On the asymptotics of solutions of Volterra integral equations

D. R. Yafaev

1. Introduction

It is well-known (see e.g. [1]) that a solution of a Volterra integral equation

$$v(t)-\int_{t_0}^{\infty} G(t, \tau)v(\tau)\,d\tau = v_0(t) \tag{1.1}$$

exists, is unique, and may be obtained by a series of iterations

$$v = \sum_{n=0}^{\infty} v_n, \quad v_{n+1}(t) = \int_{t_0}^{t} G(t, \tau)v_n(\tau)\,d\tau. \tag{1.2}$$

Moreover, if the kernel $G(t, \tau)$ vanishes sufficiently quickly (in a power scale) for large values of the variables, then the functions $v_n(t)$ can be bounded by a common power of t (for a polynomially bounded free term v_0). It follows that the solution $v(t)$ of (1.1) is bounded by the same power of t. On the other hand, if $G(t, \tau)$ decays slowly (or grows) as $t, \tau\to\infty$, then generally $v_{n+1}(t)\, v_n(t)^{-1}\to\infty$, $t\to\infty$. In this case the series (1.2) gives only a bound of exponential type for $v(t)$ (even if $v_0(t)$ has compact support).

In the present paper we study the behaviour of the solution of the equation (1.1) with a slowly decreasing (or growing) kernel $G(t, \tau)$. This problem is close in spirit to the investigation of the asymptotics of solutions to differential equations. Consider, for example, the simplest equation

$$-v''(t)+q(t)v(t) = 0, \quad t \geqq t_0. \tag{1.3}$$

If the function $q(t)$ has compact support, then the equation (1.3) has solutions that equal 1 or t for large t. Similarly, if $q(t)=0(t^{-2-\varepsilon})$, $\varepsilon>0$, then solutions of (1.3) approach 1 or t asymptotically. The proof of this assertion may be obtained by reduction of (1.3) to a Volterra integral equation. However, if $q(t)$ decays slower than t^{-2}, or grows as $t\to\infty$, then the terms $v_n(t)$ of the corresponding series (1.2) obey the relation $v_{n+1}(t)v_n(t)^{-1}\to\infty$, $t\to\infty$. This changes the asymptotics of $v(t)$. In such a case the asymptotics of $v(t)$ was found by Greene and Liouville (see e.g. [2]) with the help of the famous substitution

$$v(t) = q(t)^{-1/4} \exp\left[\pm \int_{t_0}^{t} q(s)^{1/2}\, ds\right] w(t). \tag{1.4}$$

For $w(t)$ this gives a differential equation with coefficients, which stabilize quickly at infinity. With this approach one can prove that if, for example, $q(t) \geqq ct^{-\alpha}$ or $q(t) \leqq -ct^{-\alpha}$, $c>0$, $\alpha<2$, then the equation (1.3) has solutions $v_{\pm}(t)$ satisfying

$$v_{\pm}(t) \sim q(t)^{-1/4} \exp\left[\pm \int_{t_0}^{t} q(s)^{1/2}\, ds\right], \quad t \to \infty. \tag{1.5}$$

The precise proof of (1.5) also requires assumptions on the behaviour of $q'(t)$ and $q''(t)$ as $t\to\infty$.

The problem of the asymptotic behaviour of the solution $v=(I-G)^{-1}v_0$ (G is a Volterra operator with a kernel $G(t, \tau)$) of the equation (1.1) is connected with the study of the kernel of the operator $(I-G)^{-1}-I$ for large values of t and τ. Similarly to differential equations, the effective construction of $(I-G)^{-1}$ is based on the right guess of the first approximation (Ansatz) to this operator. The kernel of the first approximation includes again (cf. with (1.4)) an exponential function. Our approach demands certain smoothness of the kernel, but we permit arbitrary power growth at infinity. The main advantage of the suggested Ansatz is that it automatically takes into account compensation of terms in the series

$$(I-G)^{-1} = I + \sum_{n=1}^{\infty} G^n. \tag{1.6}$$

It turns out that such compensation is governed by the diagonal values of $G(t, \tau)$. Our Ansatz reduces the problem to the construction of $(I-G_0)^{-1}$ for a kernel $G_0 = G_0(G)$ with critical decay at infinity (such decay is intermediate between the cases when the series (1.2) gives a polynomial or an exponential bound). We can then easily obtain a relatively rough estimate for the solution $v(t)$ of the equation (1.1). In particular, we find a wide class of cases when $v(t)$ is bounded by some power of t. It is essentially more difficult to obtain a precise estimate or the asymptotics of $v(t)$. This problem requires sufficiently detailed information about the operator $(I-G_0)^{-1}$. Fortunately, under rather wide assumptions on $G(t, \tau)$ the operator $(I-G_0)^{-1}$ may be constructed explicitly with the help of the Mellin transform.

The need to find the asymptotics of a solution of a Volterra equation originated in scattering theory. Results of the present paper are applied in [3] to scattering of a quantum particle by a time-dependent zero-range potential.

The paper is organized as follows. Some elementary information on Volterra integral operators is contained in section 2. The first approximation to the operator $(I-G)^{-1}$ (the main Ansatz) is constructed in section 3. We study the corresponding kernel G_0 with critical decay in section 4. In section 5 the operator $I-G_0$ is inverted. In section 6 we apply results on $(I-G)^{-1}$ to the original problem of the asymptotic behaviour of the solution of the equation (1.1).

2. Elementary information on Volterra integral operators

We shall consider Volterra integral operators G defined by

$$(Gf)(t) = \int_{t_0}^{t} G(t, \tau) f(\tau)\, d\tau. \tag{2.1}$$

The number t_0 is supposed to be fixed and positive. The function $G(t, \tau)$, $t \geqq \tau \geqq t_0$, in (2.1) is called the kernel of the operator G; $G(t, \tau)$ is assumed to be locally bounded. A Volterra operator and its kernel are always denoted by the same letter. The operators $I-G$ form a group; hence the inverse operator is defined by the series (1.6). The proof of its convergence is based on the following well-known

Lemma 1. *Let*

$$|G(t, \tau)| \leqq q_0(t) q_1(\tau). \tag{2.2}$$

Then the kernel G_n of the operator $G_n = G^n$ satisfies

$$|G_n(t, \tau)| \leqq q_0(t) q_1(\tau)[(n-1)!]^{-1} \Big[\int_{\tau}^{t} q(s)\, ds\Big]^{n-1}, \quad q = q_0 \cdot q_1, \tag{2.3}$$

and for the kernel of the operator $K = (I-G)^{-1} - I$ the estimate

$$|K(t, \tau)| \leqq q_0(t) q_1(\tau) \exp\Big[\int_{\tau}^{t} q(s)\, ds\Big] \tag{2.4}$$

holds.

Actually, (2.3) may be easily verified by induction:

$$|G_{n+1}(t, \tau)| = \Big|\int_{\tau}^{t} G_n(t, s) G(s, \tau)\, ds\Big| \leqq q_0(t)[(n-1)!]^{-1}$$
$$\times \int_{\tau}^{t} \Big(\int_{s}^{t} q(\sigma)\, d\sigma\Big)^{n-1} q(s)\, ds \cdot q_1(\tau) = q_0(t) q_1(\tau)(n!)^{-1}\Big(\int_{\tau}^{t} q(s)\, ds\Big)^{n}.$$

The estimate (2.3) implies convergence of the series (1.6). This in its turn justifies the formula (1.6). Summation of the estimates (2.3) over n proves (2.4). This concludes the proof.

The estimate (2.2) holds for any locally bounded function G if one sets e.g.

$$q_0(t) = \sup_{\tau \in [t_0, t]} |G(t, \tau)|, \quad q_1(\tau) \equiv 1.$$

Thus the series (1.6) is always convergent, but in general the bound (2.4) is only exponential. If, however, under the assumptions of Lemma 1 $q \in L_1(t_0, \infty)$, then $K(t, \tau)$ satisfies the same estimate as $G(t, \tau)$. This is convenient to describe in terms of the following

Definition. *Assume that for some real α and β*

$$|G(t, \tau)| \leqq C(t/\tau)^{\beta} \tau^{\alpha-1}. \tag{2.5}$$

The number α is called the order and β the type of the operator (or kernel) G.

By C and c we denote generic positive constants. Clearly, the order or the type are not uniquely defined: any α', $\alpha'>\alpha$, and β', $\beta'>\beta$, are respectively the order or the type of the operator G. Lemma 1 implies that if G obeys (2.5) with $\alpha<0$, then $K=(I-G)^{-1}-I$ has the same order and the same type as G. If $\alpha=0$, then the order of K also equals zero, but its type is undetermined. In case $\alpha>0$ Lemma 1 ensures only that

$$|K(t,\tau)| \leqq C(t/\tau)^{\beta}\exp[C\alpha^{-1}(t^{\alpha}-\tau^{\alpha})]. \tag{2.6}$$

The purpose of the present paper is to single out a rather wide class of kernels G, for which the bound (2.6) may be essentially improved on account of compensation of terms in the series (1.6).

Let us accept the following agreement. We say that a kernel G of the order α and the type β (or the bound (2.5)) is J times differentiable if for any $0\leqq j_1+j_2=j\leqq J$

$$\left|\frac{\partial^j G(t,\tau)}{\partial t^{j_1}\partial\tau^{j_2}}\right| \leqq C(t/\tau)^{\beta}\tau^{\alpha-1}t^{-j_1}\tau^{-j_2}. \tag{2.7}$$

The set of kernels satisfying (2.7) is denoted by $\mathscr{G}(\alpha,\beta;J)$. As was already noted, Lemma 1 ensures that the set of operators $I-G$, where $G\in\mathscr{G}(\alpha,\beta)\equiv\mathscr{G}(\alpha,\beta;0)$, is a group if $\alpha<0$. With help of the relations

$$K(t,\tau) = G(t,\tau)+\int_{\tau}^{t} G(t,s)K(s,\tau)\,ds = G(t,\tau)+\int_{\tau}^{t} K(t,s)G(s,\tau)\,ds$$

it is easy to verify a more general assertion.

Lemma 2. *The set of operators $I-G$, where $G\in\mathscr{G}(\alpha,\beta;J)$ and $\alpha<0$, is a group.*

We also note

Lemma 3. *Let $G_1\in\mathscr{G}(\alpha,\beta;J)$ with $\alpha<0$ and $G_2\in\mathscr{G}(0,\beta;J)$. Then $G_1G_2\in\mathscr{G}(\alpha,\beta;J)$ and $G_2G_1\in\mathscr{G}(\alpha,\beta';J)$ for any $\beta'>\beta$.*

Formal proofs of Lemmas 2 and 3 are straightforward, and we omit them.

3. The main Ansatz

Here we begin the construction of the operator $(I-G)^{-1}$ for $G\in\mathscr{G}(\alpha,\beta)$, where $\alpha>0$. In this section we find the first approximation to $(I-G)^{-1}$. We can then reduce the problem to the "critical" case when the estimate (2.2) holds with $q_0(t)q_1(t)=O(t^{-1})$. Our approach requires a certain smoothness of G so that actually we assume that $G \in \mathscr{G}(\alpha, \beta; J)$, where $\alpha > 0$ and J is sufficiently large (at least $J \geqq 2$). Moreover, at the diagonal $t=\tau$ we need an estimate opposite to (2.5), i.e.

$$|G(t, t)| \geqq ct^{\alpha-1}. \tag{3.1}$$

We shall seek the first approximation to the operator $(I-G)^{-1}$ in the form $I+\Phi$, where

$$\Phi(t, \tau) = a(t)b(\tau) \exp [\varphi(t)-\varphi(\tau)]. \tag{3.2}$$

The functions φ, a and b are supposed to have power behaviour and $\varphi(t) \to \infty$, as $t \to \infty$. Thus, we should choose the functions φ, a and b in such a way that the kernel of Ω defined by

$$I+\Omega = (I-G)(I+\Phi) \tag{3.3}$$

satisfies (2.2) with $q_0(t)q_1(t)=O(t^{-1})$. By (3.2), (3.3)

$$\Omega(t, \tau) = -G(t, \tau)+a(t)b(\tau)e^{\varphi(t)-\varphi(\tau)} - \int_\tau^t G(t, s)a(s)e^{\varphi(s)}\, ds\, b(\tau)e^{-\varphi(\tau)}. \tag{3.4}$$

Set

$$L_k(t, \tau) = (-1)^k\left(\varphi'(\tau)^{-1}\frac{\partial}{\partial\tau}\right)^k [G(t, \tau)a(\tau)\varphi'(\tau)^{-1}], \quad k \geqq 0, \tag{3.5}$$

and integrate in the integral in (3.4) $(n+1)$ times $(1 \leqq n \leqq J-1)$ by parts:

$$\int_\tau^t G(t, s)a(s)e^{\varphi(s)}\, ds = \sum_{k=0}^n [L_k(t, t)e^{\varphi(t)}-L_k(t, \tau)e^{\varphi(\tau)}] - \int_\tau^t \left(\frac{\partial}{\partial s} L_n(t, s)\right) e^{\varphi(s)}\, ds.$$

Insert this expression into (3.4) and choose the functions φ, a and b so that in (3.4) $-G(t, \tau)$ cancels $L_0(t, \tau)b(\tau)$ and $a(t)b(\tau) \exp [\varphi(t)-\varphi(\tau)]$ cancels $-L_0(t, t)b(\tau) \cdot \exp [\varphi(t)-\varphi(\tau)]$. By (3.5) this yields two equations

$$a(t)b(t) = \varphi'(t), \quad \varphi'(t) = G(t, t). \tag{3.6}$$

The second equation (3.6) determines uniquely (up to an insignificant (see (3.2)) constant factor) the function $\varphi(t)$. The first equation (3.6) determines b if a is given (or vice versa). So one of the functions a or b is arbitrary. The relations (3.6) ensure that both a and b do not equal zero. Under the conditions (3.6) the equality (3.4) takes the form

$$\Omega(t, \tau) = -\sum_{k=1}^n L_k(t, t)b(\tau)e^{\varphi(t)-\varphi(\tau)}+\sum_{k=1}^n L_k(t, \tau)b(\tau) + \int_\tau^t \left(\frac{\partial}{\partial s} L_n(t, s)\right) e^{\varphi(s)-\varphi(\tau)}\, ds\, b(\tau). \tag{3.7}$$

Let us estimate $\Omega(t, \tau)$. Set $n=1$ in (3.7), take an arbitrary power function $a(t)=t^\lambda$ and define

$$\zeta(t) = \int_{t_0}^{t} (\operatorname{Re} G(s, s))_+ \, ds. \tag{3.8}$$

Here, as usual, $(f)_+=f$ if $f\geqq 0$ and $(f)_+=0$ if $f\leqq 0$. By (3.1), (3.5), (3.6) the kernel $L_1(t, \tau)$ satisfies

$$|L_1(t, \tau)| \leqq C(t/\tau)^\beta \tau^{\lambda-\alpha},$$

and this estimate admits at least one differentiation. It follows that the first sum in (3.7) is bounded by

$$C(t/\tau)^{\lambda-\alpha}\tau^{-1}\exp[\zeta(t)-\zeta(\tau)],$$

and the second by $C(t/\tau)^\beta\tau^{-1}$. The last term in (3.7) does not exceed

$$C\int_\tau^t (t/s)^\beta s^{\lambda-\alpha-1}|e^{\varphi(s)-\varphi(\tau)}|\, ds\, \tau^{\alpha-1-\lambda} \leqq C_1(t/\tau)^\beta\tau^{-1}\exp[\zeta(t)-\zeta(\tau)],$$

if $\lambda<\alpha+\beta$. Thus relation (3.7) ensures

Theorem 1. *Let $G\in\mathcal{G}(\alpha, \beta; 2)$, where $\alpha>0$, and let the condition* (3.1) *hold. Define the kernel Φ by the formulae* (3.2), (3.6) *and $a(t)=t^\lambda$, $\lambda<\alpha+\beta$. Then the kernel Ω (see* (3.3)*) satisfies*

$$|\Omega(t, \tau)| \leqq C(t/\tau)^\beta\tau^{-1}\exp[\zeta(t)-\zeta(\tau)]. \tag{3.9}$$

By Lemma 1 for the kernel of $K=(I+\Omega)^{-1}-I$ the bound

$$|K(t, \tau)| \leqq C(t/\tau)^\gamma\tau^{-1}\exp[\zeta(t)-\zeta(\tau)] \tag{3.10}$$

holds. Here γ is some real number that depends on the constant C in (3.9). So Theorem 1 has the following

Corollary. *Under the assumptions of Theorem 1*

$$(I-G)^{-1} = (I+\Phi)(I+K), \tag{3.11}$$

where the kernel K satisfies (3.10) *for some γ.*

Theorem 1 leaves the number γ in (3.10) undetermined. Below (see sections 4 and 5) we shall find this number under more special assumptions on $G(t, \tau)$.

4. The improvement of the first approximation

To find the number γ in (3.10) we have to study in detail the structure of the operator Ω. Note that the number γ is of interest only in case the function $\zeta(t)$ (see (3.8)) is bounded (or increases slower than $\ln t$) at infinity. Therefore we assume that

$$\operatorname{Re} G(t, t) \leqq 0. \tag{4.1}$$

Under this assumption $\zeta(t)=0$, and the last factors in the right-hand sides of (3.9), (3.10) may be omitted. Thus both operators Ω and K have zero order; the type of Ω equals β and the type of K is indefinite. Our problem is to determine it.

Let us return to the equality (3.7). According to the conditions $G\in\mathcal{G}(\alpha,\beta;J)$ and (3.1)

$$\left|\frac{\partial^j(\varphi'(t))^{-1}}{\partial t^j}\right| \leqq Ct^{1-\alpha-j}, \quad 0\leqq j\leqq J,$$

so that the operator $\varphi'(\tau)^{-1}\dfrac{\partial}{\partial\tau}$ in (3.5) lowers the order of the kernel by α units. It follows that the kernels $L_k(t,t)b(\tau)$ and $L_k(t,\tau)b(\tau)$ in (3.7) have the order $-\alpha(k-1)$. Since by (4.1) $|\exp[\varphi(t)-\varphi(\tau)]|\leqq 1$, the k^{th} term in each of the sums in (3.7) has also the order $-\alpha(k-1)$. Moreover, the order of the last term in (3.7), defined by the integral, may be made arbitrary low by the choice of n. Thus the equality (3.7) gives us the expansion of $\Omega(t,\tau)$ in a sum of kernels of decreasing orders. Thereby the order of two first terms, corresponding to $k=1$, equals zero.

As we shall see in section 6, the precise bound, or the asymptotics, of the solution $v(t)$ of the equation (1.1) requires estimates not only for Ω but also for its derivatives. So we need to verify that $\Omega\in\mathcal{G}(0,\beta;m)$, where $m>0$ (in section 6 we set $m=2$). Unfortunately, because of the factor $\exp[\varphi(t)-\varphi(\tau)]$, differentiation of the first sum in (3.7) does not decrease its order. This trouble may be overcome on account of the factorisation of this sum into the product of factors depending only on t or τ. We shall now show that for a suitable choice of $a(t)$, which is undetermined up to now, and for a sufficiently large n the function

$$z(t)=\textstyle\sum_{k=1}^n L_k(t,t) \tag{4.2}$$

decays arbitrarily quickly at infinity (namely, $z(t)a^{-1}(t)=O(t^{-(n+1)\alpha})$). Thus the order of

$$\Omega_1(t,\tau)=z(t)b(\tau)\exp[\varphi(t)-\varphi(\tau)] \tag{4.3}$$

turns out to be negative. In fact, it depends on n and tends to $-\infty$ as $n\to\infty$. This permits us to prove that $\Omega_1\in\mathcal{G}(-\varepsilon,\beta;m)$ for $\varepsilon>0$ and $m>0$. In virtue of (3.5) the equality $z(t)=0$ may be regarded as a linear differential equation of order n for the function $a(t)$. Probably, we could have taken one of the solutions of this equation for the function $a(t)$. This requires a study of the asymptotics of its solutions. Another possibility is to construct an explicit, though approximate, solution of the equation $z(t)=0$. This way is considerably simpler, and we follow it.

To describe our construction, set

$$(D_k a)(t)=(-1)^k\left(\varphi'(s)^{-1}\frac{\partial}{\partial s}\right)^k[G(t,s)\varphi'(s)^{-1}a(s)] \quad (s=t). \tag{4.4}$$

We shall seek $a(t)$ as a sum

$$a = \sum_{j=1}^{n} a_j.$$

Then

(4.5)

$$z = \sum_{k=1}^{n} D_k \sum_{j=1}^{n} a_j = \sum_{l=1}^{n} \left(\sum_{j=1}^{l} D_{l+1-j} a_j\right) + \sum_{k+j \geqq n+2} D_k a_j, \quad 1 \leqq k, \quad j \leqq n.$$

Let $J \geqq 2n-1$. We submit the functions a_j to the conditions

$$\sum_{j=1}^{l} D_{l+1-j} a_j = 0. \tag{4.6}$$

The system (4.6) may be solved successively. For $l=1$ the equation $D_1 a_1 = 0$ in detailed notation reads $a_1'(t) = h(t) a_1(t)$, where $h(t) = G_t(t, t) G(t, t)^{-1}$. Here and in what follows we denote by $G_t(t, \tau)$ and $G_\tau(t, \tau)$ the partial derivatives of some kernel $G(t, \tau)$ with respect to its first and second argument, respectively. It follows that

$$a_1(t) = \exp\left[\int_{t_0}^{t} h(s)\, ds\right]. \tag{4.7}$$

Since under our assumptions $h(t) = O(t^{-1})$, $t \to \infty$, we can define

$$\lambda_0 = \sup_{t \geqq t_0} [t \operatorname{Re} h(t)], \quad \lambda_1 = \inf_{t \geqq t_0} [t \operatorname{Re} h(t)].$$

Then

$$c t^{\lambda_1} \leqq |a_1(t)| \leqq C t^{\lambda_0}, \tag{4.8}$$

and

$$a_1^{(j)}(t) a_1(t)^{-1} = O(t^{-j}), \quad 1 \leqq j \leqq J. \tag{4.9}$$

For $1 < l \leqq n$ we shall regard (4.6) as an equation for a_l. Set

$$r_l = \varphi' a_1^{-1} \sum_{j=1}^{l-1} D_{l+1-j} a_j, \quad \xi_l = a_1^{-1} a_l. \tag{4.10}$$

In terms of ξ_l the equation (4.6), i.e. $a_l' - h a_l = a_1 r_l$, takes the simpler form

$$\xi_l'(t) = r_l(t), \quad l \geqq 2. \tag{4.11}$$

Let us show that r_l satisfies

$$|r_l(t)| \leqq C t^{-1-(l-1)\alpha}, \tag{4.12}$$

the function $\xi_l(t)$ may be defined by

$$\xi_l(t) = -\int_t^{\infty} r_l(s)\, ds \tag{4.13}$$

and

$$|\xi_l(t)| \leqq C t^{-(l-1)\alpha}. \tag{4.14}$$

Moreover, the estimate (4.12) may be differentiated $(J-l)$ times and (4.14) $(J-l+1)$ times. For the proof assume that (4.12)—(4.14) hold for all $l < l_0$. We shall establish their validity for $l=l_0$. It suffices to prove (4.12). Actually, since $\alpha>0$ the equation (4.11) gives (4.13) and this in its turn implies (4.14). Let us start from the definition (4.10). By (4.4), (4.9) the operator $a_1^{-1}D_k a_1$ lowers the order by αk units. So on account of (4.14) for ξ_j, $j<l_0$,

$$(4.15) \qquad \left|(a_1^{-1}D_k(a_1\xi_j))(t)\right| \leqq Ct^{-(j+k-1)\alpha}.$$

The estimate (4.15) may be differentiated $(J-j-k+1)$ times. Thus, all summands $\varphi' a_1^{-1}D_{l_0+1-j}(a_1\xi_j)$ in (4.10) are bounded by $t^{\alpha-1-l_0\alpha}$ and these estimates are $(J-l_0)$ times differentiable. This concludes the proof of (4.12). The relations (4.5), (4.6) ensure that

$$a_1^{-1}z = \sum_{\substack{j+k\geqq n+2\\ 1\leqq j,\,k\leqq n}} a_1^{-1}D_k(a_1\xi_j).$$

By condition (4.15) this yields the estimate $a_1^{-1}(t)z(t)=O(t^{-(n+1)\alpha})$, which is $(J-2n+1)$ times differentiable. Now we set $\xi=\xi_2+\ldots+\xi_n$ and summarize the results obtained.

Lemma 4. *Let* $G\in\mathcal{G}(\alpha,\beta;J)$, $\alpha>0$, $\varphi'(t)=G(t,t)$, $J\geqq 2$, $1\leqq n\leqq (J+1)/2$ *and let condition* (3.1) *hold. Define* $z(t)$ *by formulae* (3.5), (4.2) *and* $a_1(t)$ *by formula* (4.7). *Then there exists a function* $\xi(t)$ *(depending on* n*),*

$$(4.16) \qquad |\xi(t)| \leqq Ct^{-\alpha},$$

such that for $a=a_1(1+\xi)$ *the function* z *satisfies*

$$(4.17) \qquad |z(t)| \leqq Ct^{\lambda_0-(n+1)\alpha}.$$

Moreover, estimates (4.16) *and* (4.17) *may be differentiated* $(J-n+1)$ *and* $(J-2n+1)$ *times, respectively.*

According to (4.8), under the assumptions of Lemma 4, the functions a and $b=\varphi' a^{-1}$ have estimates

$$(4.18) \qquad |a(t)| \leqq Ct^{\lambda_0}, \quad |b(t)| \leqq Ct^{\alpha-\lambda_1-1},$$

which are $(J-n+1)$ times differentiable. The inequalities (4.17), (4.18) imply that the kernel (4.3) satisfies

$$\left|\frac{\partial^m}{\partial t^{m_1}\partial\tau^{m_2}}\,\Omega_1(t,\tau)\right| \leqq Ct^{-(n+1)\alpha+\lambda_0+m_1(\alpha-1)}\tau^{\alpha-1-\lambda_1+m_2(\alpha-1)},$$

$0\leqq m\leqq J-2n+1$. In particular, $\Omega_1\in\mathcal{G}(-\varepsilon,\beta,m)$, if $(n-m)\alpha\geqq\lambda_0-\alpha-\beta$ and $(n-m)\alpha\geqq\lambda_0-\lambda_1+\varepsilon$.

Let us now consider the second sum in the right-hand side of (3.7). By Lemma 4 the kernel $L_k(t,\tau)b(\tau)$ (see (3.5), (3.6)) belongs to the class $\mathcal{G}(-\alpha(k-1),\beta;\,J-n-$

$-k+1)$. Since $a=a_1(1+\xi)$, the kernel of zero order is

$$L_1(t,\tau)b(\tau)=-a(\tau)^{-1}\frac{\partial}{\partial\tau}[G(t,\tau)G(\tau,\tau)^{-1}a(\tau)] \tag{4.19}$$

$$=-a_1(\tau)^{-1}\frac{\partial}{\partial\tau}[G(t,\tau)G(\tau,\tau)^{-1}a_1(\tau)]-(1+\xi(\tau))^{-1}\xi'(\tau)G(t,\tau)G(\tau,\tau)^{-1}.$$

Due to the equation $a_1'=ha_1$ the first summand here equals

$$G(\tau,\tau)^{-2}[G(t,\tau)G_\tau(\tau,\tau)-G(\tau,\tau)G_\tau(t,\tau)]\equiv\Omega_0(t,\tau). \tag{4.20}$$

The kernel Ω_0 has zero order and (4.16) ensures that the second summand in (4.19) belongs to the class $\mathscr{G}(-\alpha,\beta;J-n)$. The sum of this term and of $\sum_{k=2}^n L_k(t,\tau)b(\tau)$ will be denoted by $\Omega_2(t,\tau)$. Thus $\Omega_2\in\mathscr{G}(-\alpha,\beta;J-2n+1)$.

It remains to consider the integral of the right-hand side of (3.7):

$$\Omega_3(t,\tau)=\int_\tau^t L_{n+1}(t,s)\varphi'(s)e^{\varphi(s)}\,ds\,b(\tau)e^{-\varphi(\tau)}. \tag{4.21}$$

Let us make use of the bound

$$|L_{n+1}(t,\tau)|\leqq C(t/\tau)^\beta\tau^{-\alpha(n+1)+\lambda_0},\quad J\geqq 2n, \tag{4.22}$$

which may be differentiated $(J-2n)$ times. Inserting (4.22) into (4.21) and taking (4.1), (4.18) into account, we find that for $n\alpha>\lambda_0-\beta$, $n\alpha\geqq\lambda_0-\lambda_1+\alpha+\varepsilon$ the kernel Ω_3 has order $-\varepsilon$ and type β. Differentiation of (4.21) with respect to t shows that

$$\frac{\partial\Omega_3(t,\tau)}{\partial t}=L_{n+1}(t,t)\varphi'(t)b(\tau)e^{\varphi(t)-\varphi(\tau)}+\int_\tau^t\left(\frac{\partial}{\partial t}L_{n+1}(t,s)\right)\varphi'(s)e^{\varphi(s)}\,ds\,b(\tau)e^{-\varphi(\tau)}.$$

The first summand in the right-hand side is quite similar to $\Omega_1(t,\tau)$, and the second to $\Omega_3(t,\tau)$. The derivative of (4.21) with respect to τ is

$$\frac{\partial\Omega_3(t,\tau)}{\partial\tau}=-L_{n+1}(t,\tau)\varphi'(\tau)b(\tau)$$
$$+\int_\tau^t L_{n+1}(t,s)\varphi'(s)e^{\varphi(s)}\,ds[b'(\tau)-b(\tau)\varphi'(\tau)]e^{-\varphi(\tau)}.$$

Here $L_{n+1}(t,\tau)\varphi'(\tau)b(\tau)$ is analogous to terms in the second sum in (3.7), and the second summand in the right-hand side has the same structure as $\Omega_3(t,\tau)$. Thus, the same considerations as above permit us to estimate the derivatives of $\Omega_3(t,\tau)$. Omitting at this place simple calculations, we formulate only the result. The kernel $\Omega_3(t,\tau)$ belongs to $\mathscr{G}(-\varepsilon,\beta;m)$ if $m\leqq J-2n+1$, $n\alpha>\lambda_0-\beta$, $(n-m)\alpha\geqq\lambda_0-\alpha-\beta$ and $(n-m)\alpha\geqq\lambda_0-\lambda_1+\alpha+\varepsilon$.

Let us summarize our results. Denote by $[\cdot]$ the integral part of a positive number. Set $\hat\Omega=-\Omega_1+\Omega_2-\Omega_3$ and choose n so that

$$\nu_0\equiv\alpha^{-1}\max\{\lambda_0-\beta,\ \lambda_0-\lambda_1+\alpha\}<n\leqq J/2 \tag{4.23}$$

(this implies of course that $J \geqq 2[\nu_0]+2$). Then the kernel $\hat{\Omega}$ has negative order and type β. The kernel $\hat{\Omega}$ belongs to $\mathscr{G}(-\varepsilon, \beta; m)$, where $\varepsilon > 0$, $1 \leqq m \leqq J-2n+1$, if

$$(4.24) \qquad \nu \equiv \alpha^{-1} \max \{\lambda_0-\alpha-\beta,\ \lambda_0-\lambda_1+\alpha\} < n-m.$$

Let now $J \geqq 2\nu+4$. Then the inequalities $1 \leqq m \leqq J-2n+1$ and $m+\nu < n$ hold for $n=[(J+\nu-1)/3]+1$, $m=[(J-2\nu-1)/3]$. In this case $\hat{\Omega} \in \mathscr{G}(-\varepsilon, \beta;\ [(J-2\nu-1)/3])$, $\varepsilon > 0$, and the bounds (4.18) for the functions a, b are differentiable at least $[(2J-\nu+1)/3]$ times. Thus we obtain

Theorem 2. *Let $G \in \mathscr{G}(\alpha, \beta; J)$, $\alpha > 0$, let the conditions* (3.1), (4.1) *hold and let the kernels Ω and Ω_0 be defined by formulae* (3.3) *and* (4.20), *respectively. If $J \geqq 2[\nu_0]+2$, then there exists a kernel Φ of the type* (3.2) *such that*

$$(4.25) \qquad \Omega(t, \tau) = \Omega_0(t, \tau) + \hat{\Omega}(t, \tau),$$

where $\hat{\Omega}$ has negative order and type β. If $J \geqq 2\nu+4$, then (4.25) *holds with $\hat{\Omega} \in \mathscr{G}(-\varepsilon, \beta;\ [(J-2\nu-1)/3])$, $\varepsilon > 0$.*

The functions a, b and φ in (3.2) *are connected by the equations* (3.6) *and $a = a_1(1+\xi)$, where a_1 is defined by* (4.7) *and ξ satisfies* (4.16). *Moreover,* (4.16) *may be differentiated $(J-[\nu_0])$ times if $J \geqq 2[\nu_0]+2$ and $[(2J-\nu+1)/3]$ times if $J \geqq 2\nu+4$.*

Remark 1. Under the assumptions of Theorem 2 the functions a and b satisfy (4.18), which are differentiable $(J-[\nu_0])$ times if $J \geqq 2[\nu_0]+2$ and $[(2J-\nu+1)/3]$ times if $J \geqq 2\nu+4$.

Remark 2. Let the function $h(t) = G_t(t, t)G(t, t)^{-1}$ satisfy for some μ the condition

$$(4.26) \qquad h(t) - \mu t^{-1} \in L_1(t_0, \infty).$$

Then (see (4.7)) $a_1(t) = ct^{\mu}(1+o(1))$ so that one can take $\lambda_0 = \lambda_1 = \lambda = \operatorname{Re} \mu$ in Theorem 2.

Remark 3. One can omit condition (4.1) in Theorem 2. Then (for example, in the case $J \geqq 2\nu+4$) the assertion of Theorem 2 remains true if we replace the assertion $\hat{\Omega} \in \mathscr{G}(-\varepsilon, \beta; [(J-2\nu-1)/3])$ by the estimates

$$\left| \frac{\partial^m}{\partial t^{m_1} \partial \tau^{m_2}} \hat{\Omega}(t, \tau) \right| \leqq C(t/\tau)^{\beta} \tau^{-1-\varepsilon} t^{-m_1} \tau^{-m_2} \exp[\zeta(t) - \zeta(\tau)],$$

where $0 \leqq m_1 + m_2 = m \leqq [(J-2\nu-1)/3]$.

By the equality (3.3) the construction of $(I-G)^{-1}$ is reduced to the inversion of the operator $I+\Omega$ ($\Omega = -G_0$ in the notation of the Introduction). The operators Ω_0 and consequently Ω always have zero order. Thus Lemma 1 ensures only that the kernel $K=(I+\Omega)^{-1}-I$ also has zero order, but its type remains undetermined. The meaning of the equality (4.25) is that it distinguishes a truly singular (of zero order) part Ω_0 of the operator Ω. The summand Ω_0 can be expressed simply in terms of the original kernel G. Under some additional assumptions this allows us to construct the operator $(I+\Omega_0)^{-1}$ and to evaluate the type of the operator K.

5. Inversion of the singular part

In this section we need more special assumptions on $G(t,\tau)$. We accept now

The condition of asymptotic homogeneity. Let

$$G(t,\tau)=g(t/\tau)\tau^{\alpha-1}+\tilde{G}(t,\tau),\quad \alpha>0, \tag{5.1}$$

where $g(z)$ is a J_1 times differentiable function, $J_1\geqq 3$,

$$|g^{(j)}(z)|\leqq Cz^{\beta-j},\quad 0\leqq j\leqq J_1,\quad z\geqq 1, \tag{5.2}$$

$g(1)\neq 0$, and the kernel $\tilde{G}$ belongs to the class $\mathcal{G}(\alpha-\varepsilon,\beta;J_2)$, $\varepsilon>0$, $J_1\geqq J_2\geqq 1$.

The conditions of Theorem 2 are also supposed to hold. Note, however, that (3.1) follows from the above assumption. Under the conditions of this section the expression for the singular part of Ω may be further simplified. Actually, by easy computations we get according to (4.20) that

$$\Omega_0(t,\tau)=\omega(t/\tau)\tau^{-1}+\tilde{\Omega}(t,\tau), \tag{5.3}$$

where $\omega(z)=g(1)^{-1}\ [zg'(z)-\varkappa g(z)]$, $\varkappa=g'(1)g(1)^{-1}$ and $\tilde{\Omega}\in\mathcal{G}(-\varepsilon,\beta;J_2-1)$. Moreover, in this case (4.26) is fulfilled with $\mu=\varkappa$ so that one can take $\lambda_0=\lambda_1=\lambda=\operatorname{Re}\varkappa$ and (see (4.23), (4.24))

$$\nu_0=\max\{(\lambda-\beta)/\alpha,1\},\quad \nu=\max\{(\lambda-\beta)/\alpha-1,1\}. \tag{5.4}$$

Now we insert (5.3) into (4.25) and collect $\tilde{\Omega}$ and $\hat{\Omega}$ together. Then $\tilde{\Omega}+\hat{\Omega}\equiv\Omega_r\in\mathcal{G}(-\varepsilon,\beta;J^0)$, where $J^0=\min\{J_2-1,\ [(J-2\nu-1)/3]\}$. Set $\Omega_s(t,\tau)=\omega(t/\tau)\tau^{-1}$. Then the equality (4.25) reads

$$\Omega=\Omega_s+\Omega_r. \tag{5.5}$$

The summand Ω_s has zero order, and it determines the truly singular part of the kernel Ω. Since the order of Ω_r is negative, the inversion of $I+\Omega$ may be easily reduced (see below) to the inversion of $I+\Omega_s$. It is important for us that with the help of the Mellin transform one can construct the operator $(I+\Omega_s)^{-1}$ explicitly.

Recall some basic facts about the Mellin transform (see e.g. [4]). For any function $f(z)$, defined for $z\geqq 1$ and increasing as $z\to\infty$ not quicker than some power of z (i.e. $|f(z)|\leqq Cz^{\sigma_0}$), we set

$$\hat{f}(p)=\int_1^\infty z^{-p-1}f(z)\,dz.$$

The Mellin transform $\hat{f}(p)$ is defined at any rate in the half-plane $\operatorname{Re} p > \sigma_0$, is an analytic function there and $\hat{f}(p) \to 0$ as $\operatorname{Re} p \to \infty$. If, moreover,

$$\left|\left(z\frac{d}{dz}\right)^k f(z)\right| \leqq C z^{\sigma_0} \tag{5.6}$$

for $k=1$, then $\hat{f}(p) = O(|p|^{-1})$ as $p\to\infty$, $\operatorname{Re} p \geqq \sigma$ for any $\sigma > \sigma_0$. If the estimate (5.6) holds for $0 \leqq k \leqq n$, where $n \geqq 2$, then

$$\hat{f}(p) = \textstyle\sum_{k=1}^{n-1} f_k p^{-k} + O(|p|^{-n}), \quad p\to\infty, \quad \operatorname{Re} p \geqq \sigma > \sigma_0, \tag{5.7}$$

$f_0 = f(1)$. Conversely, if a function $\hat{f}(p)$ is analytic in the half-plane $\operatorname{Re} p > \sigma_0$ and satisfies (5.7) for some $n \geqq 2$, then $\hat{f}(p)$ is the Mellin transform of a function $f(z)$ such that $f^{(k)}(z) = O(z^{\sigma-k})$ for any $\sigma > \sigma_0$ and $0 \leqq k \leqq n-2$. Note that the Mellin transform of

$$\int_1^z f_1(z/\zeta) f_2(\zeta)\, d\zeta/\zeta$$

is $\hat{f}_1(p)\hat{f}_2(p)$.

For kernels (5.1) we shall find the type of the operator $K = (I+\Omega)^{-1}$ in terms of some parameters of the Mellin transform $\hat{g}(p)$ of the function $g(z)$. By (5.2) the function $\hat{g}(p)$ is analytic in the half-plane $\operatorname{Re} p > \beta$ and satisfies there (5.7) with $n = J_1 \geqq 3$. Thus there are only a finite number of its zeros in the half-plane $\operatorname{Re} p \geqq \beta + \varepsilon$ for any $\varepsilon > 0$. Denote by p_0 the maximum of the real parts of these zeros. If $\hat{g}(p)$ does not have any zero for $\operatorname{Re} p > \beta$, then we set $p_0 = \beta$. To compute the type of K we introduce

$$\delta_0 = \max\{p_0, \lambda\}, \quad \lambda = \operatorname{Re} \varkappa. \tag{5.8}$$

Let us seek the operator $(I+\Omega_s)^{-1}$ in the form $I+\Psi$, where

$$\Psi(t,\tau) = \psi(t/\tau)\tau^{-1}. \tag{5.9}$$

The equality $\Omega_s + \Psi + \Omega_s\Psi = 0$ ensures that

$$\omega(t/\tau) + \psi(t/\tau) + \int_\tau^t \omega(t/\sigma)\psi(\sigma/\tau)\, d\sigma/\sigma = 0.$$

After the change of variables $\sigma = \tau\zeta$, $t = z\tau$ we get

$$\omega(z) + \psi(z) + \int_1^z \omega(z/\zeta)\psi(\zeta)\, d\zeta/\zeta = 0.$$

The Mellin transform of this equation yields

$$\hat{\omega}(p) + \hat{\psi}(p) + \hat{\omega}(p)\hat{\psi}(p) = 0,$$

so that

$$\hat{\psi}(p) = -\hat{\omega}(p)[1+\hat{\omega}(p)]^{-1}.$$

Since $\hat{\omega}(p) = -1 + g(1)^{-1}(p-\varkappa)\hat{g}(p)$, it follows that

$$\hat{\psi}(p) = -1 + g(1)(p-\varkappa)^{-1}\hat{g}(p)^{-1}.$$

Thus the function $\hat{\psi}(p)$ is regular in the half-plane $\operatorname{Re} p > \delta_0$. The relation (5.7), where $n = J_1$, for $\hat{g}(p)$ yields (5.7) with $n = J_1 - 1$ for $\hat{\psi}(p)$. So $\hat{\psi}(p)$ is the Mellin transform of a function $\psi(z)$, which satisfies

$$|\psi^{(j)}(z)| \leqq C z^{\delta - j}, \quad 0 \leqq j \leqq J_1 - 3,$$

for any $\delta > \delta_0$. By (5.9) it follows that $\Psi \in \mathscr{G}(0, \delta; J_1 - 3)$.

Let us return to the construction of $(I + \Omega)^{-1}$. The relations (5.5) and $(I + \Psi) \cdot (I + \Omega_s) = I$ show that

$$(I + \Omega)^{-1} = (I + \Psi)[I + \Omega_r(I + \Psi)]^{-1}.$$

Let the numbers ν_0 and ν be defined by (5.4). If $J \geqq 2[\nu_0] + 2$, then Ω_r has negative order and type δ. By Lemma 3 the same is true for the operator $\Omega_r(I + \Psi)$. Now Lemma 2 ensures that

$$[I + \Omega_r(I + \Psi)]^{-1} - I = K_0 \in \mathscr{G}(-\varepsilon, \delta), \ \varepsilon > 0.$$

Applying Lemma 3 once more we find that

$$(I + \Omega)^{-1} = (I + \Psi)(I + K_0) = I + K, \tag{5.10}$$

where $K \in \mathscr{G}(0, \delta)$. Quite similarly, if the conditions of Theorem 2 hold for $J \geqq 2\nu + 4$, then the operator K in (5.10) belongs to the class $\mathscr{G}(0, \delta; J_0)$, where $J_0 = \min\{J_1 - 3, J_2 - 1, [(J - 2\nu - 1)/3]\}$.

We summazize our results in the following

Theorem 3. *Let $G \in \mathscr{G}(\alpha, \beta; J)$, $\alpha > 0$, $\operatorname{Re} G(t, t) \leqq 0$. Assume that the condition of asymptotic homogeneity holds. Let further Φ be the same as in Theorem 2, let δ_0 be defined by* (5.8) *and let δ be any number greater than δ_0. Then the operator $(I - G)^{-1}$ is represented by formula* (3.11), *where $K \in \mathscr{G}(0, \delta)$ if $J \geqq 2[\nu_0] + 2$ and $K \in \mathscr{G}(0, \delta; J_0)$ if $J \geqq 2\nu + 4$.*

6. Asymptotics of solutions

Here we use the results of Theorems 1 and 3 on the structure of the operator $(I - G)^{-1}$ to obtain a bound, or asymptotics, of the solution $v(t)$ of the integral equation (1.1). In virtue of the formula (3.11) the function $v(t)$ admits the representation

$$v(t) = w(t) + a(t) e^{\varphi(t)} \int_{t_0}^{t} e^{-\varphi(\tau)} b(\tau) w(\tau)\, d\tau, \tag{6.1}$$

where $w = v_0 + K v_0$. At first we shall obtain a bound for $v(t)$ under the conditions of Theorem 1. Assume that for some real γ_0 the free term v_0 satisfies

$$|v_0(t)| \leqq C t^{\gamma_0} \exp[\zeta(t)]. \tag{6.2}$$

By (3.10) it follows that a similar bound holds for $w(t)$. The equality (6.1) ensures now that

$$|v(t)| \leqq Ct^{\gamma} \exp [\zeta(t)], \tag{6.3}$$

where γ is some real number. In particular under the additional assumption (4.1) the solution $v(t)$ is polynomially bounded.

This assertion may be essentially improved if the conditions of Theorem 3 are fulfilled. Assume now that $G\in\mathscr{G}(\alpha, \beta; J)$, where $J\geqq 2\nu+7$, and let the condition of asymptotic homogeneity hold for $J_1=5, J_2=3$. Then by Theorem 3 $K\in\mathscr{G}(0, \delta; 2)$. For v_0 suppose that

$$|v_0^{(j)}(t)| \leqq Ct^{\delta_0-j}, \quad 0 \leqq j \leqq 2. \tag{6.4}$$

The equality $w=v_0+Kv_0$ ensures that a similar bound is valid for w:

$$|w^{(j)}(t)| \leqq Ct^{\delta-j}, \quad 0 \leqq j \leqq 2, \quad \delta > \delta_0. \tag{6.5}$$

Set

$$P_0 = (\varphi')^{-1} bw, \quad p_1 = (\varphi')^{-1} p_0'$$

and integrate twice by parts in (6.1):

$$v(t) = w(t) - a(t)\big(p_0(t)+p_1(t)\big) + a(t)e^{\varphi(t)} \times\Big[\big(p_0(t_0)+p_1(t_0)\big)e^{-\varphi(t_0)} + \int_{t_0}^{t} e^{-\varphi(\tau)} p_1'(\tau)\, d\tau\Big]. \tag{6.6}$$

By the equality $ab=\varphi'$ the functions $w(t)$ and $a(t)p_0(t)$ cancel each other on the right-hand side of (6.6). The constant $p_0(t_0)+p_1(t_0)$ will be denoted by l. So (6.6) reads

$$v(t) = -a(t)p_1(t) + a(t)e^{\varphi(t)} \Big[l + \int_{t_0}^{t} e^{-\varphi(\tau)} p_1'(\tau)\, d\tau\Big]. \tag{6.7}$$

According to the bounds (4.18) and (6.5) the first summand in (6.7) satisfies

$$|a(t)p_1(t)| \leqq Ct^{\delta-\alpha}. \tag{6.8}$$

Let us consider the cases $\delta_0-\lambda-\alpha\geqq 0$ and $\delta_0-\lambda-\alpha<0$ separately. If $\delta_0-\lambda-\alpha\geqq 0$, then by (4.1) and $p_1'(t)=0(t^{\delta-\lambda-\alpha-1})$

$$\Big|e^{\varphi(t)} \int_{t_0}^{t} e^{-\varphi(\tau)} p_1'(\tau)\, d\tau\Big| \leqq Ct^{\delta-\lambda-\alpha}. \tag{6.9}$$

Since $a(t)=O(t^{\lambda})$, the relations (6.7)—(6.9) ensure that

$$v(t) = O(t^{\delta-\alpha}). \tag{6.10}$$

If $\delta_0-\lambda-\alpha<0$ and $g_0=\operatorname{Re} g(1)<0$, then

$$\operatorname{Re}\varphi'(t) = g_0 t^{\alpha-1} + O(t^{\alpha-1-\varepsilon}) \leqq 2^{-1} g_0 t^{\alpha-1}. \tag{6.11}$$

Thus in this case the bound (6.9) remains true. Moreover, by (6.11) the function $e^{\varphi(t)}$ vanishes quicker than any power of t^{-1} as $t\to\infty$. So the bound (6.10) holds again.

In case $\delta_0-\lambda-\alpha<0$, $g_0=0$ we assume additionally that $G(t, t)$ is purely imaginary. Then the integral over (t_0, t) in (6.7) may be treated as a difference of integrals over (t_0, ∞) and (t, ∞). Each of these integrals is absolutely convergent. The integral over (t_0, ∞) changes only the constant l. The integral over (t, ∞) is $O(t^{\delta-\lambda-\alpha})$. Under these circumstances the solution $v(t)$ has the asymptotics

$$v(t) = la(t)e^{\varphi(t)}+O(t^{\delta-\alpha}).$$

Since the difference $a(t)-a_1(t)$ is $O(t^{\delta-\alpha})$, we can replace here $a(t)$ by $a_1(t)$. In explicit notation it means that

$$v(t) = l\exp\left\{\int_{t_0}^{t}[G_t(s, s)G(s, s)^{-1}+G(s, s)]\,ds\right\}+O(t^{\delta-\alpha}). \tag{6.12}$$

We collect the results on the asymptotic behaviour of $v(t)$ in

Theorem 4. 1) *Let* $G\in\mathcal{G}(\alpha, \beta; 2)$, $\alpha>0$, *let condition* (3.1) *hold and let* $v_0(t)$ *satisfy* (6.2) *for some* γ_0. *Then for the solution* $v(t)$ *of the integral equation* (1.1) *the estimate* (6.3) *holds with some real* γ. *In particular, under the additional assumption* (4.1) *the function* $v(t)$ *is polynomially bounded if* $v_0(t)$ *has this property.*

2) *Let* $G\in\mathcal{G}(\alpha, \beta; J)$, $\alpha>0$, $J\geqq 2\nu+7$, *let the condition of the asymptotic homogeneity hold for* $J_1=5$, $J_2=3$ *and let* $v_0(t)$ *satisfy* (6.4). *Then*

a) *the bound* (6.10) *is valid if* $g_0=\operatorname{Re}g(1)<0$ *or* $g_0=0$, $\operatorname{Re}G(t, t)\leqq 0$, $\alpha\leqq\delta_0-\lambda$,

b) *the asymptotics* (6.12) *is valid if* $g_0=0$, $\operatorname{Re}G(t, t)=0$, $\alpha>\delta_0-\lambda$.

It is sufficient in Theorem 4, part 1, to assume smoothness of $G(t, \tau)$ only for $t\geqq\tau\geqq t_1$, where t_1 is arbitrarily large. Actually, splitting the integral in (1.1) into the sum of integrals over (t_0, t_1) and (t_1, t) we can reduce equation (1.1) to a similar equation for $t\geqq t_1$ and the new free term

$$v_1(t) = v_0(t)+\int_{t_0}^{t_1}G(t, \tau)v(\tau)\,d\tau.$$

The same consideration can of course be applied also to the conditions of the second part of Theorem 4.

In conclusion we note that under the assumptions of Theorem 4 the estimation of absolute values (see Lemma 1) of the terms of series (1.2) gives only the inequality

$$|v(t)| \leqq Ct^{\beta}\exp[d(t)], \tag{6.13}$$

where at any rate

$$d(t) \geqq \int_{t_0}^{t}|G(s, s)|\,ds.$$

The improvement of (6.13) takes place in Theorem 4 due to mutual compensation of terms in (1.2). This is connected with restrictions on the values of the argument of $G(t, t)$. So the condition $\operatorname{Re} G(t, t) \leqq 0$ ensures that $v(t)$ is polynomially bounded. If the values of $G(t, t)$ lie outside some sector $|\arg z| \leqq \theta$, $\theta \in (0, \pi/2)$, then by (6.3)

$$(6.14) \qquad |v(t)| \leqq Ct^{\gamma} \exp \left[\cos \theta \int_{t_0}^{t} |G(s, s)|\, ds\right].$$

Since the last integral increases as t^{α}, both bounds (6.13) and (6.14) are of the same exponential order. However, the constant multiplier of t^{α} in (6.14) is smaller than that in (6.13).

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D. R. Yafaev
Leningrad Branch of the
Mathematical Institute (LOMI)
Fontanka 27
Leningrad, 191011
USSR