepancy will be small.

Lemma 3.3. *Assume that* (2.4) *holds. Then there exists a point* $q' \in [e^{-2}, e^{-1/2}]$ *such that*

$$\{re^{i\theta}: |\log(r/q')| < \theta(q') \leqq 1/8, \quad |\theta| < \theta(q')/3\} \subset D. \tag{3.15}$$

We shall choose $q = q' \exp(\theta(q'))$. It is easy to check that $q \leqq 1$.

Proof of Lemma 3.3. We first find t_0 such that $m_0 = \theta(t_0) = \min \theta(t)$, $t \in [e^{-3/2}, e^{-1}]$. If (3.15) holds with $q' = t_0$, there is nothing more to prove. If this is not the case, there exists t_1 with $|\log(t_1/t_0)| \leqq m_0$ and $\theta(t_1) = m_1 \leqq 2m_0/3$. Then we can either choose $q' = t_1$ or there exists t_2 with $|\log(t_2/t_1)| \leqq m_1$ and $\theta(t_2) = m_2 \leqq 2m_1/3$. Inductively, we construct sequences $\{t_n\}$ and $\{m_n\}$ such that

$$|\log t_{n+1}/t_n)| \leqq m_n \quad \text{and} \quad \theta(t_{n+1}) = m_{n+1} \leqq 2m_n/3, \quad n = 0, 1, \ldots.$$

If there is a first index N such that (3.15) holds with $q' = t_N$, the lemma is proved. If no such index exists, it is easy to see that $\lim_{n\to\infty} t_n = T$ which is such that $|\log(T/t_0)| \leqq 3m_0$ and $\theta(T) = \lim_{n\to\infty} \theta(t_n) = 0$ (we recall that $\theta(\cdot)$ is continuous!). But this is impossible since D is connected. Thus, there exists a first index N and the lemma is proved.

If $\theta(q') = m$, we define

$$D_1 = (D \cap \{|z| < q'\}) \cup \{re^{i\theta}: q' \leqq r < q, \ |\theta| < m/3\}.$$

We first note that there is an absolute constant $C > 0$ such that

$$\omega_q(q'e^{i\theta}, D_1) \geqq C^{-1}, \quad |\theta| \leqq m/4. \tag{3.16}$$

To see this, let V be harmonic in $\{re^{i\theta}: |\log(r/q')| < m, |\theta| < m/3\}$ with boundary values 1 on $r = q$ and 0 on the rest of the boundary. We have constructed q' in such a way that this (logarithmic) rectangle is contained in D_1. From the maximum principle, we see that $\omega_q(\cdot, D_1) \geqq V$ in the rectangle and (3.16) follows.

We claim that

$$\omega_q(0, D) \leqq 2C\omega_q(0, D_1). \tag{3.17}$$

The proof is short: if $\mu(0, d\theta)$ is the harmonic measure of $D \cap \{|z| < q'\}$ on $|z| = q'$, we have

$$\omega_q(0, D) = \int_{|\theta|<m/2} \omega_q(q'e^{i\theta}, D)\,\mu(0, d\theta) \leqq 2\int_{|\theta|<m/4} \mu(0, d\theta)$$
$$\leqq 2C\int_{|\theta|<m/4} \omega_q(q'e^{i\theta}, D_1)\,\mu(0, d\theta) \leqq 2C\omega_q(0, D_1).$$

If $r<q<R$, we have $\omega_R(r, D)\leqq\omega_q(r, D)\omega_R(q, D)$ and it follows from (3.17) that

$$\omega_R(0, D) \leqq \omega_q(0, D)\,\omega_R(q, D) \leqq 2C\omega_R(q, D)\,\omega_q(0, D_1). \tag{3.18}$$

Hence, it is sufficient to estimate $\omega_q(0, D_1)$.

We define D_2 as the Steiner symmetrization of D_1. The symmetrization does not change anything near q': since we know that $\theta_1(r)=2m/3$ for $q'\leqq r<q$ and $\theta_1(r)\geqq 2m/3$ for $q'e^{-m}<r<q'$ and that $\cos(m/3)\geqq e^{-m}$ (cf. (2.4) and (3.15)), we have

$$D_2\cap\{q' < |z| < q\} = D_1\cap\{q' < |z| < q\}.$$

Steiner symmetrization preserves the measure of the set. Hence one more application of (3.15) shows that

$$|D_2| = |D_1| \leqq \big|D\cap\{|z| < q\}\big|,$$

which is required in Lemma 2 (cf. (2.5)).

If $\varrho=q\cos(m/3)$, we define H to be harmonic in $D_2\cap\{z=x+iy\colon x<\varrho\}$ with boundary values 1 on $x=\varrho$ and 0 on the rest of the boundary. From Theorem 7 in Baernstein [2], we see that

$$\omega_q(x, D_1) \leqq H(x), \quad 0 < x \leqq \varrho. \tag{3.19}$$

Remark. Baernstein's theorem deals with circular symmetrization. It is easy to see that the argument also works for Steiner symmetrization.

We know that D_2 is simply connected. Again applying the distortion theorem, we obtain

$$H(r) \leqq \omega_\varrho(r, D_2) \leqq C\exp\big(-\pi I_2(r, \varrho)\big), \quad d_2 \leqq r \leqq \varrho. \tag{3.20}$$

Since $\varrho>q'$, we have $I_2(\varrho, q)\leqq I_2(q', q)=3/2$. Combining this fact with (3.19) and (3.20), we find that

$$\omega_q(0, D_1) \leqq C\exp\big(-\pi I_2(d_2, q)+3/2\big),$$

where d_2 is the radius in the largest disk centered at the origin which is contained in D_2. If we define $\theta_2(r)=\theta(r)$, $r>q$, our final estimate is given by (3.18) and says that

$$\omega_R(0, D) \leqq C(\Psi)\exp\big(-\pi I(d_2, R)\big).$$

We have proved Lemma 2.

Remark. The reason for our special choice $\Psi(r)=r^2$, $0<r<1$, is that we have to do a Steiner symmetrization near the origin: this operation does not change the area of $D\cap\{|z|<q'\}$ which can also be written $\int_0^{q'} r\theta(r)\,dr$. This means that we can control condition (1.1).

4. Proof of Corollary 4

It follows from (2.1) that

$$\lim_{R\to\infty} \Phi(R)\,\omega_R(0, D) = 0. \tag{4.1}$$

This is the basic fact needed in the proof. From our assumptions and from the maximum principle, it is clear that

$$u(z) \leqq M(R)\,\omega_R(z, D), \quad z\in D\cap\{|z| < R\}.$$

From (1.3), we see that there is a constant c and a sequence $\{R_j\}$ tending to infinity such that

$$u(z) \leqq c\Phi(R)\,\omega_R(z, D), \quad z\in D\cap\{|z| < R\}, \quad R\in\{R_j\}. \tag{4.2}$$

From Harnack's inequality, we deduce that if $z\in D$ is given, there is a number $C(z, d)$ such that

$$\omega_R(z, D) \leqq C(z, D)\,\omega_R(0, D). \tag{4.3}$$

Combining (4.2) and (4.3), letting $R_j\to\infty$ and using (4.1), we find that $u(z)\leqq 0$. The Corollary is proved.

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