On the multiplicative properties of the de Rham-Witt complex. II.

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Introduction

The purpose of this paper is to present a Künneth formula for the cohomology of the de Rham—Witt complex (cf. [15]) for smooth and proper varieties over a perfect field k. To explain the details of this let me recall the form the Künneth formula takes for the crystalline cohomology. In this case the cohomology of a smooth and proper variety X over k is a certain complex of W-modules $R\Gamma(X/W)$ where W is the ring of Witt vectors of k. The Künneth formula then takes the form $R\Gamma(X/W)\otimes_W^L R\Gamma(Y/W)=R\Gamma(X x_k Y/W)$. The reason for the appearance of the tensor product is that it is the product universal for the properties of the cup product in crystalline cohomology namely W-bilinearity. The Künneth formula is proved by first, with the aid of the cup product, defining a morphism $R\Gamma(X/W)\otimes_W^L R\Gamma(Y/W)\to R\Gamma(X x_k Y/W)$ and then noting that as both sides of this morphism are complexes whose cohomology are finitely generated W-modules to prove that it is an isomorphism it suffices to show that $W/p\otimes_W^L(-)$ applied to it is. Finally, $W/p\otimes_W^L((-)\otimes_W^L(-))=(W/p\otimes_W^L(-))\otimes_W^L(W/p\otimes_W^L(-))$ and $W/p\otimes_W^L R\Gamma(Z/W)\doteq H_{DR}(Z/k)$ for any smooth k-variety Z. That the Künneth morphism is an isomorphism is now clear as the Künneth morphism in de Rham-cohomology is. From our point of view this proof has one drawback. It needs the rather precise piece of information that $R\Gamma(Z/W)$ has finitely generated cohomology when Z is a smooth and proper variety over k. This may be rectified as follows. For a general morphism $M\to N$ in the derived category of W-complexes it is true that if $W/p\otimes_W^L(-)$ applied to it is an isomorphism then its completion also is, where the completion functor, $(\hat{-})$, is defined to be the composite $R\varprojlim\{W/p^n\otimes_W^L,\pi\}$, π being the morphisms induced by projections $W/p^{n+1}\to W/p^n$. We hence get a Künneth formula involving the completed tensor product, $((-)\otimes_W^L(-))\hat{}$, this time there being no need to assume that X and Y be proper. This version of the Künneth formula is then completed by a calculation of $(M\otimes_W^L N)\hat{}$

when M and N has finitely generated cohomology namely in that case it equals $M \otimes_W^L N$. The plan of proof for the present Künneth formula is very similar. The de Rham—Witt complex is a module over the Raynaud ring R, the W-ring generated by F, d and V and having certain relations. The multiplication in the de Rham—Witt complex fulfills, apart from W-bilinearity, certain relations wrt the operations of the elements of R. We define a product $(-)*_R(-)$ on R-modules universal for these relations. The reduction modulo p is now replaced by reduction modulo V and dV and a suitable completion functor is defined. As the reduction of the cohomology of the de Rham—Witt complex is the Hodge-cohomology we conclude the proof of the Künneth isomorphism by the Künneth formula for the Hodge-cohomology. This first step can be done in very great generality. Apart from some minor technical conditions it is proved for smooth varieties over a perfect scheme of characteristic $p>0$. The last step, the computation of the Künneth formula is less satisfactory. Ideally we would like to compute it completely for complexes with coherent cohomology (cf. [16: I, Déf. 3.9]). The actual result is a far cry from the ideal. We will be able to compute it only in some very special cases, which still leads to some interesting consequences however.

Here follows a more detailed description of the contents of the particular chapters.

In chapter I we develop the necessary R-module formalism. Proposition 1.1 is our version of Nakayama's lemma which will enable the reduction to the Hodge-cohomology. Two points should be noted. Firstly, we will more generally want to be able to transfer properties of the reduction modulo V and dV to the completion hence the very general form of the proposition. Secondly, contrary to the case of W-complexes, the Nakayama's lemma is not true without any boundedness restrictions on the complexes involved. An R-complex may be regarded as a double complex and the lemma will be true if the complexes involved are bounded in any of the four directions. In section 2 we show that the completion functor behaves as expected notably that the completion of a complex is complete. In section 3 we define $(-)*_R(-)$ and obtain some of its properties the main result being that $R*_R R$ is free on a countable number of generators. In section 4 we continue this study. The main result is that the reduction modulo V and dV of $(-)*_R^L(-)$ is what is expected and needed. In section 5 we define the right adjoint of $(-)*_R(-)$, the internal Hom-functor, and study its properties which will be needed in some of the explicit computations. Theorem 6.2 is the main result of the chapter. It says that a certain subcategory of $D(R)$ admits a structure of a rigid tensor category and that reduction modulo V and dV is a tensor functor. In section 8 we study the relation with crystalline cohomology or what amounts to the same, the simple complex associated to an R-complex.

One general comment on chapter *I* should be made. All our results are valid in a perfectly ringed topos satisfying a certain technical condition. The proofs of our assertions are not very different from what they would have been in the pointual case with one notable exception. Sometimes we want to compute Ext: s by resolving the first variable by free modules. This is, of course, not possible in general. If we simultaneously resolve the second variable by flasque modules it is possible however.

In chapter II we prove the Künneth formula. With the ground already covered this is very simple indeed. Using the internal Hom-formalism we also get a new proof of the duality formula for the cohomology of the de Rham—Witt complex which also works over any perfect base.

In chapter III we get down to some explicit calculations. First we prove in Proposition 1.1 that a complex of R-modules bounded from above is coherent iff its cohomology consists of coherent R-modules. As the category of coherent complexes is a triangulated subcategory of $D(R)$ this implies that the category of coherent R-modules is an abelian subcategory of R-mod a fact which, strangely enough, seems to be difficult to prove directly. Apart from the case when one of the varieties is ordinary (cf. Prop. 2.1) the terms in the Künneth formula are very difficult to compute. We do it in some particular cases by the following method. We know the associated simple complex by the Künneth formula for the crystalline cohomology and the reduction modulo V and dV by the Künneth formula for the Hodge-cohomology. The results of chapter I enable us to obtain these formulas purely algebraically. By the tensor formalism we can also compute the fixed points under F, the logarithmic cohomology. Combining this we obtain our computations. As one particular consequence of these partial computations we can prove that $((-)*_R^L(-))^\wedge$ has amplitude $[-2, 0]$ on the category of coherent complexes. Finally, we apply our results to compute the cohomology of the product of the Igusa-surface with itself. As a consequence of this we obtain an example of a smooth and proper morphism $f\colon X\to W$ such that the higher direct images in the flat topology $R^if_*\hat{G}_m$ are not pro-representable for $i=3, 4$. In the course of proving this we obtain for a smooth and proper morphism $f\colon X'\to \operatorname{Spec} k$ the relations between the $R^if_*\hat{G}_m$ and the $H^i(X, W\mathcal{O})$ without assuming that the $R^if_*\hat{G}_m$ are smooth, a result which is hopefully of independent interest (cf. [2]).

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Table of contents

Table of notations

p, k — a fixed prime number and a fixed perfect field of char. p.
$\tau_{\leq i}$, $\tau_{>i}$, $t_{\leq i}$, $t_{>i}$ — see (0: 1.1)
pointual topoi, surjections between topoi — see (0: 2)
$M[X]$, M a graded group sheaf, X a graded sheaf — see (0: 2)
$(S, \mathcal{O}_S)$ — see (0: 5)
W_S, W_{nS}, R_S, R_n, Z_n, B_n, gr^n, gr_1^n, $R.$, inv-drw-S — see (0: 5)
$D_c(R)$, $D_{\mathrm{perf}}(R)$, $D^{*,0}(R)$, $(\hat{-})$ — see (0: 5)
$W\Omega^{\cdot}_{X/S}$ — see (0: 7)
domino, $\mathbf{U}_i$, T^iM, k as R-module, $F^{\infty}BM$, $V^{-\infty}ZM$, Couer M, $\mathrm{dom}^i\, M$, A^0M, A_s^1M, A_n^1M, A_f^2M, $E_{r/s}$, slope 0, positive slope, semi-simple torsion — see (0: 8)
$D^i(M)$, A^2M — see (0: 9)
G_s, G_f, G a formal group — see (0: 10)
$(-)*_R(-)$ — see (I: 3)
$*$-flat R-modules, $W_S[F, F^{-1}]$ — see (I: 4)
$(-)\hat{*}_R^L(-)$ — see (I: 4.7.1)
$\underline{\mathrm{Hom}}_R^!(-, -)$ — see (I: 5.1)
F-crys-n, R-mod-n — see (I: 7.2)
$\mathrm{Kün}_i^R(-, -)$ — see (III: 3.1)

0. Conventions and preliminary results

1. By the word "functor" we will always mean a covariant functor. To any contravariant functor $F\colon A \to B$ there is associated a functor $A^{\mathrm{op}} \to B$ which we, by abuse of language, will denote F. Similarly for multifunctors of mixed variance. We will use the usual sign conventions for complexes (cf. [1: XVII, 1]). Note that for a contravariant additive functor $A \to B$ we obtain two different associated functors $C(A)^{\mathrm{op}} \to C(B)$ which however are canonically isomorphic (cf. [1: XVII, Lemme 1.1.5.4 ii]). This should cause no confusion.

If A is an abelian category with countable sums and $X^{\cdot\cdot}$ is a double complex in A then $\mathrm{s}(X^{\cdot\cdot})$ will be the associated simple complex whose component in degree n is $\sum_{i+j=n} X^{i,j}$ and similarly for multiple complexes. $\mathrm{s}(-)$ will then commute with direct limits. Dually, $\mathrm{s}'(-)$ will denote the simple associated complex using products. If $T(-,-)$: $A\times B\to C$ is a biadditive functor between abelian categories, if C has countable sums and if $T(-,-)$ commutes with sums then for $X^{\cdot}\in C(A)$, $Y^{\cdot}\in C(B)$, $T(X^{\cdot}, Y^{\cdot})$ will use $\mathrm{s}(-)$ in its definition. Similarly for multifunctors. Dually for multifunctors commuting with products we will use $\mathrm{s}'(-)$. Hence if $S(-,-)$: $A_1\times A_2\to A$ commutes with sums and $G(-,-,-):=T(S(-,-),-)$ then $G(X^{\cdot}, Y^{\cdot}, Z^{\cdot})=T(S(X^{\cdot}, Y^{\cdot}), Z^{\cdot})$ for $X^{\cdot}\in C(A_1)$, $Y^{\cdot}\in C(A_2)$ and $Z^{\cdot}\in C(B)$. Similarly and dually.

Example: Using all conventions introduced we see that $\mathrm{Hom}_{\mathbf{Z}}(X^{\cdot}\otimes_{\mathbf{Z}} Y^{\cdot}, Z^{\cdot})= \mathrm{Hom}_{\mathbf{Z}}(X^{\cdot}, \mathrm{Hom}_{\mathbf{Z}}(Y^{\cdot}, Z^{\cdot}))$ for $X^{\cdot}, Y^{\cdot}\in C(\mathbf{Z})^{\mathrm{op}}$ and $Z^{\cdot}\in C(\mathbf{Z})$.

For a complex $X^{\cdot}$ in an abelian category put

$$\tau_{\leqq i}X^{\cdot} := (\dots \xrightarrow{d} X^{i-1} \xrightarrow{d} Z^i \longrightarrow 0..)\,\tau_{>i}X^{\cdot} := X^{\cdot}/\tau_{\leqq i}X^{\cdot} \tag{1.1}$$

$$t_{>i}X^{\cdot} := (\dots 0 \longrightarrow X^{i+1} \xrightarrow{d} X^{i+2} \xrightarrow{d} ..)\,t_{\leqq i}X^{\cdot} := X^{\cdot}/t_{>i}X^{\cdot}$$

If $T(-,-)$ is a biadditive functor $A\times B\to C$ between abelian categories then we clearly have canonical isomorphisms of bicomplexes when $X^{\cdot}\in C(A)$ and $Y^{\cdot}\in C(B)$;

$$\varinjlim_i \{T(\tau_{\leqq i}X^{\cdot}, Y^{\cdot})\} = T(X^{\cdot}, Y^{\cdot}) \tag{1.2}$$

$$\varinjlim_i \{T(t_{>i}X^{\cdot}, Y^{\cdot})\} = T(X^{\cdot}, Y^{\cdot})$$

and if C has countable sums

$$\varinjlim_i \{\mathrm{s}(T(\tau_{\leqq i}X^{\cdot}, Y^{\cdot}))\} = \mathrm{s}(T(X^{\cdot}, Y^{\cdot})) \tag{1.3}$$

$$\varinjlim_i \{\mathrm{s}(T(t_{>i}X^{\cdot}, Y^{\cdot}))\} = \mathrm{s}(T(X^{\cdot}, Y^{\cdot})).$$

Similarly for multifunctors and dually, if C has countable products:

$$\varprojlim_i \{\mathrm{s}'(T(\tau_{>i}X^{\cdot}, Y^{\cdot}))\} = \mathrm{s}'(T(X^{\cdot}, Y^{\cdot})) \tag{1.4}$$

$$\varprojlim_i \{\mathrm{s}'(T(t_{\leqq i}X^{\cdot}, Y^{\cdot}))\} = \mathrm{s}'(T(X^{\cdot}, Y^{\cdot})).$$

Let us also recall

Definition 1.5. *If A and B are abelian categories, T: $D^b(A)\to D(B)$ is an exact functor and d is an interval in $\mathbf{Z}$ then T is said to have amplitude d if, for $X\in A$,*

$H^i(T(X))=0$ if $i \notin d$. *T is said to be of finite amplitude if it has amplitude d for some finite d. Similarly for multifunctors.*

2. In case a topos T has sufficiently many points one usually constructs the canonical flasque resolution of a sheaf by considering the morphism ${}_P\mathscr{S} \xrightarrow{t} T$ where P is some conservative set of points of T and φ the pointual topos. The only properties of ${}_P\mathscr{S}$ and t that are used is that epimorphisms split in ${}_P\mathscr{S}$ and that t is a surjection (i.e. t^* is conservative). In general Barr's theorem (cf. [18: 7.51]) confirms that for any topos T there is a surjection t: $P \to T$ where all epimorphisms in P split. I will refer to such a topos P as an acyclic topos. Again we may define the canonical flasque resolution wrt t of an abelian sheaf M in T which will be denoted $C^*(M)$ even though this time it is somewhat less canonical. As there are two slightly different ways of constructing such a $C^*(M)$ we will, to be definite, choose the one which is the cobarresolution associated to the pair of adjoint functiors (t^*, t_*) (cf. [18: 8.20]).

We will in the following, unless otherwise mentioned, consider only graded rings, modules and morphisms. Note that this means that functors generally are applied degree wise e.g. if M is a graded group in a topos S and f: $S \to T$ a morphism of topoi then f_*M is the graded group in T whose i:*th* component is f_*M^i. If M is again a graded group sheaf and X a graded sheaf (i.e. a disjoint union over Z of sheaves) then $M[X]$ will denote the internal copower of M over X with $M[X]^i = \sum_{i=j+k} M^j[X^k]$. If M is a graded module over the graded ring A then we have a canonical epimorphism $A[M'] \to M$ of A-modules where $M' := \cup_i M^i$ as a graded sheaf. As in [14: I, Lemma 4.6 1] we can construct the canonical free resolution of any $X \in C^-(A)$.

A functor T from A-modules to B-modules where A and B are rings in a topos S will be said to be internally additive if it commutes with internal sums over arbitrary sheaves and internally right exact if it commutes with all small internal (or local in the terminology of [1: V, 8]) direct limits. Clearly every internally additive right exact functor is internally right exact.

3. Let F: $(S, \mathcal{O}_S) \to (T, \mathcal{O}_T)$ be a morphism of ringed topoi. I will prove the following version of "trivial duality" for lack of reference (cf. [14: II, Prop. 5.10]).

Lemma 3.1. *Let* $M \in D^-(\mathcal{O}_T)$ *and* $N \in D^+(\mathcal{O}_S)$. *Then*

(3.1.1) $$Rf_* R\,\underline{\mathrm{Hom}}_{\mathcal{O}_S}(Lf^*M, N) = R\,\underline{\mathrm{Hom}}_{\mathcal{O}_T}(M, Rf_*N).$$

Furthermore, if f_* is of finite cohomological dimension and f of finite Tor-dimension M can be arbitrary.

Proof. Let us agree to let f^* denote the inverse image of ringed topoi and f^{-1} the usual inverse image. Let Y be a group in T, M an $\mathcal{O}_T$-module and N an $\mathcal{O}_S$-module. Then

$$\begin{aligned}&\mathrm{Hom}_{T-ab}(Y, f_* \underline{\mathrm{Hom}}_{\mathcal{O}_S}(f^*M, N)) = \mathrm{Hom}_{S-ab}(f^{-1}Y, \underline{\mathrm{Hom}}_{\mathcal{O}_S}(f^*M, N))\\ &= \mathrm{Hom}_{\mathcal{O}_S}(f^{-1}Y \otimes_{\mathbf{Z}} f^*M, N) = \mathrm{Hom}_{\mathcal{O}_S}(f^*(Y \otimes_{\mathbf{Z}} M), N) = \mathrm{Hom}_{\mathcal{O}_T}(Y \otimes_{\mathbf{Z}} M, f_*N)\\ &= \mathrm{Hom}_{T-ab}(Y, \underline{\mathrm{Hom}}_{\mathcal{O}_T}(M, f_*N)).\end{aligned}$$

As this is natural in Y we get the module version of (1.1.1) and as f_* commutes with products we get the underived version of (1.1.1). To get (1.1.1) we may assume that M has f^*-acyclic components and N injective. Then $\mathrm{Hom}_{\mathcal{O}_T}(f^*M, N)$ is flasque as in the proof of (I: 5,2) and we are through.

4. Let again f: $S \to T$ be a morphism of topoi and let C be a small category. We then have morphisms $S^C \xrightarrow{l} S(T^C \xrightarrow{l} T)$ where the direct images are the $\varprojlim_C$ and the inverse images are the constant objects functor. Clearly the following diagram commutes.

$$\begin{tikzcd} S^C \arrow[r, "f^C"] \arrow[d, "l"] & T^C \arrow[d, "l"] \\ S \arrow[r, "f"] & T \end{tikzcd} \tag{4.1}$$

which gives

$$R\varprojlim_C \circ Rf^C_* = R(\varprojlim_C \circ f^C_*) = Rf_* \circ R\varprojlim_C. \tag{4.2}$$

Furthermore, if $M \in T^C - ab$ then $\varprojlim^i_C(M)$ is the sheaf associated to $(U \mapsto H^i(l^*U, M))$. If p: $T^C/l^*U \to T^C$ is the projection then $\Gamma(l^*U, M) = \Gamma(T^C/l^*U, M)$ and as the projection $T^C/l^*U \to \varphi$ factors as $T^C/l^*U = (T/U)^C \to S^C \to S$ we get

$$R\Gamma(l^*U, M) = R\varprojlim_C(R\Gamma^C(l^*U, M)). \tag{4.3}$$

On the other hand, if $c \in C$, then $T^C \to T(X \mapsto X(c))$ is the direct image of a morphism of topoi and we also get a diagram corresponding to (4.1). As this functor is exact we get for $N \in D^+(S^C - ab)$

$$R^i f^C_* N(c) = R^i f_* N(c). \tag{4.4}$$

Applying this to $T/U \to S$ and using (4.3) we get a spectral sequence

$$\varprojlim^i_C \{c \mapsto H^j(U, M(c))\} \Rightarrow H^{i+j}(l^*U, M). \tag{4.5}$$

In case $C = \mathbf{N}$ this degenerates to a short exact sequence

$$0 \to \varprojlim^1 \{H^{i-1}(U, M)\} \to H^i(l^*U, M) \to \varprojlim \{H^i(U, M)\} \to 0. \tag{4.6}$$

Definition 4.7. *A topos S is said to be of finite cohomological dimension if there is a set S^{gen} of generators and an integer N such that $\Gamma(U, -)$ has cohomological dimension $\leqq N$ for all $U \in S^{\mathrm{gen}}$.*

Remark: i) Strictly speaking such a topos should rather be said to be uniformly locally of finite cohomological dimension which explains why I have chosen the less appropriate expression. Note that BG for a discrete group G is of finite cohomological dimension in this sense even if $\Gamma(BG, -)$ may well have infinite cohomological dimension.

ii) Typical examples of topoi of finite cohomological dimension are the Zariski-topos of a scheme whose underlying space is locally Noetherian of finite Krull dimension (cf. [12: Thm. 3.6.5]), most étale topoi of interest etc. As any topos locally isomorphic to one of finite cohomological dimension also is, one may for instance take the classifying topos of some group in one of the above-mentioned topoi.

iii) In all of the cases in the present paper there will be a fixed prime p such that it will suffice that $\Gamma(U, -)$ be of finite cohomological dimension when restricted to $\mathbf{Z}_{(p)}$-sheaves.

Lemma 4.8. *Let S be a topos of finite cohomological dimension and C a small category such that $\Gamma(\varphi^C, -)$ has finite cohomological dimension then $\varprojlim_C$: $S^C-ab\to S-ab$ has finite cohomological dimension.*

Indeed, using the notations of (4.7) and supposing that $H^i(S^C, -)=0$ for $i>M$, then (4.5) shows that $H^i(l^*U, -)=0$ if $i>M+N$ and $U\in S^{\mathrm{gen}}$. When taking the sheaf associated to $(U\mapsto H^i(l^*U, M))$ we need only let U run through S^{gen} and thus $\varprojlim^i_C(-)$ is zero for $i>M+N$.

Let us also note that (3.1.1) applied to the morphism l: $S^C\to S$, a ring $\mathcal{O}_S$ in S, a C-system of rings $\mathcal{O}$ and a morphism $l^*\mathcal{O}_S\to\mathcal{O}$ gives us for $M\in D^-(\mathcal{O}_S)$ and $N\in D^+(\mathcal{O})$

(4.9) $$R\varprojlim_C R\,\underline{\mathrm{Hom}}_{\mathcal{O}}(Ll^*M, N) = R\,\underline{\mathrm{Hom}}_{\mathcal{O}_S}(M, R\varprojlim_C N).$$

If $\mathcal{O}=l^*\mathcal{O}_S$ and S and $\Gamma(\varphi^C, -)$ has finite cohomological dimension then (4.8) shows that (4.9) remains true for arbitrary M.

5. For the rest of this paper, unless otherwise mentioned, $(S, \mathcal{O}_S)$ will be a ringed topos with S of finite cohomological dimension and with $\mathcal{O}_S$ a perfect ring of characteristic $p>0$. $W\mathcal{O}_S$ will denote the ring of Witt vectors and the $W_n\mathcal{O}_S$ the rings of truncated Witt vectors. R_S will denote the graded $W\mathcal{O}_S$-ring with generators F, V in degree 0 and d in degree 1 and relations (cf. [16]) $FV=VF=p$, $Fa=a^\sigma F$, $aV=Va^\sigma(a\in W\mathcal{O}_S)$, $FdV=d$, $d^2=0$ and $da=ad(a\in W\mathcal{O}_S)$ where σ: $W\mathcal{O}_S\to W\mathcal{O}_S$ is the ring-automorphism induced, by functoriality, from the p: *th* power-mapping on $\mathcal{O}_S$. $R_{nS}:=R_S/dV^nR_S+V^nR_S$, $\hat{R}_S:=\varprojlim R_{nS}$. As in [16: I, 1.1.4—5 and 1.3.3] one sees that

(5.1) $$R_S = \sum_{n>0} W\mathcal{O}_S V^n \oplus \sum_{n\geqq 0} W\mathcal{O}_S F^n \oplus \sum_{n>0} W\mathcal{O}_S\, dV^n \oplus \sum_{n\geqq 0} W\mathcal{O}_S F^n\, d,$$
$$R_{nS} = \sum_{n>m>0} W_{n-m}\mathcal{O}_S V^m \oplus \sum_{m\geqq 0} W_n\mathcal{O}_S F^m \oplus \sum_{n>m>0} W_{n-m}\mathcal{O}_S\, dV^m \oplus \sum_{m\geqq 0} W_n\mathcal{O}_S F^m\, d$$

By analogy with [15] we define right R_S-modules Z_1 and B_1 as the kernel and cokernel respectively of multiplication by d to the left on R_1. I claim that F: $R_2 \to R_1$ induced by multiplication to the left by F on R induces an isomorphism C^{-1}: $R_1 \xrightarrow{\sim} \sigma_* Z_1/\sigma_* B_1$. This and relations to follow can be proved in two ways. Either by direct inspection using (5.1) or by observing, as has done Illusie, that if $E(t)$ are the "formes entières" of [15: I, 2] then $R_S \to E(t)$, $(1 \mapsto t)$ presents R_S as a direct factor, as R_S-modules, of $E(t)$ and then the desired relations follow from the ones already proved for the de Rham—Witt complex. I will have nothing further to say concerning the proofs but just refer to the result for the de Rham—Witt complex. Again by analogy we define Z_n and B_n recursively by the exact sequences

(5.2)
$$0 \to B_1 \to B_{n+1} \xrightarrow{C} B_n \to 0$$
$$0 \to B_1 \to Z_{n+1} \xrightarrow{C} Z_n \to 0,$$

where C is the inverse of C^{-1}. We let π_n: $R_{n+1} \to R_n$ be the projections and ϱ_n: $R_n \to R_{n+1}$ the morphisms induced by multiplication by p. We put

(5.3)
$$0 \to gr^n \to R_{n+1} \to R_n \to 0$$
$$0 \to R_n \to R_{n+1} \to gr_1^n \to 0$$

and then we have exact sequences (cf. [8], [15] and [16]):

(5.4.1) $$0 \to B_1 \to Z_1 \xrightarrow{C} R_1 \to 0$$

(5.4.2) $$0 \to \sigma_*^n R_1/B_n \to gr^n \to \sigma_*^n R_1/Z_n(-1) \to 0$$

(5.4.3) $$0 \to B_n(1) \to gr_1^n \to Z_n \to 0$$

(5.4.4) $$0 \to Z_1 \to \sigma_* R_1 \to B_1(1) \to 0$$

(5.4.5) $$0 \to \sigma_* R_1/Z_1 \to \sigma_*^{n+1} R_1/Z_{n+1} \to \sigma_*^n R_1/Z_1 \to 0$$

(5.4.6) $$0 \to \sigma_* R_1/Z_1 \to \sigma_*^{n+1} R_1/B_{n+1} \to \sigma_*^n R_1/B_n \to 0$$

(5.4.7) $$0 \to R \xrightarrow{(V,-F)} R \oplus R \xrightarrow{F+V} R \to B_1(1) \to 0$$

(5.4.8) $$0 \to R(-1) \xrightarrow{(F,-d)} R(-1) \oplus R \xrightarrow{dV^2+V} R \to Z_1 \to 0$$

(5.4.9) $$0 \to R(-1) \xrightarrow{(F^n,-F^n d)} R(-1) \oplus R \xrightarrow{dV^n+V^n} R \to R_n \to 0.$$

((5.4.5.-6) are not to be found in the references and *I* leave their verification to the reader. They are clearly "dual" to (5.2).)

If $X^\cdot$ is a complex of R-modules then we will say that $X^\cdot$ is bounded (from above, from below, to the right, to the left, horizontally resp. vertically) if there is an $N \in \mathbf{N}$ such that $H^i(X)^j = 0$ unless $-N \leqq i, j \leqq N$ ($i \leqq N, i \geqq -N, j \leqq N, j \geqq -N$, $-N \leqq j \leqq N$, resp. $-N \leqq i \leqq N$). We will say that $X^{\cdot}$ is bounded in one direction if $X^{\cdot}$ is bounded either to the left, to the right, from above or from below. $D^{*,0}(R_S)$ will denote the derived category of complexes bounded in different directions according to $*$ being $\emptyset$ (no vertical restriction), $+$ (from below), $-$ (from above) and b (vertically) and 0 being $\emptyset$ (no horizontal restriction), l (to the left), r (to the right) and h (horizontally).

Recall ([8: III, Def. 2.1]) that an inverse de Rham—Witt system on S is a system

$$(5.5)\qquad \begin{gathered}(M_n, \pi: M_n \to M_{n-1}, F: M_n \to \sigma_* M_{n-1}, d: M_n \to M_n(1)(1),\\ V: \sigma_* M_{n-1} \to M_n),\end{gathered}$$

where M_n is a $W_n\mathcal{O}$-module, (π, F, d, V) are $W_n\mathcal{O}$-homomorphisms with the relations $\pi F = F\pi$, $d\pi = \pi d$, $\pi V = V\pi$, $FV = p = VF$, $FdV = d$, $d^2 = 0$. (R_n, π, F, d, V) is such a system denoted $R.$ and, by functoriality, for any R-module M $R.\otimes_R M := (R_n \otimes_R M, \pi, F, d, V)$ is an inverse de Rham—Witt system. Its left derived functor

$$(5.6)\qquad R.\otimes_R^L(-): D^-(R) \to D^-(\text{inv—drw—}S)$$

has amplitude $[-2, 0]$ by (5.4.9) and hence extends to $D(R)$.

If $M.$ is an inverse de Rham—Witt system then $\varprojlim \{M_n, \pi\}$ is in a natural way an R-module, the (F, d, V) being induced by the (F, d, V) of $M.$, giving a functor $\varprojlim$: inv-drw$-S \to R_S$-mod. The forgetful functor from inv-drw$-S$ to inverse systems of graded sheaves commutes with inverse limits so the free objects functor exists and inv-drw$-S$ has a set of generators. As inv-drw$-S$ evidently is an Ab-5-category we see, by the criterion of Grothendieck [12: Thm. 1.10.1], that inv-drw$-S$ has sufficiently many injectives and hence that $\varprojlim$ derives to

$$(5.7)\qquad R\varprojlim: D^+(\text{inv-drw}-S) \to D^+(R_S).$$

We have forgetful functors inv-drw$-S \to S^N-ab$ $(M. \mapsto (M_n, \pi))$ and R-mod$\to$ $S-ab$ and the following diagram commutes

$$\begin{array}{ccc} \text{inv-drw}-S & \xrightarrow{\varprojlim} & R\text{-mod} \\ \downarrow & & \downarrow \\ S^N-ab & \xrightarrow{\varprojlim} & S-ab. \end{array}$$

Lemma 5.8. *The following diagram commutes*

$$(5.8.1)\qquad \begin{array}{ccc} D^+(\text{inv-drw}-S) & \xrightarrow{R\varprojlim} & D^+(R) \\ \downarrow & & \downarrow \\ D^+(S^N-ab) & \xrightarrow{R\varprojlim} & D^+(S-ab). \end{array}$$

What needs to be shown is that if $M.$ is an injective object in inv-drw$-S$ then (M_n, π) is $\varprojlim$-acyclic. It is clear that the system $(\overline{M}_n, \pi) = \left(\sum_{i \leqq n} M_i, \pi\right)$ of "discontinous sections" has a structure of inverse de Rham—Witt system such that the canonical embedding $M. \to \overline{M}$ is a morphism in inv-drw$-S$. Hence $M.$ is a factor in $\overline{M}.$ and therefore has surjective transition morphisms π. I claim that M_n and $\operatorname{Ker} \pi$: $M_{n+1} \to M_n$ are flasque for all n. To see this it suffices to embed $M.$ in some object in inv-drw$-S$ with this property as $M.$ is injective. Choose an acyclic topos T and a surjection t: $T \to S$. Ring T with the perfect ring $t^*\mathcal{O}_S$ thus making t a flat morphism of ringed topoi. We get functors t_*: inv-drw$-T \to$ inv-drw$-S$ and t^*: inv-drw$-S \to$ inv-drw$-T$. By acyclicity of T and left exactness of t_* every $N \in$ inv-drw$-S$ in the image of t_* has N_n and $\operatorname{Ker} \pi_n$ flasque for all n. Now $M.$ embeds in $t_* t^* M.$. It is now clear that for all $U \in S H^i(U, M.) = 0$ for $i > 0$ and as π_n is surjective and $\operatorname{Ker} \pi_n$ flasque, $\Gamma(U, M.)$ has surjective transition maps. (4.6) now shows that $H^i(l^*U, M.) = 0$ for $i > 0$ which implies that $\varprojlim{}^i M. = 0$ for $i > 0$.

Corollary 5.8.2. $\varprojlim$: inv-drw$-S \to R_S$-mod *has finite cohomological dimension and hence* $R\varprojlim$ *extends to all of* D(inv-drw$-S$).

This follows from Lemma 4.8.

If $M \in D(R_S)$ we define the completion $\hat{M}$ of M, by

$$\hat{M} := R\varprojlim (R. \otimes^L_R M). \tag{5.9}$$

There is a canonical morphism

$$M \to \hat{M} \tag{5.10}$$

defined as follows: We may assume that M has $R. \otimes_R (-)$-acyclic components and therefore that $R. \otimes^L_R M = R. \otimes_R M$. We have the projection morphism $M \to (R. \otimes M, \pi)$ and the induced morphism $M \to \varprojlim (R. \otimes_R M)$ is evidently an R-morphism. Now define (5.10) as the composite $M \to \varprojlim (R. \otimes_R M) \to R\varprojlim (R. \otimes_R M)$. We will say that M is complete if (5.10) is an isomorphism.

It is clear that without the assumption of finite cohomological dimension on S all these results and definitions make sense and are true on $D^+(R_S)$.

Lemma 5.11. *Let* f: $(T, \mathcal{O}_T) \to (V, \mathcal{O}_V)$ *be a morphism of ringed topoi where* $\mathcal{O}_T$ *and* $\mathcal{O}_V$ *are perfect rings of characteristic p. Then* Rf_*: $D^+(R_T) \to D^+(R_S)$ *commutes with completions and hence takes complete complexes to complete complexes.*

Indeed, (4.2) and (5.8) show that it suffices to show that $Rf_*(R. \otimes^L_R (-)) = R. \otimes^L_R Rf_*(-)$. As the functors $M. \mapsto M_n$ form a conservative set of exact functors it suffices to show that $Rf_*(R_n \otimes^L_R (-)) = R. \otimes^L_R Rf_*(-)$. This is just the projection formula which in this case follows from (5.4.9).

Lemma 5.12. *Let* $(\hat{-})$ *be an endofunctor of an S-enriched category C and* c: $id \to (\hat{-})$ *a natural transformation such that if* $X \in C$ *then* $c = \hat{c}$: $\hat{X} \to \hat{\hat{X}}$. *If* $X, Y \in C$

then the following diagram commutes:

$$\begin{tikzcd} \underline{\mathrm{Hom}}_C(\hat{X}, Y) \arrow[r, "c^*"] \arrow[d, "c_*"] & \underline{\mathrm{Hom}}_C(X, Y) \arrow[d, "c_*"] \arrow[dl, "(\hat{-})"'] \\ \underline{\mathrm{Hom}}_C(\hat{X}, \hat{Y}) \arrow[r, "c^*"] & \underline{\mathrm{Hom}}_C(\hat{X}, \hat{Y}). \end{tikzcd} \tag{5.12.1}$$

Indeed, the right hand triangle is just naturality of c. Let $\varphi\colon \hat{X}\to Y$. Then $c_*(\varphi)=c\varphi$ which equals $\hat{\varphi}c$ by naturality of c. On the other hand, by functoriality of $(\hat{-})$, $\widehat{(c^*(\varphi))}=\hat{\varphi}\hat{c}$.

Definition 5.13. *A complex M of R_S-modules is said to be coherent (perfect) if $R_1\otimes_R^L M$ is coherent (perfect) and M is complete. The category of coherent (perfect) complexes will be denoted $D_c(R)$ $(D_{\mathrm{perf}}(R))$. Here a complex of $\mathcal{O}_S$-modules is said to be perfect if there is a finite interval d in $\mathbf{Z}$ such that the complex is locally isomorphic to a complex concentrated in d whose components are free $\mathcal{O}_S$-modules on finite sets.*

Remark: The notion of coherentness and perfectness for R-complexes has little to do with the usual notions. They should rather be thought of as giving the right notions on the "ringed" protopos $(S, R._S)$. Indeed, it follows from (I: 1.1) that if $\mathcal{O}_S$ is a coherent ring and $M\in D^-(R_S)$ is coherent then $R_n\otimes_R^L M$ is coherent. Note however that even if M is perfect $R_n\otimes_R^L M$ will rarely be. Note also that our notion of a perfect $\mathcal{O}_S$-complex differs slightly from the usual notion in requiring uniform boundedness. Of course, if the final object of S is quasi-compact there is no difference.

Finally, if $f\colon (S, \mathcal{O}_S)\to(T, \mathcal{O}_T)$ is a morphism of perfectly ringed topoi of characteristic p let us agree to write $f_n\colon (S, W_n\mathcal{O}_S)\to(T, W_n\mathcal{O}_T)$ resp. $f.\colon (S^N, W.\mathcal{O}_S)\to(T^N, W.\mathcal{O}_T)$ for the induced morphisms of ringed topoi.

6. Let $W_n\mathcal{O}_S[d]$ be the ring of dual numbers where d has degree 1. As usual, if M, N are $W_n\mathcal{O}_S[d]$-modules we may endow $\underline{\mathrm{Hom}}_{W_n\mathcal{O}_S}(M, N)$ and $M\otimes_{W_n\mathcal{O}_S} N$ with canonical $W_n\mathcal{O}_S[d]$-module structures and we let $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_S}(-, -)$ and $(-)\otimes_{W_n\mathcal{O}_S}^L(-)$ denote the corresponding derived functors. As $W_n\mathcal{O}_S[d]$ is flat over $W_n\mathcal{O}_S$ the underlying $W_n\mathcal{O}_S$-complexes of $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_S}(M, N)$ and $M\otimes_{W_n\mathcal{O}_S}^L N$ for $M\in D(W_n\mathcal{O}_S[d])$ $N\in D^+(W_n\mathcal{O}_S[d])$ (resp. $N\in(W_n\mathcal{O}_S[d])$) are simply $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_S}(M, N)$ and $M\otimes_{W_n\mathcal{O}_S}^L N$ now computed in $D(W_n\mathcal{O}_S)^{\mathrm{op}}\times D(W_n\mathcal{O}_S)$ (resp. $D(W_n\mathcal{O}_S)\times D(W_n\mathcal{O}_S)$). A complex of $W_n\mathcal{O}_S[d]$-modules will be said to be perfect if its underlying $W_n\mathcal{O}_S$-complex is perfect. The category of perfect complexes will be denoted $D_{\mathrm{perf}}(W_n\mathcal{O}_S[d])$.

Left multiplication by d on R_n gives $R_n\otimes_R M$ a structure of $W_n\mathcal{O}_S[d]$-module for any R_S-module M and hence gives a functor

$$R_n\otimes_R^L(-)\colon D(R_S)\to D(W_n\mathcal{O}_S[d]).$$

7. Recall [15] that for $f\colon X\to S$ a scheme smooth over a perfect scheme S there is defined a sheaf of graded commutative differential algebras $W\Omega^{\cdot}_{X/S}$ which is an R_S-module and such that the following relations hold:

(7.1) $$F(\omega_1\cdot\omega_2)=F\omega_1\cdot F\omega_2,\quad \omega_1\cdot V\omega_2=V(F\omega_1\cdot\omega_2)\quad \omega_1,\omega_2\in W\Omega^{\cdot}_{X/S}.$$

Furthermore, (cf. [16: II, Thm. 1.2])

(7.2) $$R_n\otimes^L_R W\Omega^{\cdot}_X=R_n\otimes_R W\Omega^{\cdot}_X=: W_n\Omega^{\cdot}_X$$
$$W\Omega^{\cdot}_X=\varprojlim\{W_n\Omega^{\cdot}_X\}.$$

We also know that $W_n\Omega^i_X$ is coherent as $W_n\mathcal{O}_X$-modules for all i and n and $W_1\Omega^{\cdot}_X=\Omega^{\cdot}_{X/S}$. From (4.6), (7.2) and [6: Thm. 1.3.1] it then follows, as $(|X|, W_n\mathcal{O}_X)$ is a scheme, that $W\Omega^{\cdot}_X=R\lim\{W_n\Omega^{\cdot}_X\}=R\varprojlim(R.\otimes^L_R W\Omega_X)=\widehat{W\Omega^{\cdot}_X}$ i.e. that $W\Omega^{\cdot}_X$ is complete as R-module in the ringed topos $(|X|, f^*\mathcal{O}_S)$. Thus by (5.11) $Rf_*W\Omega^{\cdot}_X$ is complete.

8. Recall ([16]) that if $S=\operatorname{Spec} k$ where k is a perfect field of characteristic $p>0$ an R_S-module is said to be coherent if it is a sucessive extension of R-modules of the following types:

I. *An R-module which is finitely generated as W-module.*

II. *A degree shift of the R-modules*

$$\mathbf{U}_i\colon \prod_{j\geqq 0} kV^j \xrightarrow{d} \prod_{j\geqq 0} kdV^{i+j}\quad\text{where}\quad dV^{-k}:=F^k d\quad\text{for}\quad k\geqq 0.$$

Recall also (loc. cit.) that an R-module M is said to be a domino if it is a successive extension of $\mathbf{U}_i$: s and that in that case one puts $\dim_k M:=\dim_k M^0/VM^0$ which is always finite.

In [8] a canonical filtration $0\subseteqq T^2M\subseteqq T^1M\subseteqq T^0M=M$ of a coherent R-module by coherent submodules such that T^2M is a successive extension of shifts of $\mathbf{U}_i$: s and the R-module k $(F=d=V=0)$, T^1M/T^2M is a successive extension of R-modules of finite length as W-modules, with $d=0$ and F bijective and F-crystals of positive slope i.e. a finitely generated free W-module with F and V topologically nilpotent in the p-adic topology and $d=0$ and M/T^1M is a slope zero (or unit root) F-crystal i.e. a finitely generated free W-module with F bijective and $d=0$.

Following [16], we put, for any R-module M:

(8.1) $$F^\infty BM:=\bigcup_n \operatorname{Im} F^n d$$
$$V^{-\infty}ZM:=\bigcup_n \operatorname{Ker} dV^n$$
$$\text{Coeur } M^i:=V^{-\infty}ZM^i/F^\infty BM^i$$
$$\operatorname{dom} M^i:=M^i/V^{-\infty}ZM^i\xrightarrow{d} F^\infty BM^{i+1}$$

which all are R-modules. We will call and denote (cf. [8]) $A_f^2M^i:=\text{Coeur } T^2M^i$ the nilpotent torsion (in degree i) of M, $A_s^1M^i:=p\text{-tors}\,(T^1M/T^1M)^i$ the semi-simple torsion (in degree i) of M, $A_n^1M^i:=(T^1M/T^2M)^i/p\text{-tors}$ the positive slope part (in degree i) of M and $A^0(M):=M/T^1M$ the slope zero part of M (in degree i).

Let us also agree to call a coherent R-module elementary if it is one of the following types of modules: Slope zero, semi-simple torsion killed by p, positive slope, k or U_0. It is then clear that any coherent R-module is a successive extension of shifts of elementary modules. (Note that we have exact sequences $0\to\mathbf{U}_i\to\mathbf{U}_{i-1}\to k\to 0$ and $0\to k(-1)\to\mathbf{U}_i\to\mathbf{U}_{i+1}\to 0$.) Also put $E_{j/i+j}:=R^0/F^i-V^j$, $i>0$.

9. In [8] a contravariant functor $D(-)$ from complexes with coherent cohomology to complexes with coherent cohomology is defined by putting $D^i(M):=H^i(D(M))$ for such a complex and $A^2(N):=T^2N$ for a coherent R-module N.

Theorem 9.1. i) *There is a natural exact sequence of R-modules*

$$0\to A^0(D^i(M))\to D^0(H^{-i}(M))\to D^2(H^{1-i}(M))\to A^2(D^{i+1}(M))\to 0$$

ii)
$$A_n^1(D^i(M))^j=D^1(A_n^1(H^{1-i}(M)^{-j-1}))(-1)$$
$$A_s^1(D^i(M))^j=D^1(A_s^1(H^{1-i}(M)^{-j}))$$

iii)
$$A_f^2(D^i(M))=D^2(A_f^2(H^{2-i}(M)^{-j-1}))(-1).$$

If M is a coherent R-module then

$$D^0(M)=\operatorname{Hom}_W(A^0(M),W),\quad (F,d,V)=(F^{-1*}\sigma_*,0,pF^{-1}),$$
$$D^1(A_n^1(M))=\operatorname{Hom}_W(A_n^1(M),W)(1),\quad (F,d,V)=(V^*\sigma_*,0,F^*\sigma_*^{-1}),$$
$$D^1(A_s^1(M))=\operatorname{Hom}_W(A_s^1(M),K/W),\quad (F,d,V)=(F^{-1*}\sigma_*,0,pF^{-1}),$$
$$D^2(A_f^2(M))=\operatorname{Hom}_W(A_f^2(M),K/W)(1),\quad (F,d,V)=(V^*\sigma_*,0,F^*\sigma_*^{-1}),$$
$$D^2(\mathbf{U}_i)=\mathbf{U}_{-i}(2),\quad D^1(\mathbf{U}_i)=0,$$
$$D^i(M)=0,\quad i\neq 0,1,2.$$

10. If G is a finite type formal group over a perfect field k then G_{red} is a subformal group scheme. Put

$$G_s:=G_{\text{red}}\quad G_f:=G/G_{\text{red}}. \tag{10.1}$$

Note that G_s is a smooth formal group scheme and G_f a finite group scheme.

I. The tensor formalism of perfect complexes

Proposition 1.1. *Let A be a thick subcategory of $W\mathcal{O}_S$-mod (ungraded) stable under σ_* and let $M\in D(R_S)$.*

i) *If M is bounded from the left or from above and $H^i(R_1\otimes_R^L M)^j\in A$ for $(i,j)\in\{(i,j)\colon (i+j=r;\ i\geqq s)\vee(i+j=r-1;\ i\geqq s+1)\vee(i+j=r+1;\ i\geqq s+1)\}$ for some $r,s\in\mathbf{Z}$ then, for all n, $H^i(R_n\otimes_R^L M)^j\in A$ for $i+j=r$; $i\geqq s$.*

ii) *If M is bounded from the right or from below and $H^i(R_n\otimes_R^L M)\in A$ for $(i,j)\in\{(i,j)\colon(i+j=r;\ i\leqq s)\vee(i+j=r+1;\ i\leqq s)\vee(i+j=r-1;\ i\leqq s-1)\}$ for some $r,s\in\mathbf{Z}$ then, for all n, $H^i(R_n\otimes_R^L M)^j\in A$ for $i+j=r$; $i\leqq s$.*

iii) *Under the conditions of* i) *the conclusion remains valid for R_n replaced by gr^n.*

Let us first agree to write $B\leqq C$ if B and C belong to some Abelian category and B is a subquotient of C. This is clearly a transitive relation and $B\leqq 0$ implies that $B=0$. Let us also, for simplicity of notation, put $R_1^{i,j}, Z_1^{i,j}, B_1^{i,j}$ etc$:=H^i(R_1\otimes_R^L M)^j$, $H^i(Z_1\otimes_R^L M)^j$, $H^i(B_1\otimes_R^L M)^j$ etc.

Lemma 1.1.1. *Let $M\in D(R)$. If $R_1^{i,j}=0$ then $B_1^{i+1,j}\leqq Z_1^{i,j+1}$, $B_1^{i,j+1}\leqq Z_1^{i,j+1}$, $Z_1^{i,j}\leqq B_1^{i,j}$, $Z_1^{i,j}\leqq B_1^{i+1,j-1}$.*

This follows immediately from the long exact sequences associated to (0: 5.4.1) and (0: 5.4.4).

Lemma 1.1.2. *Let $M\in D(R)$.*

i) *If M is bounded to the left or from above and $R_1^{i,j}=0$ for $i+j=r$ or $r-1$, $i\geqq s$ then $Z_1^{i,j}=B_1^{i,j}=0$; $i\geqq s$.*

ii) *If M is bounded to the right or from below and $R_1^{i,j}=0$ for $i+j=r$ or $r-1$; $j\geqq r-s$ then $Z_1^{i,j}=B_1^{i,j}=0$ for $i+j=r$; $i\leqq s$.*

Indeed, (0: 5.4.8) and (0: 5.4.7) show that $Z_1^{i,r-i}=B_1^{i,r-i}=0$ for $i\gg 0$ (resp. $i\ll 0$). On the other hand, by lemma 1.1.1, if $i\geqq s$ then $Z_1^{i,r-i}\leqq B_1^{i,r-i}\leqq Z_1^{i+1,r-(i+1)}\leqq B_1^{i+1,r-(i+1)}\leqq Z_1^{i+2,r-(i+2)}\leqq\ldots\leqq 0$ (resp. if $i\leqq s$ then $B_1^{i,r-i}\leqq Z_1^{i,r-i}\leqq B_1^{i-1,r-(i-1)}\leqq Z_1^{i-1,r-(i-1)}\leqq\ldots\leqq 0$) and hence, by the remark made above,

$$Z_1^{i,r-i}=B_1^{i,r-i}=0 \quad\text{if}\quad i\geqq s \quad (\text{resp.}\quad i\leqq s).$$

We can now prove the proposition assuming first that $A=0$. (0: 5.2) shows that if $B_1^{i,j}=0$ then, for all n, $B_n^{i,j}=0$ and (0: 5.2) again shows that if $B_1^{i,j}=Z_1^{i,j}=0$ then $Z_n^{i,j}=0$ for all n. By (0: 5.4.2) $Z_n^{i+1,j-1}=B_n^{i+1,j}=R_1^{i,j}=R_1^{i,j-1}=0$ implies that $(gr^n)^{i,j}=0$ and (0: 5.4.3) shows that if $Z_n^{i,j}=B_n^{i,j+1}=0$ then $(gr_1^n)^{i,j}=0$. Lemmas 1.1.1-2, (0: 5.3) and induction on n enable us to conclude the proof. In case $A\neq 0$ we just redo the argument calculating modulo A.

Remark: It should be noted that even when S is the spectrum of a perfect field there is a non-zero $M \in D_c(R_S)$ s.t. $R_1 \otimes_R^L M = 0$.

Corollary 1.1.3. *Let $M, N \in D(R)$ and suppose that they both are bounded in the same direction and that there is a morphism $f\colon M \to N$ such that $R_1 \otimes_R^L f$ is an isomorphism in $D(\mathcal{O}_S)$. Then f is an isomorphism upon completing.*

This follows immediately from the proposition applied to a mapping cone of f and all pairs (r, s) of integers.

Corollary 1.1.4. i) *Let N be an R_1-acyclic R-module. Then N is acyclic for all the R-modules appearing in* (0: 5.4.1-9) *and the exact sequences* (0: 5.2-3, 5.4.1-9) *remain exact when tensored with N.*

ii) *Let N be as in* i) *and $T \in S$. If N is $\Gamma(T, -)$-acyclic then so is N tensored with any of the R-modules occuring in* (0: 5.2-3, 5.4.1-9).

Proof: Z_1- and B_1-acyclicity follow from lemma 1.1.2. Acyclicity for the rest of the modules follows from (0: 5.2, 5.4.2-3) and then the sequences obviously remain exact when tensored with N. As for ii) we note that all the R-modules involved have finite resolutions by finite free modules. As N is acyclic for these modules the resolutions remain exact when tensored with N and then we get an exact sequence

$$0 \to N(i_1)^{r_1} \to N(i_2)^{r_2} \to \dots N(i_n)^{r_n} \to X \otimes_R N \to 0,$$

where X is one of the R-modules in question. This shows that $X \otimes_R N$ is $\Gamma(T, -)$-acyclic.

Proposition 2.1. *Suppose $M \in D(R)$. Then the morphism $M \to \hat{M}$ induces an isomorphism*

$$R_\cdot \otimes_R^L M \xrightarrow{\sim} R_\cdot \otimes_R^L \hat{M}. \tag{2.1.1}$$

In particular $\hat{M} = \hat{\hat{M}}$.

Fix X an acyclic topos and a surjection $\varphi\colon X \to S$ and ring X by $\varphi^*\mathcal{O}_S$. The projection formula $R_n \otimes_R^L \varphi_* \varphi^* N = \varphi_* \varphi^* (R_n \otimes_R^L N)$ shows that if N is an R_1-acyclic module then so are the components of $\tau_{\leqq m} C^*(M)$ (cf. [1: XVII, 4.2.9]). As $R_\cdot \otimes_R (-)$ has finite amplitude we may first replace M by a complex M' with $R_\cdot \otimes_R (-)$-acyclic components. If S has finite cohomological dimension $\leqq A$ and S^{gen} is the set of generators required in (0: 4.7) then $s(\tau_{\leqq A} C^*(M))$ will be a complex whose components are $R_\cdot \otimes_R (-)-$ and S^{gen}-acyclic. As $(\hat{-})$ has finite amplitude we may hence assume that M is an $R_\cdot \otimes_R (-)-$ and S^{gen}-acyclic R-module. I claim that

$$0 \to M_\cdot(-1) \xrightarrow{(F^n, -F^n d)} M_\cdot(-1) \oplus M_\cdot \xrightarrow{dV^n + V^n} M_\cdot \to M_n \to 0 \tag{2.1.2}$$

is an exact sequence of pro-objects, where $M.:=R.\otimes_R M$ and $M_n:=R_n\otimes_R M$. (This is due to Illusie.) Indeed, suppose it true for M a free R-module (not necessarily S^{gen}-acyclic). We may then take a free resolution $f\colon F^\cdot\to M$. Put $Y:=C(f)$. Then $Y.:=R.\otimes_R Y$ and $Y_n:=R_n\otimes_R Y$ are acyclic as M is R_n-acyclic. Hence the simple complex associated to

(2.1.3) $$0\to Y.(-1)\to Y.(-1)\oplus Y.\to Y.\to Y_n\to 0$$

is acyclic and the second spectral sequence of (2.1.3) plus exactness of (2.1.2) for free modules show exactness of (2.1.2) for M. As for exactness of (2.1.2) for a free module we may, as internal sums are exact, assume that M is free on one generator in which case it is a simple calculation (cf. [16: III, 3.3.3]). Let us return to the case of M being $R.\otimes_R(-)-$ and S^{gen}-acyclic. If we can show that $M.$ is $\varprojlim$-acyclic the $\varprojlim$ applied to (2.1.2) is still exact which by (0: 5.4.9) is exactly what we want to show. By (0: 4.6) it suffices to show that M_n is S^{gen}-acyclic for all n and that $\Gamma(T,M_{n+1})\to\Gamma(T,M_n)$ is surjective for all n and $T\in S^{\text{gen}}$. M_n is S^{gen}-acyclic by (1.1.4) and, again by (1.1.4), we have an exact sequence $0\to gr^n\otimes_R M\to M_{n+1}\to M_n\to 0$ with $gr^n\otimes_R M$ S^{gen}-acyclic, which shows that $\Gamma(T,M_{n+1})\to\Gamma(T,M_n)$ is surjective for $T\in S^{\text{gen}}$.

Definition 3.1. *Let $M,N\in R$-mod. $M*_R N$ is defined to be the R-module generated by $m*n$ ($m\in M$, $n\in N$) using the relations;*

i) $$Vm*n=V(m*Fn),\quad m*Vn=V(Fm*n)$$

ii) $$F(m*n)=Fm*Fn$$

iii) $$d(m*n)=dm*n+(-1)^{\deg m}m*dn$$

iv) $$(m_1+m_2)*n=m_1*n+m_2*n,\quad \lambda m*n=\lambda(m*n),\quad \lambda\in WO_S$$
$$m*(n_1+n_2)=m*n_1+m*n_2,\quad m*\lambda n=\lambda(m*n),\quad \lambda\in WO_S.$$

The existence of $M*_R N$ is obvious and it is clear that $(-)*_R(-)$ is an internally biadditive and right exact functor.

Proposition 3.2. *$R*_R R$ is a free R-module on generators*

(3.2.1) $$F^i*1,\ i>0;\quad 1*F^i,\ i\geq 0,$$
$$F^i d*1,\ i>0;\quad 1*F^i d,\ i\geq 0.$$

From (0: 5.1) and (3.1 iv) it follows that $R*_R R$ is generated, as an R-module, by

$$\begin{array}{llll} V^i*V^j, & F^i*V^j, & V^i*F^j, & F^i*F^j,\\ dV^i*V^j, & F^i d*V^j, & dV^i*F^j, & F^i d*F^j,\\ V^i*dV^j, & F^i*dV^j, & V^i*F^j d, & F^i*F^j d,\\ dV^i*dV^j, & F^i d*dV^j, & dV^i*F^j d, & F^i d*F^j d. \end{array}$$

By i), ii) and iii) we get

$$V^i * V^j = V^i p^j(1 * F^{i-j}) \quad \text{if} \quad i \geqq j; \quad V^j p^i(F^{j-i} * 1) \quad \text{if} \quad i < j$$

$$F^i * V^j = V^j(F^{i+j} * 1)$$

$$V^i * F^j = V^i(1 * F^{i+j})$$

$$F^i * F^j = F^i(1 * F^{j-i}) \quad \text{if} \quad j \geqq i; \quad F^j(F^{i-j} * 1) \quad \text{if} \quad j < i$$

$$dV^i * V^j = V^j(F^{j-i} d * 1) \quad \text{if} \quad j \geqq i; \quad p^j dV^i(1 * F^{i-j}) - V^i(1 * F^{i-j} d) \quad \text{if} \quad j < i$$

$$F^i d * V^j = V^j(F^{i+j} d * 1)$$

$$dV^i * F^j = dV^i(1 * F^{i+j}) - p^j V^i(1 * F^{i+j} d),$$

(3.2.2)

$$F^i d * F^j = F^j(F^{i-j} d * 1) \quad \text{if} \quad i \geqq j; \quad F^i d(1 * F^{j-i}) - F^i p^{j-i}(1 * F^{j-i} d) \quad \text{if} \quad i < j$$

$$V^i * dV^j = V^i(1 * F^{i-j} d) \quad \text{if} \quad i \geqq j; \quad p^i dV^j(F^{j-i} * 1) - V^j(F^{j-i} d * 1) \quad \text{if} \quad i < j$$

$$F^i * dV^j = dV^j(F^{i+j} * 1) - p^i V^j(F^{i+j} d * 1)$$

$$V^i * F^j d = V^i(1 * F^{i+j} d)$$

$$F^i * F^j d = F^i(1 * F^{j-i} d) \quad \text{if} \quad j \geqq i; \quad F^j d(F^{i-j} * 1) - F^j p^{i-j}(F^{i-j} d * 1) \quad \text{if} \quad j <$$

$$dV^i * dV^j = dV^i(1 * F^{i-j} d) \quad \text{if} \quad i \geqq j; \quad -dV^j(F^{j-i} d * 1) \quad \text{if} \quad i < j$$

$$F^i d * dV^j = -dV^j(F^{i+j} d * 1)$$

$$dV^i * F^j d = dV^i(1 * F^{i+j} d)$$

$$F^i d * F^j d = F^i d(1 * F^{j-i} d) \quad \text{if} \quad j \geqq i; \quad -F^j d(F^{i-j} d * 1) \quad \text{if} \quad j \leqq i.$$

This shows that $R *_R R$ is generated by (3.2.1). To show that they freely generate it one must show that every morphism of sheaves $\bigcup_{i>0} F^i * 1 \cup F^i d * 1 \cup \bigcup_{i \geqq 0} 1 * F^i \cup 1 * F^i d \to M$ where M is an R-module extends to a morphism $R *_R R \to M$ of R-modules. By (3.1 iv) and (3.2.2) one can define this extension on $r * s$ $(r, s \in R)$ and it only remains to show that (3.1 i—iv) are fulfilled. This is a very tedious and completely straightforward verification which I leave to the reader.

Corollary 3.2.3. *If M is an R-module then*

$$R *_R M = \bigoplus_{i>0} V^i(1 * M) \oplus (\bigoplus_{i \geqq 0} F^i * M) \oplus (\bigoplus_{i>0} dV^i(1 * M)) \oplus (\bigoplus_{i \geqq 0} F^i d * M).$$

Indeed, there is an evident additive morphism from the right hand side to the left hand side. Both are internally additive and right exact so to show that this morphism is an isomorphism we may assume that $M = R$ but then it follows easily from the proposition.

Proposition 3.3. *Let* $M, N \in R$*-mod and suppose that* F *is bijective on* N*. Then*

$$M *_R N = M \otimes_{W\mathcal{O}_S} N. \tag{3.3.1}$$

Proof: By (3.1 iv) there is a morphism of W_S-modules

$$M \otimes_{W\mathcal{O}_S} N \to M *_R N \tag{3.3.2}$$
$$m \otimes n \mapsto m * n.$$

On the other hand, we may put an R-module structure on $M \otimes_{W\mathcal{O}} N$ by

$$F(m \otimes n) = Fm \otimes Fn, \quad d(m \otimes n) = dm \otimes n + (-1)^{\deg m} m \otimes dn, \tag{3.3.3}$$
$$V(m \otimes n) = Vm \otimes F^{-1} n.$$

It is easy to see that $m \otimes n \in M \otimes_{W\mathcal{O}} N$ fulfills (3.1 i—iv) so that there exists a morphism of R-modules

$$M *_R N \to M \otimes_W N \tag{3.3.4}$$
$$m \otimes n \mapsto m \otimes n$$

and again it is easy to see that (3.3.2) and (3.3.4) are inverses of each other.

Let us also record that for $M, N \in R$-mod there is a canonical isomorphism;

$$M *_R N \to N *_R M \tag{3.4}$$
$$m * n \mapsto (-1)^{\deg m \cdot \deg n} n * m.$$

4. We will now study the derivation of the functor $(-) *_R (-)$.

Definition 4.1. *An* R*-module* M *is said to be* $*$*-flat if* $M *_R (-)$ *(and hence* $(-) *_R M$*) is an exact functor.*

Lemma 4.2. i) Every free R-module M is $*$-flat and every $W\mathcal{O}_S$-flat R-module having a bijective F is $*$-flat.

ii) Let M and N be R-complexes such that N is bounded from above with $*$-flat components. If either M or N is acyclic then $M *_R N$ is acyclic.

That a free R-module is $*$-flat is clear as $(-) *_R (-)$ is internally additive and (3.2.3) shows that a one generator free module is $*$-flat. The second assertion of i) follows from Prop. 3.3. As for ii) note first that by (0: 1.3) and exactness of inductive limits we may assume that the non-acyclic component is bounded in which case it is obvious.

Definition 4.3. $W\mathcal{O}[F, F^{-1}]$ *is defined to be the* R*-module* $\sum_{i \in \mathbf{Z}} W\mathcal{O}_S F^i$ *concentrated in degree* 0*, with* $d=0$ *and* F *and* V *having the obvious action.*

Lemma 4.4. Suppose that M is a complex of R-modules bounded from above and to the right. Then there exists a quasi-isomorphism $X \to M$ such that X is bounded to the right and from above and has components of the type

$$R[T] \oplus W\mathcal{O}[F, F^{-1}][T'] \quad T, T' \in S.$$

Indeed, consider the category $A_N (N \in \mathbf{Z})$ of R-modules M such that $M^i = 0$ for $i > N$ and F is bijective on M^N. It suffices to show that A_N has sufficiently many objects of the required form as A_N is a thick subcategory of R-mod. By (4.2 i), if $M \in A_N$ then put $F(M) = R[\bigcup_{i<N} M^i] \oplus W\mathcal{O}[F, F^{-1}][M^N]$. As F is bijective on R^1 (i.e. $R \in A_1$) $F(M) \in A_N$ and we have an epimorphism of R-modules $F(M) \to M$.

Proposition 4.5. *$(-) *_R^L (-)$ is defined on $D(R) x D^-(R) \cup D^-(R) x D(R)$. It takes $D^-(R) x D^-(R)$ to $D^-(R)$; $D^r(R) x D^{-,r}(R) \cup D^{-,r}(R) x D^r(R)$ to $D^r(R)$ and $D^1(R) x D^{-,1}(R) \cup D^{-,1}(R) x D^1(R)$ to $D^1(R)$.*

The first statement follows from lemma 4.2. The statement concerning $D^-(R)$ is obvious. The one concerning $D^r(R)$ follows from lemmas 4.2 and 4.4 and the one concerning $D^1(R)$ is clear as taking free resolutions preserves boundedness to the left.

Lemma 4.6. *Let $M, N \in R$-mod. The mapping*

$$\varphi\colon M \otimes_{W\mathcal{O}} N \to M *_R N$$

$$m \otimes n \mapsto m * n$$

induces an isomorphism

(4.6.1) $$(R_1 \otimes_R M) \otimes_{\mathcal{O}} (R_1 \otimes_R N) \xrightarrow{\sim} R_1 \otimes_R (M *_R N).$$

Proof. $\varphi(Vm \otimes n) = V\varphi(m \otimes Fn)$, $\varphi(dVm \otimes n) = \pm dV\varphi(m \otimes Fn) \pm V(\varphi(m \otimes Fdn))$ etc. which shows that φ does indeed induce a morphism (4.6.1). To show that we have an isomorphism we reduce, by internal additivity and right exactness to $M = N = R$ where we use (3.2) and (0: 5.1). to explicitly calculate both sides.

Remark: For $n > 1$ it is in general not true that

$$(R_n \otimes_R M) \otimes_{W_n \mathcal{O}} (R_n \otimes_R M) = R_n \otimes_R (M *_R N).$$

Proposition 4.7. Let $M \in D(R)$, $N \in D^-(R)$. Then

(4.7.1.) $$(R_1 \otimes_R^L M) \otimes_{\mathcal{O}}^L (R_1 \otimes_R^L N) = R_1 \otimes_R^L (M *_R^L N).$$

Suppose first that $M \in D^-(R)$. Then we may assume that M and N are free complexes and hence so is $M *_R N$ by (3.2). By (0: 5.1) $R_1 \otimes_R M$ and $R_1 \otimes_R N$ are free $\mathcal{O}_S$-complexes. Thus $R_1 \otimes_R^L M \otimes_{\mathcal{O}}^L R_1 \otimes_R^L N = R_1 \otimes_R M \otimes_{\mathcal{O}} R_1 \otimes_R N$ and $R_1 \otimes_R^L (M *_R^L N) = R_1 \otimes_R (M *_R N)$. We now conclude by Lemma 4.6. In particular this shows that if M is an R_1-acyclic module and F a free R-module then $M *_R F$

is R_1-acyclic. As R_1 has finite Tor-dimension $M\in D(R)$ can be assumed to have R_1-acyclic components and N to have free components. Thus $M*_R N$ has R_1-acyclic components and $R_1\otimes_R N$ has $\mathcal{O}_S$-free components so the argument above still works.

Definition 4.7.1. *Let* $M, N\in D(R)$ *be such that* $M*_R^L N$ *is defined. Put*

$$M\hat{*}_R^L N := (M*_R^L N)\hat{}\quad (:= R\varprojlim (R_\cdot\otimes_R^L(M*_R^L N))). \tag{4.7.2}$$

Theorem 4.8. *Assume that* $M\in D(R)$ *and* $N\in D^-(R)$ *(or vice versa).*

i) *If* M *and* N *are both bounded either from above, or to the left or to the right, then so is* $M\hat{*}_R^L N$. *Furthermore,*

$$R_1\otimes_R^L(M\hat{*}_R^L N) = (R_1\otimes_R^L M)\otimes^L(R_1\otimes_R^L N).$$

ii) *If* $M\in D^+(R)$, $R_1\otimes_R^L M$ *is bounded to the right,* $N\in D^b_{\mathrm{perf}}(R)$ *and* M *is complete then* $M\hat{*}_R^L N\in D^+(R)$.

iii) *If* $M, N\in D^b_{\mathrm{perf}}(R)$ *then so is* $M\hat{*}_R^L N$.

Indeed, i) follows from (4.5), the fact that completions preserve boundedness in any direction and (4.7) combined with (2.1). To prove ii) we will need the following result which we record for future use:

Lemma 4.9. i) *If* $M\in D^-(R)$ *and* $R_1\otimes_R^L M$ *is bounded to the left then so is* $\hat{M}$.
ii) *If* $M\in D^+(R)$ *and* $R_1\otimes_R^L M$ *is bounded to the right then so is* $\hat{M}$.

This follows immediately from (1.1) and the fact that lim preserves boundedness to the right or to the left. The lemma now implies that M and N are bounded to the right and hence so is $M\hat{*}_R^L N$. As $R_1\otimes_R^L(M\hat{*}_R^L N)=(R_1\otimes_R^L M)\otimes^L(R_1\otimes_R^L N)$, the assumptions imply that $R_1\otimes_R^L(M\hat{*}_R^L N)$ is bounded from below and hence (1.1) together with the fact that $\varprojlim$ preserves boundedness from below and Prop. 2.1 show that $M\hat{*}_R^L N\in D^+(R)$. iii) now follows from ii) and i).

5. The next step is to define the adjoint of $(-)*_R(-)$, the internal Hom-functor.

Definition 5.1. *For* $M, N\in R$-mod *put*

$$\underline{\mathrm{Hom}}^!_R(M, N) := \underline{\mathrm{Hom}}_R(R*M, N) \tag{5.1.1}$$

$\underline{\mathrm{Hom}}_R(R*M, N)$ (and hence $\underline{\mathrm{Hom}}^!_R(M, N)$) will be given the R-module structure induced by the right R-module structure on $R*_R M$ induced by right multiplication on R.

Lemma 5.2. i) *If* X *is a complex of* R-*modules and* Y *is a complex of injective* R-*modules then* $\underline{\mathrm{Hom}}^!_R(X, Y)$ *is acyclic if either of* X *and* Y *is.* $R\underline{\mathrm{Hom}}^!_R(-, -)$ *is hence defined on* $D(R)^{\mathrm{op}}xD^+(R)$.

ii) *If M is either $R[T]$ or $W\mathcal{O}[F, F^{-1}][T]$ and I is a flasque R-module then* $R\underline{\mathrm{Hom}}^!_R(M, I)=\underline{\mathrm{Hom}}^!_R(M, I)$.

Suppose that X is acyclic. If Y is bounded then i) is clear as $R*_R(-)$ is exact by (3.2.3). In general $\underline{\mathrm{Hom}}^{!\cdot}_R(X, Y)=\varprojlim \{\underline{\mathrm{Hom}}^{!\cdot}_R(X, t_{<n}Y)\}$. We thus have the following general situation: An inverse system of double complexes $F_\cdot^{\cdot}$ such that $F_\cdot^{i,j}$ is essentially constant and $s'(F_\cdot^{\cdot})$ is acyclic. Furthermore, as $\underline{\mathrm{Hom}}^!_R(M, N)=\underline{\mathrm{Hom}}_R(R*_RM, N)$ $\underline{\mathrm{Hom}}^!_R(M, N)$ is, by [1, V, Prop. 4.10], flasque when N is injective, and so $F_n^{i,j}$ is flasque. I claim that in this case $\varprojlim \{s'(F_\cdot^{\cdot})\}$ is acyclic. Indeed, as $\varprojlim$ has finite amplitude there are two convergent spectral sequences $\varprojlim^i \{H^j(s'(F_\cdot^{\cdot}))\}\Rightarrow R^*\varprojlim \{s'(F_\cdot^{\cdot})\}$ and $\varprojlim^i \{s'(F_\cdot^{\cdot})^j\}\Rightarrow R^*\varprojlim \{s'(F_\cdot^{\cdot})\}$. The first shows that the second converges to zero and it only remains to show that $s'(F_\cdot^{\cdot})^j$ is $\varprojlim$-acyclic for all j. By the assumptions it is the product of essentially constant systems with flasque components and (0: 4.6) gives $\varprojlim$-acyclicity. The case of Y being acyclic is shown similarly or using that Y is actually contractible.

By definition $R\,\underline{\mathrm{Hom}}^!_R(M, I)=R\,\underline{\mathrm{Hom}}_R(R*_RM, I)$ and, by (3.2), if M is free then so is $R*_RM$ on a sheaf T', say. Hence $R\,\underline{\mathrm{Hom}}^!_R(M, I)=R\prod_{T'}I$. $\prod_{T'}(-)$ is simply $j_{T'*}j^*_{T'}(-)$, $j_{T'}\colon S_{/T'}\to S$, so $R\prod_{T'}I=\prod_{T'}I$ as $j^*_{T'}I$ is flasque by [1: V, Prop. 4.11]. As for $M=W\mathcal{O}[F, F^{-1}][T]$ (3.3) shows that $\underline{\mathrm{Hom}}^!_R(M, N)=\underline{\mathrm{Hom}}_R(R\otimes_{W\mathcal{O}}M, N)=\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, N)$ for any R-module N. As R is flat (free even) as a $W\mathcal{O}$-module $R\,\underline{\mathrm{Hom}}^!_R(M, I)=R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, I)$ and M is free as $W\mathcal{O}$-module. We now redo the argument for $R[T]$.

Remark: i) could also be proved as in [14: II, 3.1].

Proposition 5.3. $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ *takes* $D^+(R)^{\mathrm{op}}xD^-(R)$ *to* $D^+(R)$, $D^1(R)^{\mathrm{op}}\times xD^{+,r}(R)$ *to* $D^r(R)$ *and* $D^r(R)^{\mathrm{op}}xD^1(R)$ *to* $D^1(R)$.

Indeed the first statement is obvious, for the second we need to show that every complex bounded from below and to the right has an injective resolution with the same properties. Let M be an R-module such that $M^i=0$ $i>N$. I claim that if I is its injective envelope then $I^i=0$, $i>N$. In fact, $\sum_{i>N}I^i$ is a submodule of I whose intersection with M is trivial and as $M\hookrightarrow I$ is an essential extension $\sum_{i>N}I^i$ is zero. It is now clear that the desired resolution can be found. For the last statement we reduce, using (0: 1.4) as in (5.2), to the case of the complex in the first variable being bounded from above. We then take a resolution as in Lemma 4.4 in the first variable and a canonical flasque resolution in the second (which preserves boundedness to the left) and then use (5.2 ii).

Proposition 5.4. *If* $M, N, P\in R$*-mod then*

(5.4.1) $$\underline{\mathrm{Hom}}_R(M*_RN, P)=\underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}^!_R(N, P)).$$

Proof: Consider an R-morphism $\varphi\colon M\to\underline{\mathrm{Hom}}^!_R(N,P)$. It gives rise to φ': $M\otimes_W N\to P$; $m\otimes n\mapsto\varphi(m)(1*n)$. One has $\varphi'(Vm\otimes n)=\varphi(Vm)(1*n)=\varphi(m)(V*n)=\varphi(m)V(1*Fn)=V\varphi(m)(1*Fn)=V\varphi'(m\otimes Fn)$ and similarly for the other relations. φ' thus induces an R-morphism $\varphi'\colon M*_R N\to P$. Conversely, given φ': $M*_R N\to P$ an R-morphism we obtain for $m\in M\varphi_m\colon R\otimes_W N\to P$; $r\otimes n\mapsto\varphi'(rm*n)$ and one verifies immediately that for variable m it gives an R-morphism $\varphi\colon M\to\underline{\mathrm{Hom}}_R(R*_R N,P)$ and that these two processes are inverses of each other.

Remark: The existence of an internal right adjoint of $(-)*_R(-)$ follows from commutation with internal direct limits and the form (5.1.1) is forced upon us by putting $M=R$ in (5.4.1).

Definition 5.5. *Let* $M, N\in R$*-mod. The morphism*

$$R_n\otimes_R\underline{\mathrm{Hom}}^!_R(M,N)\to\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n\otimes_R M,R_n\otimes_R N)\tag{5.5.1}$$

is defined to be the adjoint of $R_n\otimes_R M\otimes_{W_n\mathcal{O}}R_n\otimes_R\underline{\mathrm{Hom}}^!_R(M,N)\xrightarrow{\varphi}R_n\otimes_R(M*_R\underline{\mathrm{Hom}}^!_R(M,N))\xrightarrow{R_n\otimes_R ev}R_n\otimes_R N$ *where* $ev\colon M*_R\underline{\mathrm{Hom}}^!_R(M,N)\to N$ *is the composite of* (3.4) *and the adjoint of* $id\colon\underline{\mathrm{Hom}}^!_R(M,N)\to\underline{\mathrm{Hom}}^!_R(M,N)$ *and* φ *is induced from* $N_1\otimes_W N_2\to N_1*_R N_2$ *as in* (4.6.1).

Proposition 5.6. *Let* $X\in D(R)$, $Y\in D^-(R)$ *and* $Z\in D^+(R)$ *then*

$$\begin{aligned}R\,\underline{\mathrm{Hom}}_R(X*^L_R Y,Z)&=R\,\underline{\mathrm{Hom}}_R(X,R\,\underline{\mathrm{Hom}}^!_R(Y,Z))\\ R\,\underline{\mathrm{Hom}}^!_R(X*^L_R Y,Z)&=R\,\underline{\mathrm{Hom}}^!_R(X,R\,\underline{\mathrm{Hom}}^!_R(Y,Z)).\end{aligned}\tag{5.6.1}$$

Indeed, we may assume that Y is a free complex and Z is injective. Note that (5.4.1) shows that $\underline{\mathrm{Hom}}^!_R(M,I)$ is injective if M is $*$-flat and I is injective. As products of injectives are injective $\underline{\mathrm{Hom}}^{!\cdot}_R(Y,Z)$ is an injective complex and the first part of (5.6.1) now follows from (5.4.1) and (0: 1). The second part follows in the same way replacing X by $R*_R X$.

Proposition 5.7. *Let* $X\in D^-(R)$, $Y\in D^+(R)$. *The morphism* (5.5.1) *extends to*

$$R_n\otimes^L_R R\,\underline{\mathrm{Hom}}^!_R(X,Y)\to R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n\otimes^L_R X,R_n\otimes^L_R Y).\tag{5.7.1}$$

i) *If* $n=1$ *then* (5.7.1) *is an isomorphism.*

ii) *Suppose that* F *is bijective on* $H^*(Y)$ *and that* $H^i(Y)$ *is bounded horizontally for all* i *then* (5.7.1) *is an isomorphism.*

Let us first define (5.7.1). As R_n has finite Tor-dimension we may assume that Y has R_n-acyclic components. Fixing, again, an acyclic topos and a surjection onto S we see, as above, that $\mathrm{s}(C^*(Y))$ has R_n-acyclic and flasque components and hence we may assume to begin with that Y has. Similarly, we may assume that X has free components. By (5.2 ii) $R\,\underline{\mathrm{Hom}}^!_R(X,Y)=\underline{\mathrm{Hom}}^{!\cdot}_R(X,Y)$ and, by assumption,

$R_n\otimes^L_R X=R_n\otimes_R X$ and $R_n\otimes^L_R Y=R_n\otimes_R Y$. Hence, using (5.5.1), we may define (5.7.1) as $R_n\otimes^L_R R\,\underline{\mathrm{Hom}}^!_R(X, Y)\to R_n\otimes_R R\,\underline{\mathrm{Hom}}^!_R(X, Y)=R_n\otimes_R \underline{\mathrm{Hom}}^{!\cdot}_R(X, Y)\to \underline{\mathrm{Hom}}^{\cdot}_{W_n\mathcal{O}}(R_n\otimes_R X,\ R_n\otimes_R Y)=\underline{\mathrm{Hom}}^{\cdot}_{W_n\mathcal{O}}(R_n\otimes^L_R X,\ R_n\otimes^L_R Y)\to R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n\otimes^L_R X, R_n\otimes^L_R Y)$.

To show i) we may assume that X is a free R-module on a sheaf T and Y is an R-module which is R_1-acyclic and flasque. Then $R_1\otimes_R Y$ is flasque by Cor. 1.1.4 and $R_1\otimes_R X=R_1[T]$ is a free $\mathcal{O}_S$-module by (0: 5.1). Thus, by the analogue of Lemma 5.2, $R\,\underline{\mathrm{Hom}}_{\mathcal{O}}(R_1\otimes_R X, R_1\otimes_R Y)=\underline{\mathrm{Hom}}_{\mathcal{O}}(R_1\otimes_R X, R_1\otimes_R Y)$. We thus want to show that $R_1\otimes^L_R \prod_T \underline{\mathrm{Hom}}^!_R(R, Y)=\prod_T \underline{\mathrm{Hom}}\,(R_1, R_1\otimes_R Y)$. To be able to see this we need a description of $\underline{\mathrm{Hom}}^!_R(R, Y)=\underline{\mathrm{Hom}}_R(R*_R R, Y)$. Note that $\underline{\mathrm{Hom}}^!_R(R, Y)$ has a structure of R-bimodule, the first structure being the natural one and the second being induced by right multiplication on R.

Lemma 5.7.2. i) *Let $M\in R$-mod. Every $\varphi\in\underline{\mathrm{Hom}}^!_R(R, M)=\underline{\mathrm{Hom}}_R(R*_R R, M)$ can be uniquely written as a product*

(5.7.3) $$\prod_{i>0} V^i m_i+\prod_{i\geq 0} m_{-i}F^i+\prod_{i>0} dV^i n_i+\prod_{i\geq 0} n_{-i}F^i d$$

where $V^i m$, mF^i, $dV^i m$ and $mF^i d$ are the homomorphisms taking the value m on $F^i d*1$, $-(1*F^i d)$, F^i*1 resp. $1*F^i$ and is 0 on the other basis elements (cf. (3.2)). Any such product may be multiplied by $\lambda\in W$ and F, d and V to both the right and the left, where, apart from the usual relations, we also have

(5.7.4) $$Fm=F(m)F,\quad mV=VF(m),\quad VmF=V(m)$$
$$dm=d(m)+(-1)^{\deg m} md$$

(E.g. $V(mF^2)=VmF^2=VmFF=V(m)F$, $F(mF)=FmF=F(m)F^2$).

In this way (λ, F, d, V) of the first R-module structure on $\underline{\mathrm{Hom}}^!_R(R, M)$ is given by left multiplication by (λ, F, d, V) and the second by right multiplication by (λ, V, d, F).

ii) *If M is flasque and acyclic for R_1 and has F injective then*

(5.7.5) $$R_n\otimes^L_R \underline{\mathrm{Hom}}^!_R(R, M)=\prod_{n>i>0} V^i(R_{n-i}\otimes_R M)\oplus\prod_{i\geq 0}(R_n\otimes_R M)F^i$$
$$\oplus\prod_{n>i>0} dV^i(R_{n-i}\otimes_R M)\oplus\prod_{i\geq 0}(R_n\otimes_R M)F^i d,$$

where the tensor product is taken wrt the first structure.

i) follows from (3.2) and the relations (3.1). As for ii) the proof is a simple calculation almost word for word as (0: 5.1) and [16: II, Thm. 1.2], the only point being that the assumptions together with (1.1.4) and the fact, proved in (1.1.4), that $(V^n R+dV^n R)\otimes_R M$ is flasque allow us to compute at the presheaf level.

Let us return to the relation $R_1\otimes^L_R \prod_T \underline{\mathrm{Hom}}^!_R(R, Y)=\prod_T \underline{\mathrm{Hom}}_{\mathcal{O}}(R_1, R_1\otimes_R Y)$. As F is injective on R we may assume that apart from being flasque and R_1-acyclic Y also has F injective. (5.7.2 i) shows that $\underline{\mathrm{Hom}}^!_R(R, Y)$ is the product of flasque sheaves and hence is flasque. The projection formula for j_T: $S/T\to S$ then shows that

$R_1 \otimes_R^L \prod_T \underline{\mathrm{Hom}}_R^!(R, Y) = R \prod_T (R_1 \otimes_R^L \underline{\mathrm{Hom}}_R^!(R, Y))$. By (5.7.2 i) $\underline{\mathrm{Hom}}_R^!(R, Y)$ is R_1-acyclic and as it is flasque, $R_1 \otimes_R \underline{\mathrm{Hom}}_R^!(R, Y)$ is flasque by (1.1.4). Thus, $R_1 \otimes_R^L \prod_T \underline{\mathrm{Hom}}_R^!(R, Y) = \prod_T R_1 \otimes_R \underline{\mathrm{Hom}}_R^!(R, Y)$ and we are reduced to showing that $R_1 \otimes_R \underline{\mathrm{Hom}}_R^!(R, Y) = \underline{\mathrm{Hom}}_{\mathcal{O}}(R_1, R_1 \otimes_R Y)$. The left hand side is described by (5.7.5) and (0: 5.1) shows that the right hand side has the same description and it is easy to see that under these identifications (5.5.1) is the identity.

In proving ii) we may assume that Y is an R-module with F bijective and bounded horizontally. By filtering by $\sum_{i>N} Y^i$ we may assume that Y is concentrated in a single degree and by shifting that this degree is 0. By truncating a resolution by modules of the type $W\mathcal{O}[F, F^{-1}][T]$, $T \in S$, and dévissage we may assume that Y is R_1-acyclic and p is injective and by taking a canonical flasque resolution we may assume that Y is flasque. As in the proof of i) we reduce ourselves to showing that $R_n \otimes_R \underline{\mathrm{Hom}}_R^!(R, Y) = R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, R_n \otimes_R Y)$ and that $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, R_n \otimes_R Y)$ is flasque. Let us first convince ourselves that $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, R_n \otimes_R Y) = \underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, R_n \otimes_R Y)$ and that it is flasque. By (0: 5.1) $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, M) = \prod_{n>i>0} V^i\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, M) \oplus \prod_{i>0} \underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\mathcal{O}, M) F^i \oplus \prod_{n>i>0} dV^i\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}} (W_{n-i}\mathcal{O}, M) \oplus \prod_{i>0} \underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\mathcal{O}, M) F^i d$ where M is a W_n-module and $V^i\varphi$, φF^i, $dV^i\varphi$ and $\varphi F^i d$ are the morphisms taking the value $\varphi(1)$ on dV^i, $F^i d$, V^i resp. F^i. This represents $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, -)$ as the composite of $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, -)$:s and a product. It is therefore clear that it suffices to show that $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, R_n \otimes_R Y) = \underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, R_n \otimes_R Y)$ and that it is flasque. As F is bijective and $d=0$ on Y $R_n \otimes_R Y = Y/p^n Y$. Thus $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, R_n \otimes_R Y) = p^{n-i}(Y/p^n) = Y/p^{n-i}$ as Y is p-torsion free. Y/p^{n-i} is flasque because of the exact sequence $0 \to Y \xrightarrow{p^{n-i}} Y \to Y/p^{n-i} \to 0$ and flasqueness of Y. Furthermore, the exact sequence $\ldots \to W_n\mathcal{O} \xrightarrow{p^i} W_n\mathcal{O} \xrightarrow{p^{n-i}} W_n\mathcal{O} \xrightarrow{p^i} W_n\mathcal{O} \to W_{n-i}\mathcal{O} \to 0$ shows that, for $j>0$, $\underline{\mathrm{Ext}}^j_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, M) = p_i(M)/p^{n-i}M$ if j is odd; $p^{n-1}(M)/p^i M$ if j is even. This shows that, for $j>0$, $\underline{\mathrm{Ext}}_{W_n\mathcal{O}}(W_{n-i}\mathcal{O}, Y/p^n Y) = 0$. We have just described $\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n, R_n \otimes_R Y)$ as

$$\prod_{n>i>0} V^i Y/p^{n-i} Y \oplus \prod_{i>0} (Y/p^n Y) F^i \oplus \prod_{n>i>0} dV^i Y/p^{n-i} Y \oplus \prod_{i>0} (Y/p^n Y) F^i d.$$

By (5.7.5) $R_n \otimes_R \underline{\mathrm{Hom}}_R^!(R, Y)$ has the same description and it is easily seen that under these identifications (5.5.1) is, up to a sign, the identity.

Lemma 5.8. *Let $M, N \in R$-mod. Then $\underline{\mathrm{Hom}}_R^!(M, N) = \underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_R^!(R, N))$, where the first R-module structure on $\underline{\mathrm{Hom}}_R^!(R, N)$ is used to compute $\underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_R^!(R, N))$ and the second to give this sheaf the structure of R-module inherent in $\underline{\mathrm{Hom}}_R^!(M, N)$.*

Indeed,

$$\underline{\mathrm{Hom}}_R^!(M, N) = \underline{\mathrm{Hom}}_R(R * M, N) = \underline{\mathrm{Hom}}_R(M * R, N) = \underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_R^!(R, N))$$

and the rest of the lemma is obvious.

Remark: In the case where $S=\operatorname{Spec} k$, k a perfect field, (5.7.3) and [8: III, Prop. 3.5] show that $\underline{\mathrm{Hom}}^!_R(R, W)=\check{R}$ in the terminology of [8]. The lemma then shows that $R\,\underline{\mathrm{Hom}}^!_R(-, W)=R\,\underline{\mathrm{Hom}}_R(-, \check{R})$. The results of [8] then make it eminently reasonable that $R\,\underline{\mathrm{Hom}}^!_R(-, W)$ should be dualizing in the general case. (5.8) also enables us to compare the results of [8] and the present paper. Finally, the expression of $R\,\underline{\mathrm{Hom}}_R(-, \check{R})$ as $R\,\underline{\mathrm{Hom}}^!_R(-, W)$ could be said to throw some further light on [8].

Proposition 5.9. *If* $X\in D(R)$, $Y\in D^+(R)$ *then*

$$R\,\underline{\mathrm{Hom}}^!_R(X, Y)^\wedge = R\,\underline{\mathrm{Hom}}^!_R\,\hat{X}, \hat{Y}), \tag{5.9.1}$$

$$R\,\underline{\mathrm{Hom}}_R(X, \hat{Y}) = R\,\underline{\mathrm{Hom}}_R(\hat{X}, \hat{Y}). \tag{5.9.2}$$

Proof: Put $M^! := R\,\underline{\mathrm{Hom}}^!_R(R, M)$ for $M\in D^+(R)$. Then $R\,\underline{\mathrm{Hom}}^!_R(X, Y)= R\,\underline{\mathrm{Hom}}_R(X, Y^!)$. This should be interpreted as follows: $Y^!$ is a complex of R-bimodules and $R\,\underline{\mathrm{Hom}}_R(X, Y^!)$ should be computed by replacing $Y^!$ by a complex of bimodules whose components are injective when considered as R-modules through the first structure. However, as $\underline{\mathrm{Hom}}^!_R(R, -)$ has an exact left adjoint, $R*_R(-)$, $Y^!$ has injective components if Y has and the desired equality follows from Lemma 5.8. As R_n has a resolution by finite free R-modules $R.\otimes^L_R R\,\underline{\mathrm{Hom}}_R(X, Y^!)= R\,\underline{\mathrm{Hom}}_R(X, R.\otimes^L_R Y^!)$ and by (0: 4, 9) $R\,\underleftarrow{\lim}\,(R\,\underline{\mathrm{Hom}}_R(X, R.\otimes^L_R Y^!))=R\,\underline{\mathrm{Hom}}_R(X, R\,\underleftarrow{\lim}\,(R.\otimes^L_R Y^!))$. Thus $R\,\underline{\mathrm{Hom}}^!_R(X, Y)^\wedge=R\,\underline{\mathrm{Hom}}_R(X, (Y^!)^\wedge)$. The following lemma then finishes the proof.

Lemma 5.9.3. *Let* $X\in D(R)$, $Y\in D^+(R)$.

i) $$(Y^!)^\wedge = (\hat{Y})^!$$

ii) $$R\,\underline{\mathrm{Hom}}_R(X, \hat{Y}) = R\,\underline{\mathrm{Hom}}_R(\hat{X}, \hat{Y}).$$

The analogue of (5.7.5) for the second structure shows that, when M is $R.$-acyclic and flasque and has F injective, $R.\otimes^L_R M^! = \prod_{i\geq 0} V^i(R.\otimes_R M)\oplus\prod_{i>0}(R._{-i}\otimes_R M)F^i\oplus\prod_{i\geq 0} dV^i(R.\otimes_R M)\oplus\prod_{i>0}(R._{-i}\otimes_R M)F^i d$. This analogue may either be proved directly or by noting that (3.4) exchanges the two structures of $\underline{\mathrm{Hom}}_R(R*_R R, M)$. We thus see through (1.1.4) that $R.\otimes^L_R M^!$ is $\underleftarrow{\lim}$-acyclic and $R\,\underleftarrow{\lim}\,(R.\otimes^L_R M^!)= \prod_{i\geq 0} V^i\hat{M}\oplus\prod_{i>0}\hat{M}F^i\oplus\prod_{i\geq 0} dV^i\hat{M}\oplus\prod_{i>0}\hat{M}F^i d$ as $\underleftarrow{\lim}$ commutes with products. This is, however, nothing but $\underline{\mathrm{Hom}}^!_R(R, \hat{M})$. On the other hand, as for $T\in SR\Gamma(T, \hat{M})=R\Gamma(T, M)^\wedge$ (0: 5.11) and $R\Gamma(T, M)=M(T)$ and by Cor. 1.1.4 $R.\otimes^L_R M(T)=R.\otimes_R M(T)$ has surjective transition maps and hence is $\underleftarrow{\lim}$-acyclic, we have $R\Gamma(T, \hat{M})=\Gamma(T, \hat{M})$ so $\hat{M}$ is flasque and thus, by (5.2 ii), $\hat{M}^!= \underline{\mathrm{Hom}}^!_R(R, \hat{M})=M^{!\wedge}$. Having i) for $R.$-acyclic and flasque R-modules with F injective we easily get it for $X\in D^+(R)$. Incidentally, it is clear that this isomorphism is characterized by the commutativity of

$$
\begin{tikzcd}
\hat{Y}^! \arrow[r] \arrow[dr] & \hat{Y}^{!\wedge} \\
\hat{Y}^! \arrow[u] \arrow[r] & \hat{Y}^{!\wedge} \arrow[u]
\end{tikzcd}
\tag{5.9.4}
$$

As for ii), by (2.1) it suffices to show that if Y is complete then $X \to \hat{X}$ induces an isomorphism $R\,\underline{\mathrm{Hom}}_R(\hat{X}, Y) \to R\,\underline{\mathrm{Hom}}_R(X, Y)$. This follows immediately from (0: 5.12.1) once we have verified the condition of (0: Lemma 5.12). We may as usual assume that X is a flasque and R_1-acyclic R-module and then we need to verify that a sequence $\{x_m\}$ in X is equal in $R_n \otimes_R X$ to the constant sequence $\{x_n\}$. This follows from (2.1.1).

Proposition 5.10. i) *If* $M \in D^+_{\mathrm{perf}}(R)$, $N \in D^{+,l}(R)$ *and* $R_1 \otimes^L_R N \in D^-(\mathcal{O})$ *then* $R\,\underline{\mathrm{Hom}}^!_R(M, N)^\wedge \in D^-(R)$.

ii) $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ takes $D^b_{\mathrm{perf}}(R)^{\mathrm{op}} x D^b_{\mathrm{perf}}(R)$ to $D^b_{\mathrm{perf}}(R)$.

Indeed, (5.3) and (4.9) show that $R\,\underline{\mathrm{Hom}}^!_R(M, N)$ and hence $R\,\underline{\mathrm{Hom}}^!_R(M, N)^\wedge$ is bounded to the right and (2.1) and Prop. 5.7. i) show that $R_1 \otimes^L_R R\,\underline{\mathrm{Hom}}^!_R(M, N)^\wedge \in D^-(\mathcal{O})$ so by (1.1) $R\,\underline{\mathrm{Hom}}^!_R(M, N)^\wedge \in D^-(R)$. ii) now follows from i) and (5.3) and (5.9) which shows that $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ takes complete complexes to complete complexes.

Definition 6.1. *If* $M \in D(R)$ *put*

$$D(M) := R\,\underline{\mathrm{Hom}}^!_R(M, W\mathcal{O}) \tag{6.1.1}$$

where $W\mathcal{O}$ *has the obvious* R*-module structure.*

The following theorem is the main result of this chapter:

Theorem 6.2. i) $(-)\hat{*}^L_R(-)$ is a coherently commutative and associative (additive) functor $D^b_{\mathrm{perf}}(R) x D^b_{\mathrm{perf}}(R) \to D^b_{\mathrm{perf}}(R)$.

ii) $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ *is an (additive) functor*

$$D^b_{\mathrm{perf}}(R)^{\mathrm{op}} x D^b_{\mathrm{perf}}(R) \to D^b_{\mathrm{perf}}(R).$$

iii) $$W\mathcal{O}\,\hat{*}^L_R(-) = \mathrm{id} \quad \text{on} \quad D^b_{\mathrm{perf}}(R).$$

iv) $$R\,\underline{\mathrm{Hom}}_R(W\mathcal{O}, R\,\underline{\mathrm{Hom}}^!_R(-, -)) = R\,\underline{\mathrm{Hom}}_R(-, -) \quad \text{on} \quad D^b_{\mathrm{perf}}(R)^{\mathrm{op}} x D^b_{\mathrm{perf}}(R).$$

v) $$R\,\underline{\mathrm{Hom}}^!_R((-)\hat{*}^L_R(-), -) = R\,\underline{\mathrm{Hom}}^!_R(-, R\,\underline{\mathrm{Hom}}^!_R(-, -)) \quad \text{on} \quad D^b_{\mathrm{perf}}(R)^{\mathrm{op}} x D^b_{\mathrm{perf}}(R)^{\mathrm{op}} x D^b_{\mathrm{perf}}(R).$$

vi) *For all finite sets I*

$$i{\in}I^{\hat{*}_R^L} R\,\underline{\mathrm{Hom}}^!_R(-,-) = R\,\underline{\mathrm{Hom}}^!_R\big(i{\in}I^{\hat{*}_R^L}(-),\, i{\in}I^{\hat{*}_R^L}(-)\big)$$

vii) $\mathrm{id}=D(D(-))$

viii) $R_1\otimes^L_R(-)\colon D^b_{\mathrm{perf}}(R)\to D^b_{\mathrm{perf}}(\mathcal{O}[d])$ *is a conservative tensor functor.*

Remark: This shows that, in the terminology of [5: II], $D^b_{\mathrm{perf}}(R)$ admits a structure of a rigid additive tensor category.

i) and ii) are clear. For iii) we have $W\mathcal{O}*^L_R X=W\mathcal{O}\otimes^L_{W\mathcal{O}}X=X$. Hence $W\mathcal{O}\hat{*}^L_R X=\hat{X}$ but any $X\in D^b_{\mathrm{perf}}(R)$ is complete. If $X, Y, Z\in D^b_{\mathrm{perf}}(R)$ then $R\,\underline{\mathrm{Hom}}_R(X*^L_R Y, Z)=R\,\underline{\mathrm{Hom}}_R(X\hat{*}^L_R Y, Z)$ by (5.9.2) and by (5.6.1) $R\,\underline{\mathrm{Hom}}_R(X*^L_R Y, Z)=R\,\underline{\mathrm{Hom}}_R\big(X, R\,\underline{\mathrm{Hom}}^!_R(Y, Z)\big)$. This shows that $R\,\underline{\mathrm{Hom}}^!_R(-,-)$ is an internal Hom in the sense of tensor categories. v) is now formal from this (cf. [5: II, 1.6.3]), I now claim that $R_1\otimes^L_R(-)\colon D^b_{\mathrm{perf}}(R)\to D^b_{\mathrm{perf}}(\mathcal{O})$ preserves the adjunction units. As, by (4.8 i), it is a tensor functor; this is, however, purely formal.

The morphisms vi) and vii) are now defined by the general theory of tensor categories. As $R_1\otimes^L_R(-)$ preserves adjunction units and internal Hom: s, (5.7), applying it to the morphisms of vi) and vii) gives the corresponding morphisms in $D^b_{\mathrm{perf}}(\mathcal{O})$ where they clearly are isomorphisms. We now conclude by Cor. 1.1.3. that viii) is clear and iv) follows from (5.6.1).

Remark: It is easy to see directly that v) is valid only under the assumption that the complex in the last variable be complete. One of the isomorphisms obtainable from vi) and vii) can also be defined directly, namely, $D(X\hat{*}^L_R Y)=R\,\underline{\mathrm{Hom}}^!_R(X, D(Y))$ for $X\in D(R)$, $Y\in D^-(R)$.

Proposition 7.1. *Let* $X\in D(R)$. *Then* $R_n\otimes^L_R D(X)=R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(R_n\otimes^L_R X, W_n\mathcal{O})$ in $D(W_n\mathcal{O}[d])$.

This follows directly from (5.7 ii)).

Definition 7.2. i) An R-module of level n ($n\geq 0$) is an R-module concentrated in degrees 0 to n with F bijective in degree n. The category of R-modules of level n will be denoted R-mod-n. R-mod-n is evidently an Abelian subcategory of R-mod closed under kernels, cokernels, extensions and internal sums and products.

ii) An F-crystal of level n on S is an ungraded $W\mathcal{O}_S$-module together with a σ-linear map F and a σ^{-1}-linear map V such that $VF=FV=p^n$. The category of F-crystals of level n (on S) will be denoted F-crys-n. It is clearly equal to the category of (ungraded) $W\mathcal{O}[F, V]\,(n)$-modules where $W\mathcal{O}[F, V]\,(n)$ is the $W\mathcal{O}$-ring with generators F and V and relations $Fa=a^\sigma F$, $aV=Va^\sigma$, $FV=VF=p^n$ $a\in W\mathcal{O}$ (cf. [16: I, 4.1]).

Lemma 7.3. i) *If* $M\in R$-mod-m *and* $N\in R$-mod-n *then* $M*_R N\in R$-mod-$(m+n)$ *and* $\underline{\mathrm{Hom}}^!_R(M, N)(-m)\in R$-mod-$(m+n)$.

ii) *If* $M\in F$-crys-m, $N\in F$-crys-n *then* $M\otimes_{W\mathcal{O}}N$ *and* $\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, N)$ *may be given structures of F-crystals of level* $m+n$ *by* $F(m\otimes n)=Fm\otimes Fn$, $V(m\otimes n)=Vm\otimes Vn$ and $Ff(m)=F(f(Vm))$, $Vf(m)=V(f(Fm))$.

As ii) is obvious let us consider i). It is clear that every $T\in R$-mod-t is the image of $L=R[X]\oplus W\mathcal{O}[F, F^{-1}][Y]$ where X is concentrated in degrees 0 to $t-1$ and Y is concentrated in degree t. Furthermore, as it has been noted in Lemma 4.4, $L\in R$-mod-t. As $(-)*_R(-)$ is right exact and commutes with internal sums and $\underline{\mathrm{Hom}}^!_R(-, -)$ is left exact and commutes with internal products we may assume that M is either $R(-i)$, $0\leq i<m$, or $W\mathcal{O}[F, F^{-1}](-m)$. The first case is now clear from (3.2.3) resp. (5.7.3). We learn from Prop. 3.3 and the proof of Lemma 5.2 that $W\mathcal{O}[F, F^{-1}]*_R(-)=W\mathcal{O}[F. F^{-1}]\otimes_{W\mathcal{O}}(-)$ resp. $\underline{\mathrm{Hom}}^!_R(W\mathcal{O}[F, F^{-1}], -)=\underline{\mathrm{Hom}}_{W\mathcal{O}}(W\mathcal{O}[F, F^{-1}], -)$ which again makes everything clear.

Lemma 7.4. i) $(-)*_R(-)$ *derives to a functor* $D^-(R\text{-mod-}m)xD(R\text{-mod-}n)\cup D(R\text{-mod-}m)xD^+(R\text{-mod-}n)\to D(R\text{-mod-}(m+n))$ *whose composite with* $D(R\text{-mod-}(n+m))\to D(R\text{-mod})$ *is* $(-)*^L_R(-)$ *and hence will also be denoted* $(-)*^L_R(-)$. $\underline{\mathrm{Hom}}^!_R(-, -)(-m)$ *derives to a functor* $D(R\text{-mod-}m)^{\mathrm{op}}xD^+(R\text{-mod-}n)\to D(R\text{-mod-}(m+n))$ *whose composite with* $D(R\text{-mod-}(m+n))\to D(R\text{-mod})$ *is* $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ *and hence will also be denoted* $R\,\underline{\mathrm{Hom}}^!_R(-, -)$.

ii) $(-)\otimes_{W\mathcal{O}}(-)$ *derives to a functor* $D^-(F\text{-crys-}m)xD(F\text{-crys-}n)\cup D(F\text{-crys-}m)xD^-(F\text{-crys-}n)\to D(F\text{-crys-}(m+n))$ *whose composite with* $D(F\text{-crys-}(m+n))\to D(W\mathcal{O}\text{-mod})$ is $(-)\otimes^L_{W\mathcal{O}}(-)$ *and hence will also be denoted* $(-)\otimes^L_{W\mathcal{O}}(-)$. $\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$ *derives to a functor* $D(F\text{-crys-}m)^{\mathrm{op}}xD^+(F\text{-crys-}n)\to D(F\text{-crys-}(m+n))$ *whose composite with* $D(F\text{-crys-}(m+n))\to D(W\mathcal{O}\text{-mod})$ *is* $R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$ *and hence will also be denoted* $R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$.

Proof: That $(-)*_R(-)$ derives as stated is clear as, by the proof of Lemma 4.4, there are sufficiently many $*$-flat objects in R-mod-t for all $t\geq 0$. This also makes the second statement concerning $(-)*_R(-)$ clear. As for $\underline{\mathrm{Hom}}^!_R(-, -)$, R-mod-n is evidently a Grothendieck category so it has sufficiently many injectives and $\underline{\mathrm{Hom}}^!_R(-, -)$ derives as stated. We also have a natural transformation from this derived functor composed with $D(R\text{-mod-}(m+n))\to D(R\text{-mod})$ to $R\,\underline{\mathrm{Hom}}^!_R(-, -)$. To see that this is a natural equivalence we may, as in the proof of Prop. 5.3, assume that the complex in the first variable is bounded from above and then compute $R\,\underline{\mathrm{Hom}}^!_R(-, -)$ by taking a resolution by modules of the type $R[X]\oplus W\mathcal{O}[F, F^{-1}][Y]$, X concentrated in degrees 0 to $m-1$ and Y in degree m, in the first variable and a canonical flasque resolution in the second. These will be resolutions in R-mod-m resp. R-mod-n and thus will also compute the derived functor of $\underline{\mathrm{Hom}}^!_R(-, -)$.

Remark: Injective objects in R-mod-t will rarely be injective in R-mod. An injective object in R-mod-0, for instance, will be injective as R-module only when it is zero.

In ii) the case of $(-)\otimes_{W\mathcal{O}}(-)$ is similar to that of $(-)*_R(-)$ as $W\mathcal{O}[F, V](n)$ is flat as $W\mathcal{O}$-module. The case of $\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$ will follow if we can show that an injective $W\mathcal{O}[F, V](n)$-module is injective as $W\mathcal{O}$-module which is true as the forgetful functor $W\mathcal{O}[F, V](n)\text{-mod}\to W\mathcal{O}\text{-mod}$ has an exact left adjoint by $W\mathcal{O}$-flatness of $W\mathcal{O}[F, V](n)$.

Definition 7.4. i) $W.\mathcal{O}\otimes_{W\mathcal{O}}(-)$ is the functor $M\mapsto\{W\mathcal{O}/p^nW\mathcal{O}\otimes_{W\mathcal{O}}M, \mathrm{proj}\}$ from F-crystals of level n to inverse systems of F-crystals of level n or, as the case may be, the same functor from $W\mathcal{O}$-modules to inverse systems of $W\mathcal{O}$-modules.

ii) $\varprojlim$ will denote the functor $\{M.\}\mapsto\varprojlim\{M.\}$ from inverse systems of F-crystals of level n to F-crystals of level n (or, as the case may be, ...).

iii) $(\hat{-})$: $D(F\text{-crys-}n)\to D(F\text{-crys-}n)$ is defined to be $R\varprojlim\{W.\mathcal{O}\otimes^L_{W\mathcal{O}}(-)\}$. (Idem for $(\hat{-})$: $D(W\mathcal{O}\text{-mod})\to D(W\mathcal{O}\text{-mod})$.)

Lemma 7.6. i) *There is a canonically defined functor* $(\hat{-})$: $D(R\text{-mod-}n)\to D(R\text{-mod-}n)$ *whose composite with* $D(R\text{-mod-}n)\to D(R\text{-mod})$ *is the* $(\hat{-})$ *of* (0: 5.9).

ii) *The canonical natural transformation* $id\to(\hat{-})$ *in* $D(F\text{-crys-}n)$ *induces an isomorphism*

$$W.\otimes^L_W(-)\to W.\otimes^L_W(\hat{-}). \tag{7.6.1}$$

In particular $(\hat{-})=(\hat{\hat{-}})$.

To prove i) it clearly suffices to show that if $M\in R$-mod-n is flasque and R_1-acyclic then $\hat{M}\in R$-mod-n. We already know that $\hat{M}$ as a complex is concentrated in degree 0. It is immediately clear that M is concentrated in degrees 0 to n so it suffices to show that F is bijective in degree n on M. As $\hat{M}=\varprojlim\{R.\otimes_RM\}$ this will be proved if we can show that F is an isomorphism of the pro-object $(R.\otimes_RM)^n$. This is a condition closed under cokernels, finite sums, and internal copowers. We may therefore assume that $M=R(-i)$ $0\leqq i<n$ or $W\mathcal{O}[F, F^{-1}](-n)$. The second case is obvious as $R.\otimes_RW\mathcal{O}[F, F^{-1}]=\sum_i W.\mathcal{O}F^i$. In the first case we want to show that F is an isomorphism of the pro-object $R^1_.$ which is an easy calculation using (0: 5.1).

ii) is proved in the same way as Prop. 2.1.

Definition 7.7. i) $W\mathcal{O}(n)$ is defined to be the F-crystal of level n whose underlying $W\mathcal{O}$-module is $W\mathcal{O}$ and which has $F:=p^n\sigma$, $V:=\sigma^{-1}$.

ii) For any $M \in C(F\text{-crys-}m)$ $M(n) := M \otimes_{W\mathcal{O}} W\mathcal{O}(n) \in C(F\text{-crys-}(m+n))$. If $n' \geqq n$ any F-crystal N of level n, may be regarded as an F-crystal of level n' with the new (F, V) being $(F, p^{n'-n}V)$. By abuse we will regard the functor thus obtained an inclusion and say that M is an F-crystal of level n'.

Remark: The reason for putting the two similar functors $(M, F, V) \mapsto (M, p^{n'-n}F, V)$ (resp. $\mapsto (M, F, p^{n'-n}V)$) on unequal footings is, of course, that F is the basic endomorphism and V should rather be regarded as putting conditions on F. When M is torsion free F determines V and this is literally true.

Proposition 7.8. *Let $M \in C(F\text{-crys-}m)$, $N \in C(F\text{-crys-}n)$ and $Q \in C(F\text{-crys-}q)$. Put $r = \max(m+n, q+n)$ and $s = \max(m, q+n)$. Supposing that Q is torsion free we have an isomorphism in $C(S-ab)$*

$$(7.8.1) \qquad \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{F\text{-crys-}r}(M \otimes_{W\mathcal{O}} N, Q(n)) = \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{F\text{-crys-}s}(M, \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(N, Q)).$$

We clearly have $\underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(M \otimes_{W\mathcal{O}} N, Q) = \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(M, \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(N, Q))$ and it only remains to show, assuming that M, N and Q are concentrated in degree 0, that $\varphi\colon M \otimes_{W\mathcal{O}} N \to Q(n)$ is a morphism in F-crys-r iff its adjoint $\varphi'\colon M \to \underline{\mathrm{Hom}}_{W\mathcal{O}}(N, Q)$ is a morphism in F-crys-s. As Q is torsion free and FV=some power of p it suffices to check this condition for F. We have $\varphi' \circ F(m)(n) = \varphi(Fm \otimes n)$ and $F \circ \varphi'(m)(n) = F\varphi(m \otimes Vn)$ whereas $\varphi \circ F(m \otimes n) = \varphi(Fm \otimes Fn)$ and $F \circ \varphi(m \otimes n) = p^n F\varphi(m \otimes n)$. Assume now that $F \circ \varphi = \varphi \circ F$. This implies by torsion freeness of Q:

$$\varphi(Fm \otimes FVn) = p^n F\varphi(m \otimes Vn) \Rightarrow p^n \varphi(Fm \otimes n) = p^n F\varphi(m \otimes Vn) \Rightarrow \varphi(Fm \otimes n) =$$

$$F\varphi(m \otimes Vn) \Rightarrow \varphi' \circ F(m)(n) = F \circ \varphi'(m)(n).$$

Conversely, if $F \circ \varphi' = \varphi' \circ F$ then $(\varphi \circ F)(m \otimes n) = \varphi(Fn \otimes Fn) = F\varphi(m \otimes VFn) = p^n F\varphi(m \otimes n) = (F \circ \varphi)(m \otimes n)$.

The moral of (7.8) is that although it does not show that F-crys-r-morphisms $M \otimes^L_W N \to Q(n)$ corresponds bijectively to F-crys-s-morphisms $M \to R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(N, Q)$ it does so for all practical purposes. Let us illustrate this by showing how to define the evaluation morphism ev: $M \otimes^L_{W\mathcal{O}} R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, N) \to N(m)$ $M \in D^-(F\text{-crys-}m)$, $N \in D^+(F\text{-crys-}n)$ as a morphism in $D(F\text{-crys-}(2m+n))$. We may replace M by a complex of free $W\mathcal{O}[F, V](m)$-modules which is, a fortiori, $W\mathcal{O}$-free and N by a torsion free flasque complex. Thus $M \otimes^L_{W\mathcal{O}} R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, N) = M \otimes_{W\mathcal{O}} \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(M, N)$ and $R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(M, N) = \underline{\mathrm{Hom}}^{\boldsymbol{\cdot}}_{W\mathcal{O}}(M, N)$. As N is torsion free we may apply the proposition and define ev as the adjoint of the identity.

Definition 7.9. i) Let M be an R-module of level n. Let M denote M considered as a complex of F-crystals of level n where F and V are defined as follows (cf. [19]):

$$V:\quad \begin{array}{ccccccccc} M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^2 & \xrightarrow{d} & \dots M^{n-1} & \xrightarrow{d} & M^n \\ \downarrow{\scriptstyle p^{n-1}} & & \downarrow{\scriptstyle p^{n-2}V} & & \downarrow{\scriptstyle p^{n-3}V} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle F^{-1}} \\ M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^2 & \xrightarrow{d} & \dots M^{n-1} & \xrightarrow{d} & M^n \end{array}$$

$$F:\quad \begin{array}{ccccccccc} M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^2 & \longrightarrow & \dots M^{n-1} & \xrightarrow{d} & M^n \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & & \downarrow{\scriptstyle p^2F} & & \downarrow{\scriptstyle p^{n-1}F} & & \downarrow{\scriptstyle p^nF} \\ M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & M^2 & \xrightarrow{d} & \dots M^{n-1} & \xrightarrow{d} & M^n. \end{array}$$

(The relations $dF=pFd$, $Vd=dpV$ show that F and V are morphisms of complexes and $VF=FV=p$ that we get complexes in F-crys-n.)

If M is a complex of R-modules of level n let $\mathrm{s}(M)$ denote the associated simple complex regarded as a complex of F-crystals of level n. $\mathrm{s}(-)$ certainly preserves quasi-isomorphisms and hence defines a functor $\mathrm{s}\colon D(R\text{-mod-}n)\to D(F\text{-crys-}n)$. We define similarly $\mathrm{s}\colon D(W_n\mathcal{O}[d]\text{-mod})\to D(W_n\mathcal{O}\text{-mod})$ and $\mathrm{s}\colon D(R\text{-mod})\to D(W\mathcal{O}\text{-mod})$.

Proposition 7.10. i) *Let* $M\in D(R\text{-mod-}m)$ *then*

$$W\mathcal{O}/p^nW\mathcal{O}\otimes^L_{W\mathcal{O}}\mathrm{s}(M)=\mathrm{s}(R_n\otimes^L_R M). \tag{7.10.1}$$

ii) $$\mathrm{s}(\hat{M})=\widehat{\mathrm{s}(M)}.$$

Proof: First assume i) and let us prove ii). As usual we may assume that M has R, $-$, $W.\mathcal{O}-$ and S^{gen}-acyclic components. Then $\mathrm{s}(M)$ has $W.\mathcal{O}$-acyclic components and, using i), $\mathrm{s}(\hat{M})=\mathrm{s}(\varprojlim(R.\otimes_R M))=\varprojlim(\mathrm{s}(R.\otimes_R M))=\varprojlim(W.\mathcal{O}\otimes_{W\mathcal{O}}\mathrm{s}(M))=\widehat{\mathrm{s}(M)}$. Note that only finite sums are in this case involved in $\mathrm{s}(-)$ so $\mathrm{s}(-)$ and $\varprojlim$ do commute.

Let us now prove i). As $p^nR_n=0$ we have a mapping $R/p^nR\to R_n$ of right R-modules. Furthermore, as R is torsion free $R/p^nR\otimes^L_R M=C(p^n,M)$, the mapping cone of p^n on $M\in D(R)$. We thus have a morphism $W\mathcal{O}/p^nW\mathcal{O}\otimes^L_{W\mathcal{O}}\mathrm{s}(M)=C(p^n,\mathrm{s}(M))=\mathrm{s}(C(p^n,M))=\mathrm{s}(R/p^nR\otimes^L_R M)\to\mathrm{s}(R_n\otimes^L_R M)$ and to show that this is an isomorphism we must show that $R/p^nR\otimes^L_R M\to R_n\otimes^L_R M$ induces an isomorphism on the associated simple complex. By (0: 1.3) we may assume that M is bounded from above and by taking a free resolution (which means that we leave R-mod-n) we may assume that that M is a free R-module. As the problem commutes with internal sums we may finally assume that $M=R$ and we are thus reduced to showing that $\mathrm{s}(R/p^nR)\to\mathrm{s}(R_n)$ is a quasi-isomorphism which is a simple calculation using (0: 5.1) (cf. [15: I, 3.17.2]).

Definition 7.11. $M \in D(F\text{-crys-}n)$ is said to be perfect if $\mathcal{O}\otimes^L_{W\mathcal{O}} M \in D_{\mathrm{perf}}(\mathcal{O})$ and $M \to \hat{M}$ is an isomorphism. The triangulated subcategory of $D(F\text{-crys-}n)$ consisting of perfect complexes will be denoted $D_{\mathrm{perf}}(F\text{-crys-}n)$.

Remark: The exact sequences $0 \to W\mathcal{O}/p \to W\mathcal{O}/p^{n+1} \to W/p^n \to 0$ show that if M is perfect then $W.\mathcal{O}\otimes^L_{W\mathcal{O}} M$ is bounded and hence $M = R\varprojlim(W.\mathcal{O}\otimes^L_{W\mathcal{O}} M)$ is. Therefore $D_{\mathrm{perf}}(F\text{-crys-}n) \subseteq D^b(F\text{-crys-}n)$.

Theorem 7.12. a) $\mathrm{s}(-)$ *maps* $D^b_{\mathrm{perf}}(R\text{-mod-}n)$ *to* $D_{\mathrm{perf}}(F\text{-crys-}n)$. b) *For* $M \in D^b_{\mathrm{perf}}(R\text{-mod-}n)$, $N \in D^b_{\mathrm{perf}}(R\text{-mod-}n)$ $\mathrm{s}(M \hat{*}^L_R N) = \mathrm{s}(M)\hat{\otimes}^L_{W\mathcal{O}}\mathrm{s}(N)$ *where* $(-)\hat{\otimes}^L_{W\mathcal{O}}(-) := ((-)\otimes^L_{W\mathcal{O}}(-))\hat{}$ *and* $\mathrm{s}(R\,\underline{\mathrm{Hom}}^!_R(M, N)(-m)) = R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(\mathrm{s}(M), \mathrm{s}(N))$ *in* $D(F\text{-crys-}(m+n))$.

Indeed, a) follows from Prop. 7.10. By the definition of $(-)*_R(-)$ we have a morphism $(-)\otimes_{W\mathcal{O}}(-) \to (-)*_R(-)$ and this gives

$$\mathrm{s}(-)\otimes_{W\mathcal{O}}\mathrm{s}(-) = \mathrm{s}((-)\otimes_{W\mathcal{O}}(-)) \to \mathrm{s}((-)*_R(-)) \tag{7.12.1}$$

and therefore a morphism $\mathrm{s}(M)\otimes^L_{W\mathcal{O}}\mathrm{s}(N) \xrightarrow{\varphi} \mathrm{s}(M *^L_R N)$ which clearly will be a morphism in $D(F\text{-crys-}(m+n))$. Applying $W\mathcal{O}/pW\mathcal{O}\otimes^L_{W\mathcal{O}}(-)$ to φ and using (7.10) and (4.7) give $\mathrm{s}(R_1\otimes^L_R M)\otimes^L \mathrm{s}(R_1\otimes^L_R N) \to \mathrm{s}((R_1\otimes^L_R M)\otimes^L(R_1\otimes^L_R N))$ which clearly is an isomorphism. The exact sequences $0 \to W\mathcal{O}/p \to W\mathcal{O}/p^{n+1} \to W\mathcal{O}/p^n \to 0$ show that $W.\mathcal{O}\otimes^L_{W\mathcal{O}}(\varphi)$ gives an isomorphism and thus applying $R\varprojlim(-)\mathrm{s}(M)\hat{\otimes}^L_W \mathrm{s}(N) = \mathrm{s}(M *^L_R N)\hat{} = \mathrm{s}(M \hat{*}^L_R N)$, the last step by (7.10 ii)). A morphism

$$\mathrm{s}(R\,\underline{\mathrm{Hom}}^!_R(M, N)(m)) \to R\,\underline{\mathrm{Hom}}_W(\mathrm{s}(M), \mathrm{s}(N)) \tag{7.12.2}$$

is defined, using (7.8), as the "adjoint" of $\mathrm{s}(-)$ applied to the evaluation mapping $M \hat{*}^L_R R\,\underline{\mathrm{Hom}}^!_R(M, N)(m) \to N(m)$ composed with $\mathrm{s}(M)\hat{\otimes}^L_W \mathrm{s}(R\,\underline{\mathrm{Hom}}^!_R(M, N)(m)) = \mathrm{s}(M \hat{*}^L_R R\,\underline{\mathrm{Hom}}^!_R(M, N)(m))$. $W\mathcal{O}/p\otimes^L_{W\mathcal{O}}(-)$ applied to (7.12.2) gives by (7.10) and (5.7) an isomorphism and (7.12.2) therefore becomes an isomorphism after completion. By (5.9), a) and the analogue of (5.9) for $R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$ both sides are already complete and (7.12.2) is an isomorphism.

Remark: The interested reader will easily find analogues of (6.2) for $(-)\hat{\otimes}^L_{W\mathcal{O}}(-)$ and $R\,\underline{\mathrm{Hom}}_{W\mathcal{O}}(-, -)$.

8. Later we will need the following result.

Lemma 8.1. *Let* $(R.\otimes^L_R M)^F$, $M \in D(R)$, *denote the complex of prosystems* $\{\mathrm{Cone}\,(R_{n+1}\otimes^L_R M \xrightarrow{F-1} R_n\otimes^L_R M)\}$ *and* $(R/p^{\cdot}\otimes^L_R M)^F$ *the complex of prosystems* $\{\mathrm{Cone}\,(R/p^{n+1}\otimes^L_R M \xrightarrow{F-1} R/p^{n+1}\otimes^L_R M)\}$. *The projection* $(R/p^{\cdot}\otimes^L_R M) \xrightarrow{F} (R.\otimes^L_R M)^F$ *is an isomorphism in* $D(\text{pro-}S\text{-ab})$.

Indeed, we reduce first to M being a free R-module and then, as the problem is stable under internal copowers, we may assume that M is R. Note also that in

$D(\text{pro-}S\text{-ab})$ $(R/p^{\cdot}\otimes_R^L M)^F=\{\mathrm{Cone}\,(R/p^{n+1}\otimes_R^L M \xrightarrow{F-1} R/p^n\otimes_R^L M)\}$. As F and 1 are of degree 0 we may consider the degrees separately. In degree 0 $R_n^0=R^0/V^n$ and $R^0/p^n\to R_n^0$ is surjective. Hence we need to prove that $F-1$ is a proisomorphism on $\mathrm{Ker}:=\mathrm{Ker}\,(R^0/p^{\cdot}\to R^0/V^{\cdot})$. As V is injective on R and $VF=p$ this system is isomorphic to $\{\ldots\to R^0/F^{n+1}\xrightarrow{V} R^0/F^n\to\ldots\}$. $F-1$: $\mathrm{Ker}_{n+1}\to\mathrm{Ker}_n$ lifts to $F-1$: $\mathrm{Ker}_{n+1}\to\mathrm{Ker}_{n+1}$ and the two cones are proisomorphic so it suffices to consider the last morphism, but F is nilpotent on R^0/F^{n+1} and thus $F-1$ is an isomorphism. In degree 1 it is clear that $F-1$ is injective on R/p^n and hence on $\mathrm{Ker}\,(R/p^{\cdot}\to R.)$. On the other hand it is easily seen from (0: 5.1) that $W\mathcal{O}dV^n\to R$ induces a surjection $W_n\mathcal{O}dV^n\to(\mathrm{Ker}\,(R/p^{\cdot}\to R.)^1/F-1)_n$ and thus that $\mathrm{Ker}\,(R/p^{\cdot}\to R.)^1/F-1$ is the zero system.

9. Let me finally say a few words on functoriality. If f: $(S,\mathcal{O}_S)\to(S',\mathcal{O}_{S'})$ is a morphism of ringed topoi where both $\mathcal{O}_S$ and $\mathcal{O}_{S'}$ are perfect rings of characteristic p and both S and S' are of finite cohomological dimension we have functors Lf_n^*: $D^-(W_n\mathcal{O}_{S'})\to D^-(W_n\mathcal{O}_S)$ (resp. $D^-(R_{S'})\to D^-(R_S)$). It is further clear that $Lf_n^*(R_n\otimes_R^L(-))=R_n\otimes_R^L(Lf^*(-))$ and

$$Lf^*((-)*_R^L(-)) = Lf^*(-)*_R^L Lf^*(-). \quad \text{If we put} \quad L\hat{f}^*(-) := (Lf^*(-))^{\wedge}$$

this shows that $L\hat{f}^*$ takes $D^b_{\mathrm{perf}}(R_{S'})$ to $D^b_{\mathrm{perf}}(R_S)$ once it is known that Lf^* preserves boundedness to the right (cf. (4.9)). This is clear because it is easily seen that $R_S=W\mathcal{O}_S\otimes_{f^{-1}W\mathcal{O}_{S'}}f^{-1}R_{S'}$ which implies that $W\mathcal{O}_{S'}$-flat $R_{S'}$-modules are f^*-acyclic which in turn implies that $W\mathcal{O}_{S'}[F,F^{-1}][X]$ is f^*-acyclic and we conclude by (4.4). (This also shows that Lf^* preserves $D^-(R\text{-mod-}n)$). We get furthermore morphisms

$$\begin{array}{ccc}
Lf^*(-)*_R^L Lf^*(-) & = & Lf^*((-)*_R^L(-)) \\
\downarrow & & \downarrow \\
L\hat{f}^*(-)*_R^L L\hat{f}^*(-) & & Lf^*((-)\,\hat{*}_R^L(-))
\end{array}$$

Applying $R_1\otimes_R^L(-)$ gives us nothing but isomorphisms and we thus get isomorphisms after completing. Hence $L\hat{f}^*(-)*_R^L L\hat{f}^*(-)=L\hat{f}^*((-)*_R^L(-))$. This shows that $L\hat{f}^*$ is a tensor functor between rigid tensor categories and [5: II, Prop. 1.9] shows that $L\hat{f}^*(R\,\underline{\mathrm{Hom}}_R^!(-,-))=R\,\underline{\mathrm{Hom}}_R^!(L\hat{f}^*(-),L\hat{f}^*(-))$.

II. The Künneth and duality formulas

0. In this chapter S will be a perfect scheme whose underlying topological space is locally Noetherian of finite Krull-dimension and hence, by [12: Thm. 3.6.5], its Zariski-topos will be of finite cohomological dimension.

Lemma 0.1. a) *Let $M \in D(R)$ be bounded in one direction and suppose that $R_1 \otimes_R^L M \in D_{qc}(\mathcal{O}_S)$. Then $R_n \otimes_R^L M \in D_{qc}(W_n \mathcal{O}_S)$. for all n.*

b) *Suppose that $M \in D(R)$ is bounded to the left or from above, that $H^i(R_1 \otimes_R^L M) = 0$ for $i > N$ and $R_1 \otimes_R^L M \in D_{qc}(\mathcal{O}_S)$. Then $H^i(\hat{M}) = 0$ for $i > N$.*

Proof: Let A be the category of $W\mathcal{O}_S$-modules killed by some power of p, p^n, and quasi-coherent as $W_n\mathcal{O}_S$-modules. Then A fulfills the condition of (I: Prop. 1.1) and a) follows immediately from this. As for b) it follows from (I: Prop. 1.1) that $H^i(R_n \otimes_R^L M) = H^i(gr^n \otimes_R^L M) = 0$ for $i > N$ and that $H^j(R_n \otimes_R^L M)$ is quasi-coherent for all j. Thus $\varprojlim^i (H^*(R. \otimes_R^L M)) = 0$ for $i > 1$ by (0: 4.6). Furthermore, as $H^{N+1}(gr^n \otimes_R^L M) = 0$, the transition maps $H^N(R_{n+1} \otimes_R^L M) \to H^N(R_n \otimes_R^L M)$ are surjective, which together with quasi-coherence and [6: Thm. 1.3.1] implies that $\varprojlim^1 (H^N(R. \otimes_R^L M)) = 0$. b) now follows from the spectral sequence

$$\varprojlim^i (H^j(R. \otimes_R^L M)) \Rightarrow H^{i+j}(\hat{M}).$$

Theorem 1.1. i) *Let $X \xrightarrow{f} S$, $Y \xrightarrow{g} S$ be schemes smooth over S and suppose that either $Rf_* W\Omega^{\cdot}_{X/S}$ or $Rg_* W\Omega^{\cdot}_{X/S}$ is bounded from above. Then*

$$Rf_* W\Omega^{\cdot}_{X/S} \hat{*}_R^L Rg_* W\Omega^{\cdot}_{X/S} = R(f x_S g)_* W\Omega^{\cdot}_{X x_S Y/S}. \tag{1.2.1}$$

ii) *Let $X \xrightarrow{f} S$ be smooth and proper of relative dimension N. Then $Rf_* W\Omega^{\cdot}_{X/S} \in D^b_{\mathrm{perf}}(R\text{-mod-}N)$ and*

$$D(Rf_* W\Omega^{\cdot}_{X/S})[-N](-N) = Rf_* W\Omega^{\cdot}_{X/S}. \tag{1.2.2}$$

Indeed, let us first define (1.2.1). The relations (0: 7.1) show that for open $U \subseteq X$, $V \subseteq Y$ multiplication in $W\Omega^{\cdot}_{U x_S V/S}$ induces a mapping $W\Omega^{\cdot}_{X/S}(U) *_R W\Omega^{\cdot}_{Y/S}(V) \to W\Omega^{\cdot}_{X x_S Y/S}(U x_S V)$. Hence, taking affine hypercoverings $U^{\cdot}$ and $V^{\cdot}$ of X resp. Y and an affine refinement $W^{\cdot}$ of $U^{\cdot} x_S V^{\cdot}$ we get $\Gamma(U^{\cdot}, W\Omega^{\cdot}_{X/S}) *_R \Gamma(V^{\cdot}, W\Omega^{\cdot}_{Y/S}) \to \Gamma(W^{\cdot}, W\Omega^{\cdot}_{X x_S Y/S})$. Using that $\Gamma(U^{\cdot}, W\Omega^{\cdot}_{X/S}) = R\Gamma(X, W\Omega^{\cdot}_{X/S})$ etc. and sheafifying we get $Rf_* W\Omega^{\cdot}_{X/S} *_R Rg_* W\Omega^{\cdot}_{Y/S} \to R(f x_S g)_* W\Omega^{\cdot}_{X x_S Y/S}$. Deriving and completing and using (cf. (0: 5.11)) that $R(f x_S g)_* W\Omega^{\cdot}_{X x_S Y/S}$ is complete we obtain (1.2.1). As the two complexes involved are complete and bounded to the left to show that (1.2.1) is an isomorphism it suffices to show that $R_1 \otimes_R^L$ (1.2.1) is an isomorphism. Using (0: 7.2) and (I: (4.7)) we conclude by the usual Künneth formula for the Hodge-cohomology. As for ii) the hypercovering argument shows that $Rf_* W\Omega^{\cdot}_{X/S} \in D(R\text{-mod-}N)$ and $R_1 \otimes_R^L Rf_* W\Omega^{\cdot}_{X/S} = Rf_* \Omega^{\cdot}_{X/S}$, which is perfect by [17: 2.5]. We also get from [6: Cor. 4.2.2] and a limit argument that $R^i f_* \Omega^{\cdot}_{X/S}=0$ if $i>N$ and clearly $R^i f_* \Omega^{\cdot}_{X/S}$ is quasi-coherent. By Lemma 0.1 $R^i f_* W\Omega^{\cdot}_{X/S}=0$, $i>N$. The spectral sequence $R^i f_* W\Omega^j_{X/S} \Rightarrow H^{i+j}(\mathrm{s}(Rf_*(W\Omega^{\cdot}_{X/S}))) = R^{i+j} f_*(W/W\mathcal{O}_S)$ shows that $R^{2N} f_*(X/W\mathcal{O}_S) = R^N f_* W\Omega^N_{X/S}/dR^N f_* W\Omega^{N-1}_{X/S}$. From [II: VI, Prop. 1.6] we get a morphism $R^{2N} f_*(X/W\mathcal{O}_S) \to W\mathcal{O}_S$ and hence a morphism of R-modules of level N Tr: $R^N f_* W\Omega^{\cdot}_{X/S} \to W\mathcal{O}_S(-N)$. As $R^i f_* W\Omega^{\cdot}_{X/S}=0$ if $i>N$ we obtain a morphism in $D(R\text{-mod-}N)$

(1.2.3) $$\mathrm{Tr}\colon\ Rf_* W\Omega^{\cdot}_{X/S} \to W_{\mathcal{O}_S}(-N)[-N].$$

From (1.2.1) we get the internal product $Rf_* W\Omega^{\cdot}_{X/S} \hat{*}^L_R Rf_* W\Omega^{\cdot}_{X/S} \to Rf_* W\Omega^{\cdot}_{X/S}$ and composing with Tr we get the pairing

(1.2.4) $$Rf_* W\Omega^{\cdot}_{X/S} \hat{*}^L_R Rf_* W\Omega^{\cdot}_{X/S} \to W_S(-N)[-N].$$

Taking the adjoint gives us (1.2.2). As all complexes involved are bounded in all directions and complete, to show that (1.2.2) is an isomorphism it suffices to show that $R_1 \otimes^L_R$ (1.2.2) is. By (0: 7.2) and (I: 5.7) this is simply the duality morphism for Hodge cohomology which is proved to be an isomorphism in [17: 2.5].

Corollary 1.2.5. *Let f and g be affine. Then* $f_* W\Omega^{\cdot}_{X/S} \hat{*}^L_R g_* W\Omega^{\cdot}_{Y/S} = (f \times_S g)_* W\Omega^{\cdot}_{X \times_S Y/S}$.

Remark: It is clear that (1.2) admits many variants. One could for instance throw in finite groups acting on f and g and (1.2.1) resp. (1.2.2) would then be equivariant. One could also replace $W\Omega^{\cdot}_{X/S}$ and $W\Omega^{\cdot}_{Y/S}$ by $W\Omega_{X/S} \otimes E$ resp. $W\Omega^{\cdot}_{Y/S} \otimes F$ where E and F are unit root crystals (cf. [7]).

III. Consequences and some calculations

0. In this chapter, unless otherwise mentioned, S will be the spectrum of a perfect field and X and Y will be smooth and proper S-schemes.

1. We will now convince ourselves that the results of [16: I—II] remain valid in a slightly more general context.

Proposition 1.1. i) *Let $M \in D^-(R)$. Then $M \in D^-_c(R)$ iff $H^i(M)$ is a coherent R-module for all i.*

ii) *Let $M \in D^b_c(R)$. In the spectral sequence*

$$E_1^{j,i} = H^i(M)^j \Rightarrow H^{i+j}(\mathrm{s}(M)) B^{i,j}_\infty \subseteq F^\infty BH^j(M) \subseteq V^{-\infty} ZH^j(M)^i \subseteq Z^{i,j}_\infty$$

("survie du coeur").

iii) *Let $M \in D^-(W)$. Then $M \in D^-_{\mathrm{perf}}(W)$ iff $H^i(M)$ is a finitely generated W-module for all i.*

The proof of i) and ii) is, up to a change of notation, the same as the proof of [16: II, Thm. 2.2, Thm. 3.4]. I will do this change for i) and leave it to the reader in the case of ii). (For a different proof of ii) see [9].) From [16: I, Cors, 3.6-7] it follows that a coherent R-module is in $D_c(R)$ considered as a complex concentrated in degree 0. As $D_c(R)$ is a triangulated subcategory of $D(R)$ we get one implication and for the other we may assume that $H^i(M)=0$ for $i>r$ and that we want to show that $H^r(M)$ is coherent. By assumption $H^i(R_n\otimes_R^L M)$ is of finite length and hence $H^i(M)=\varprojlim(H^i(R.\otimes_R^L M))$. (I: (1.1)) then shows that $H^r(M)$ is bounded horizontally. The spectral sequences

$$\text{(1.1.1)} \qquad \mathrm{Tor}_{-i}^R(R_n, H^j(M)) \Rightarrow H^{i+j}(R_n\otimes_R^L M)$$

show that $R.\otimes_R H^r(M)=H^r(R.\otimes_R^L M)$ and hence that $H^r(M)=\varprojlim(R.\otimes_R M)$ and $R_n\otimes_R M$ is of finite length for all n. In the terminology of [16] this says that $H^r(M)$ is a profinite R-module. (1.1.1) shows that $\mathrm{Tor}_1^R(R_n, H^r(M))$ is of finite length and in [16: I, Thm. 3.8] it is shown that a profinite module N, bounded in degrees, with $\mathrm{Tor}_1^R(R_1, N)$ of finite length is coherent. iii) is well known and may be proved in a way similar to i).

Proposition 1.2. i) *If $M, N\in D_c^b(R)$ and F is bijective on $H^*(M)$, then $M\hat{*}_R^L N=M\otimes_W^L N$.*

ii) If $N\in D_{\mathrm{perf}}^b(F\text{-crys-}n)$, $M\in D_{\mathrm{perf}}^b(F\text{-crys-}n)$, then $M\hat{\otimes}_W^L N=M\otimes_W^L N$.

To see i) note first that by (I: 3.3, 4.2) $M*_R^L N=M\otimes_W^L N$ so it only remains to show that $M*_R^L N=M\otimes_W^L N$ is complete. Clearly if P is an R-module with $d=0$ and F bijective and Q is any R-module then $R_n\otimes_R(P\otimes_W Q)=P\otimes_W(R_n\otimes_R Q)$. Taking $*$-flat resolutions in both variables we get $R.\otimes_R^L(M\otimes_W^L X)=M\otimes_W^L(R.\otimes_R^L X)$ for $X\in D^-(R)$. This shows that the question of completeness only depends on the structure of $H^*(M)$ as W-module. It is clear, however, that $H^i(M)$ is finitely generated as W-module (cf. [16: II: Cor. 3.8]). By dévissage we are thus reduced to $M=W$ where it is obvious. ii) is shown similarly.

Let $(S_{\mathrm{perf}}, \mathcal{O}_{S_{\mathrm{perf}}})$ be the ringed topos of sheaves on the étale site of perfect S-schemes ringed by the structure sheaf and let $f\colon (S_{\mathrm{perf}}, \mathcal{O}_{S_{\mathrm{perf}}})\to(S, \mathcal{O}_S)$ be the natural projection. Put $(-)^F:=(R.\otimes_R^L Lf^*(-))^F\colon D_c^b(R_S)\to D^b(\text{pro-}S_{\mathrm{perf}})$ (cf. (I: 9.1) and [16: IV, 3.6.3]).

Lemma 1.3. i) *$(-)^F$ has amplitude* $[0, 0]$.

ii) *If $M\in D_c^b(R)$ then $H^*(M^F)$ is a pro-quasi-algebraic group of finite type whose connected part is essentially constant.*

Indeed, by [16: I: Cors. 3.6-7], $R.\otimes_R^L M=R.\otimes_R M$ in $D(\text{pro-}S)$ if M is a coherent R-module and by [16: IV, 3.8, 3.10] $F-1$ is surjective, in pro-S, on $f^*_.(R.\otimes_R M)$. As f and hence $f_.$ is flat we get $M^F=(Lf^*_.(R.\otimes_R^L M))^F=\mathrm{Ker}\, F-1$:

$f_.^*(R.\otimes_R M)=\underline{M}^F$ in the terminology of [16: IV, 3.6.3). i) is now clear and ii) follows from [16: IV, 3.11].

Lemma 1.4. *Let* $k\subseteqq k'$ *be an inclusion of perfect fields and* $f\colon S'\to S$ *the corresponding morphism of schemes.* $L\hat{f}^*\colon D_c^b(R_S)\to D_c^b(R_{S'})$ *is a conservative functor of amplitude* $[0,0]$.

We have $L\hat{f}^*=R\varprojlim(Lf_.^*(R.\otimes_R^L(-)))$ and $R.\otimes_R^L(-)$ has amplitude $[0,0]$ on $D_c^b(R)$ by [16: I, Cors. 3.6—7], $f.$ is flat as f is and $R\varprojlim$ has amplitude $[0,0]$ on systems whose cohomology has components of finite length. This shows that $L\hat{f}^*$ has amplitude $[0,0]$ and it is conservative as $Lf^*\colon D(\mathcal{O}_S)\to D(\mathcal{O}_{S'})$ is (cf. (I: 1.1.3)).

Lemma 1.5. *The following sequences are exact for* S *a perfect scheme:*

$$0\to\hat{R}\xrightarrow{F^i-V^j}\hat{R}\to E_{j/i+j}\to 0,\quad i>0 \tag{1.5.1}$$

$$0\to\hat{R}\xrightarrow{(V,-F)}\hat{R}\oplus\hat{R}\xrightarrow{F+V}\hat{R}\to k\to 0 \tag{1.5.2}$$

$$0\to\hat{R}(-1)\xrightarrow{(V,-dV)}\hat{R}(-1)\oplus\hat{R}\xrightarrow{Fd+F}\hat{R}\to\mathbf{U}_0\to 0 \tag{1.5.3}$$

where F^i-V^j *etc. denote multiplication to the right.*

Indeed, (1.2.1) is well known in degree 0 so it suffices to show that F^i-V^j is bijective in degree 1. It follows from (0: 5.1) that V is bijective in degree 1 and that F^i is topologically nilpotent. Hence $F^i-V^j=V^j(V^{-j}F^i-1)$ is bijective. (1.5.2) is the simple complex associated to

$$\begin{tikzcd} \hat{R} \arrow[r, "F"] \arrow[d, "V"] & \hat{R} \arrow[d, "V"] \\ \hat{R} \arrow[r, "F"] & \hat{R}\,. \end{tikzcd}$$

It therefore suffices to show that F is bijective on ${}_V\hat{R}$, injective on $\hat{R}/V$ and that $\hat{R}/(F,V)=k$. However, V is injective on $\hat{R}$ and bijective in degree 1 and it suffices thus to show that F is injective on $(\hat{R}/V)^0$ and $(\hat{R}/(F,V))^0=k$. This is clear from (0: 5.1). It follows from (0: 5.1) that $\mathbf{U}_0=\hat{R}/(Fd,F)$. Exactness at the other places is shown exactly as [16: I, 3.2]. (Indeed, the isomorphism $R\to R^{\mathrm{op}}$ taking (F,d,V) to (V,d,F) transforms (1.5.3) to the completed version of (0: 5.4.9).)

If M is complete (I: 5.9.3 ii) shows that $R\operatorname{Hom}_R(\hat{R},M)=R\operatorname{Hom}_R(R,M)=M$ and hence

Corollary 1.5.4. *Let* $M\in D_c^b(R)$.

i) $R\,\underline{\operatorname{Hom}}_R(E_{j/i+j},M)=\mathbf{s}(0\to M\xrightarrow{F^i-V^j}M\to 0)\quad i>0,$

ii) $R\,\underline{\operatorname{Hom}}_R(k,M)=\mathbf{s}(0\to M\xrightarrow{(V,-F)}M\oplus M\xrightarrow{F+V}M\to 0),$

iii) $R\,\underline{\operatorname{Hom}}_R(\mathbf{U}_0,M)=R_1\otimes_R^L M[-2](1).$

Proposition 1.6. *The functor* $D(-)$: $D^b_c(R)^{\mathrm{op}} \to D^b_c(R)$ *coincides with the one of* [8].

This follows from the remark following (I: 5.8).

2. Recall that X is said to be ordinary if F is bijective on $R^*\Gamma(W\Omega^{\cdot}_X)$ and that X is said to be Hodge—Witt if $R^*\Gamma(W\Omega^{\cdot}_X)$ is finitely generated as W-module (cf. [16: IV, 4.6, 4.12]). Note that an ordinary variety is Hodge—Witt.

Proposition 2.1. i) *If X is ordinary then*

$$R\Gamma(W\Omega^{\cdot}_{X \times Y}) = R\Gamma(W\Omega^{\cdot}_X)\otimes^L_W R\Gamma(W\Omega^{\cdot}_Y).$$

ii) *(cf.* [16: IV, 4.14]*) if X and Y are ordinary then so is $X\times Y$.*
iii) *If X is ordinary and Y is Hodge—Witt then $X\times Y$ is Hodge—Witt.*

Indeed, i) follows from (II: Thm. 1.1) and (1.2 i) and ii) and iii) follow from i).

Remark: We will see later that iii) is the only case where $X\times Y$ is Hodge—Witt.

Corollary 2.1.2. *If X is ordinary there is a canonical, (non-canonically) split, short exact sequence*

(1.2.2)
$$0 \to \bigoplus_{j=j_1+j_2} R^{j_1}\Gamma(W\Omega^{\cdot}_X)\otimes_W R^{j_2}\Gamma(W\Omega^{\cdot}_Y) \to R^j\Gamma(W\Omega^{\cdot}_{X\times Y})$$
$$\to \bigoplus_{j+1=j_1+j_2} \mathrm{Tor}^W_1(R^{j_1}\Gamma(W\Omega^{\cdot}_X),\ R^{j_2}\Gamma(W\Omega^{\cdot}_Y)) \to 0.$$

3. Definition 3.1. For $M, N\in D^-(R)$ put

(3.1.1)
$$\mathrm{Kün}^R_i(M, N) := H^{-i}(M\,\hat{*}^L_R N).$$

Proposition 3.2. i) *If M, N are coherent R-modules then* $\mathrm{Kün}^R_i(M, N)$ *is a coherent R-module for every i and* 0 *if $i<0$.*
ii) *For $M, N\in D^b_c(R)$ there is a spectral sequence of R-modules*

(3.2.1)
$$E^{i,j}_2 = \bigoplus_{j=j_1+j_2} \mathrm{Kün}^R_{-i}(H^{j_1}(M),\ H^{j_2}(N)) \Rightarrow \mathrm{Kün}^R_{i+j}(M, N).$$

iii) *There is a spectral sequence of R-modules*

(3.2.2)
$$\bigoplus_{j=j_1+j_2} \mathrm{Kün}^R_{-i}(R^{j_1}\Gamma(W\Omega^{\cdot}_X),\ R^{j_2}\Gamma(W\Omega^{\cdot}_Y)) \Rightarrow R^{i+j}\Gamma(W\Omega^{\cdot}_{X\times Y}).$$

Indeed, the first part of i) follows from (I: Thm. 6.2) and Prop. 1.1. The second part will follow from the proof of ii) and iii) follows from (II: 1.1) and ii). Let us therefore prove ii). If F_1 and F_2 are free R-modules then $F_1 *_R F_2$ is also free by (I: 3.2). Hence $R.\otimes^L_R(F_1 *_R F_2) = R.\otimes_R(F_1 *_R F_2)$ and as $R.\otimes_R M$ has surjective transition maps for any R-module M $(F_1 *_R F_2)\hat{} = \varprojlim (R.\otimes_R(F_1 *_R F_2))$. Thus $F_1\hat{*}^L_R F_2 = \mathrm{Kün}^R_0(F_1, F_2)$. This shows that $M\hat{*}^L_R N$ may be computed by taking free resolutions $F^\cdot$ and $G^\cdot$ of M resp. N and then taking the simple complex associated to the double complex whose components are the $\mathrm{Kün}_0^R(F^i, G^j)$. ii) now follows from [4: XVII, 2].

Remark: Even though $(-)*_R(-)$ is, $\mathrm{Kün}_0^R(-,-)$ no doubt is not, in the terminology of [4: V, 8], left balanced as a bifunctor on R-modules.

4. We will now attempt to compute $\mathrm{Kün}_*^R(M, N)$ for some coherent R-modules M and N. To do this we will on the one hand use the spectral sequence which follows from (I: 7.12 b) and (1.2 ii)

$$\mathrm{Kün}_{-i}^R(M, N)^j \Rightarrow \mathrm{Tor}^W_{-(i+j)}(\mathbf{s}(M), \mathbf{s}(N)) \tag{4.1}$$

and on the other hand that we can compute $D(M\hat{*}_R^L N)$ and $(M\hat{*}_R^L N)^F$. Namely, by (I: Thm. 6.2) we get

$$D(M\hat{*}_R^L N) = D(M)\hat{*}_R^L D(N) \tag{4.2}$$

and if $f\colon S_{\mathrm{perf}} \to S$ is the natural projection we get from (I: 6.2, 9.1, 10) and Cor. 1.5.4 i):

$$(M\hat{*}_R^L N)^F = \mathrm{Cone}\,(F-1, \mathbf{Z}/p^\cdot \otimes_{\mathbf{Z}}^L L\hat{f}^*(M\hat{*}_R^L N)) = \mathbf{Z}/p^\cdot \otimes_{\mathbf{Z}}^L R\,\underline{\mathrm{Hom}}_R$$

$$(W\mathcal{O}_{S_{\mathrm{perf}}}, R\,\underline{\mathrm{Hom}}_R^!\,(L\hat{f}^*D(M)), L\hat{f}^*N) = \mathbf{Z}/p^\cdot \otimes_{\mathbf{Z}}^L R\,\underline{\mathrm{Hom}}_R\,(L\hat{f}^*D(M), L\hat{f}^*N).$$

We will compute the last term using Cor. 1.5.4. In fact, I will only compute the geometric points i.e. I will assume that k is algebraically closed and compute $\mathrm{Ext}_R^*(D(M), N)$ and then identify the pro-quasi-algebraic group. With some abuse of language we may then write

$$\mathrm{Kün}_{-i}^R(M, N)^F = \mathrm{Ext}_R^i(D(M), N) \tag{4.3}$$

using that $(-)^F$ is, by Lemma 1.3 i), exact and hence $H^i(P^F) = H^i(P)^F$ for $P \in D_c^b(R)$. I leave to the reader the task of making the computations in S_{perf} and hence justifying my alleged identifications.

Recall (cf. [16: IV, 3.11]) that a shifted domino in degree i and $i+1$ gives rise, through $(-)^F$, to a connected quasi-algebraic group in degree $i+1$ whose dimension is equal to the dimension of the domino, that semi-simple torsion gives rise to finite quasi-algebraic groups whose (graded) rank as such is equal to the (graded) length of the module, that slope zero gives rise to finite type torsion free étale pro-quasi-algebraic groups whose rank as such equals the W-rank of the module and that all other parts of coherent R-modules give rise to the zero pro-quasi-algebraic group.

Let now E and F be coherent R-modules of positive slope (cf. (0: 8)) concentrated in degree 0. By (0: 9.1) $D(E) = E^\vee[-1]\,(1)$, $D(F) = F^\vee[-1]\,(1)$ $(E^\vee := \mathrm{Hom}_W(E, W)$, $F^\vee := \mathrm{Hom}_W(F, W))$. Hence, using (4.2), $D(E\hat{*}_R^L F) = E^\vee \hat{*}_R^L F^\vee[-2]\,(2)$. This implies, by (0: 9.1), that $\mathrm{Kün}_i^R(E, F) = 0$, $i > 2$, as the different parts would be dual to parts in $\mathrm{Kün}_j^R(E^\vee, F^\vee)$ for negative j:s which are zero by (3.2 i). By the same reasoning $\mathrm{Kün}_2^R(E, F)$ consists solely of slope 0 and $\mathrm{Kün}_1^R(E, F)$ is of finite type as W-module (cf. [8: Rem. foll. IV, 3.5.1]).

We know from (I: 7.4) that $\mathrm{Kün}_*^R(E, F)$ is of level 2 and hence that it is zero in degrees >2 and <0 and that F is bijective in degree 2. Furthermore, $\mathbf{s}(E)\otimes_W^L \mathbf{s}(F)= E\otimes_W F$ which is an F-crystal of slopes <2. This implies by (4.1), "survie du coeur" and what has already been proved that $\mathrm{Kün}_2^R(E, F)=0$, $\mathrm{Kün}_1^R(E, F)$ is 0 except in degree 1 and $\mathrm{Kün}_1^R(E, F)^1 \subseteq E\otimes_W F$, $\mathrm{Kün}_0^R(E, F)$ is 0 except in degree 0 and 1 and $\mathrm{Kün}_0^R(E, F)^1 = d\,\mathrm{Kün}_0^R(E, F)^0$. In particular, $\mathrm{Kün}_1^R(E, F)$ is torsion free being a subgroup of $E\otimes_W F$.

Let us now further assume that $E=F=E_{1/2}$. We have $(E_{1/2}\hat{*}_R^L E_{1/2})^F = R\,\mathrm{Hom}_R\big(D(E_{1/2}), E_{1/2}\big) = R\,\mathrm{Hom}_R(E_{1/2}, E_{1/2})\ [1]\ (-1)$. (1.5.4 i) shows that $\mathrm{Hom}_R(E_{1/2}, E_{1/2}) = W(\mathbf{F}_{p^2})\cdot 1 \oplus W(\mathbf{F}_{p^2})F$ and $\mathrm{Ext}_R^1(E_{1/2}, E_{1/2}) = W\cdot 1/pW\cdot 1 \xrightarrow{\sim} G_a^{\mathrm{perf}}$. This shows that $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ contains a one-dimensional domino and no semi-simple torsion. As $E_{1/2}\otimes_W E_{1/2}$ is of slope 1, (4.1) shows that $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ is torsion and $\mathrm{Kün}_1^R(E_{1/2}, E_{1/2})^1$ is of slope zero, of rank 4 as $E_{1/2}\otimes_W E_{1/2}$ is. I now claim that $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0/V \xrightarrow{\sim} k$. Indeed, from (I: 6.2 viii) we get for $M, N\in D_c^b(R)$:

$$\text{(4.4)}\quad \mathrm{Tor}_{-i}^R\big(R_1, \mathrm{Kün}_{-j}^R(M, N)\big) \Rightarrow \bigoplus_{i+j=j_1+j_2} \mathrm{Tor}_{-j_1}^R(R_1, M)\otimes_k \mathrm{Tor}_{-j_2}^R(R_1, N).$$

Note that this is a spectral sequence of $k[d]$-modules. In particular, $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0/V = \mathrm{Tor}_0^R\big(R_1, \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})\big)^0 = \mathrm{Tor}_0^R(R_1, E_{1/2})\otimes_k \mathrm{Tor}_0^R(R_1, E_{1/2}) = E_{1/2}/V\otimes_k E_{1/2}/V \xrightarrow{\sim} k$. As $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})/V\text{-tors}$ is V-torsion free $\dim_k \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0/V = \dim_k V\text{-tors}/V + \dim_k \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0/V\text{-tors}/V$ but the domino part contributes 1 to the last term so V-tors$/V$ and hence V-tors itself is zero. Therefore $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ is a 1-dimensional domino and by (0: 8) isomorphic to $\mathbf{U}_i$ for some i. As it has already been observed d is surjective in degree 1 which shows that $i\geq 0$. Clearly $E_\infty^{0,0} = \mathrm{Ker}\big(\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0 \xrightarrow{d} \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})\big)$ in the spectral sequence (4.1) and it also clearly equals $\mathrm{Im}\big(E_{1/2}\otimes_W E_{1/2} \to \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0\big)$. We have a morphism of spectral sequences from (4.1) to

$$\text{(4.5)}\qquad \mathrm{Tor}_{-i}^R(R_1, M *_R^L N)^j \Rightarrow \mathrm{Tor}_{-(i+j)}^W\big(k, \mathbf{s}(M)\otimes_W^L \mathbf{s}(N)\big)$$

and hence $E_{1/2}\otimes_W E_{1/2} \xrightarrow{\varphi} \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ is a factor of $E_{1/2}\otimes_W E_{1/2} \to R_1\otimes_R E_{1/2}\otimes_k R_1\otimes_R E_{1/2} \xrightarrow{\sim} k$ and so is non-zero. On the other hand, φ factors through $E_{1/2} *_R E_{1/2}$ and $E_{1/2} *_R E_{1/2} \to \mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ is an R-morphism. As $FE_{1/2} = VE_{1/2}$ we get $\mathrm{Im}(VE_{1/2} *_R E_{1/2})^0 = V\,\mathrm{Im}(E_{1/2} *_R FE_{1/2})^0 = V\,\mathrm{Im}(E_{1/2} *_R VE_{1/2})^0 = V^2\,\mathrm{Im}(FE_{1/2} *_R E_{1/2})^0 = \ldots \bigcap_n V^n\,\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})^0$ which is 0 as $\mathrm{Kün}_0^R(E_{1/2}, E_{1/2})$ being a coherent R-module is separated in the V-topology. Thus $\mathrm{Im}(VE_{1/2}\otimes_W E_{1/2})$ and similarly $\mathrm{Im}(E_{1/2}\otimes_W VE_{1/2})$ is 0 and φ factors through $E_{1/2}/V\otimes_W E_{1/2}/V \xrightarrow{\sim} k$

and we see that the k-dimension of $\operatorname{Im}\varphi$ and therefore of Ker: $\text{Kün}_0^R(E_{1/2}, E_{1/2})^0 \xrightarrow{d} \text{Kün}_0^R(E_{1/2}, E_{1/2})$ is exactly 1. This shows that $i=1$ and in conclusion:

$$\begin{aligned} &\text{Kün}_0^R(E_{1/2}, E_{1/2}) \xrightarrow{\sim} \mathbf{U}_1 \\ (4.6)\qquad &\text{Kün}_1^R(E_{1/2}, E_{1/2}) \xrightarrow{\sim} W^4(-1) \\ &\text{Kün}_i^R(E_{1/2}, E_{1/2}) = 0, \quad i \neq 0, 1. \end{aligned}$$

Remark: If E is a supersingular elliptic curve $R\Gamma(W\Omega_E^{\cdot}) \xrightarrow{\sim} W \oplus E_{1/2}[-1] \oplus W[-1](-1)$ and $R\Gamma(W\Omega_{E\times E}^{\cdot}) \xrightarrow{\sim} W \oplus E_{1/2}^2 \oplus [-1](-1) \oplus W[-2](-2) \oplus E_{1/2} \hat{*}_R^L E_{1/2}[-2]$ and (4.6) gives a computation of the Hodge–Witt cohomology of $E\times E$. Of course, the same argument applied directly to the Hodge–Witt cohomology of $E\times E$ gives the result without using the Künneth-formula, the last argument being replaced by considerations of the cup-product. $H^1(W\mathcal{O}) *_R H^1(W\mathcal{O}) \to H^2(W\mathcal{O})$. The same type of considerations applied to a super-singular Abelian twofold not the product of elliptic curves show that in that case $R^2\Gamma(W\Omega^{\cdot}) \xrightarrow{\sim} \mathbf{U}_2 \oplus W(-2)$ thus giving a direct proof of [15: II, 7. 1 b)].

5. Let us now consider $\mathbf{U}_0 \hat{*}_R^L \mathbf{U}_0$. Note first that by [16: I, 3.7] and (1.5.4 iii) we have

$$\begin{aligned} &R_1 \otimes_R \mathbf{U}_0 \xrightarrow{\sim} \left(k \xrightarrow{\text{id}} k(-1)\right) \\ (5.1)\qquad &\operatorname{Tor}_1^R(R_1, \mathbf{U}_0) = 0 \\ &\operatorname{Tor}_2^R(R_1, \mathbf{U}_0) \xrightarrow{\sim} \left(k(-1) \xrightarrow{\text{id}} k(-2)\right), \end{aligned}$$

$$\begin{aligned} &\operatorname{Hom}_R(\mathbf{U}_0, \mathbf{U}_0) \xrightarrow{\sim} k \oplus k(-1) \\ (5.2)\qquad &\operatorname{Ext}_R^1(\mathbf{U}_0, \mathbf{U}_0) = 0 \\ &\operatorname{Ext}_R^2(\mathbf{U}_0, \mathbf{U}_0) \xrightarrow{\sim} k(1) \oplus k. \end{aligned}$$

Finally, as $\mathbf{s}(\mathbf{U}_0)=0$ (4.1) converges in this case to 0.

By (0:9.1) $D(\mathbf{U}_0)=\mathbf{U}_0[-2](2)$ which implies that $D(\mathbf{U}_0 \hat{*}_R^L \mathbf{U}_0)= \mathbf{U}_0 \hat{*}_R^L \mathbf{U}_0[-4](4)$. By "survie du coeur" there is no heart and by (5.2) and (4.3) we have one 1-dimensional domino and one shift one step to the right of one in $\text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)$, the shift one step to the right plus the shift two steps to the right of 1-dimensional dominoes in $\text{Kün}_2^R(\mathbf{U}_0, \mathbf{U}_0)$ and the rest of the Kün_i^R: s are zero. (4.1) then shows that $H^i(\text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0), d)=0$ if $i\neq 0$. The long exact sequence of cohomology associated to the short exact sequence of complexes:

$$\begin{aligned} 0 \to (\operatorname{dom}^0 \text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0), d) \to (\text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0), d) \to \\ (\operatorname{dom}^1 \text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)(-1), d) \to 0 \end{aligned}$$

shows that $H^*(\operatorname{dom}^1 \text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0), d)=0$ and as $\operatorname{dom}^1 \text{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)$ is a 1-dimensional domino it is necessarily isomorphic to $\mathbf{U}_0$. On the other hand,

$\mathrm{dom}^0\,\mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)$ is isomorphic to $\mathbf{U}_i$ for some i and the long exact sequence from above shows that $i \geqq 0$. If $i > 0$ then $d\colon R_1 \otimes_R \mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)^0 \to R_1 \otimes_R \mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)^1$ is 0 but (5.1) and (4.4) show that it is not. Therefore $i=0$ and $\mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0)$ is an extension of $\mathbf{U}_0$ by $\mathbf{U}_0(-1)$. (5.2) shows that such an extension is trivial so $\mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0) \xrightarrow{\sim} \mathbf{U}_0 \oplus \mathbf{U}_0(-1)$. By duality and using (0:9.1) we get

$$\begin{aligned} &\mathrm{Kün}_0^R(\mathbf{U}_0, \mathbf{U}_0) \xrightarrow{\sim} \mathbf{U}_0 \oplus \mathbf{U}_0(-1) \\ (5.3)\qquad &\mathrm{Kün}_2^R(\mathbf{U}_0, \mathbf{U}_0) \xrightarrow{\sim} \mathbf{U}_0(-1) \oplus \mathbf{U}_0(-2) \\ &\mathrm{Kün}_i^R(\mathbf{U}_0, \mathbf{U}_0) = 0 \quad i \neq 0, 2. \end{aligned}$$

Considering $\mathbf{U}_0 \hat{*}_R^L E$ where E is of positive slope concentrated in degree 0 we see, as $R_1 \otimes_R E = E/VE$, $\mathrm{Tor}_1^R(R_1, E) = E/FE(-1)$ and $\mathrm{Tor}_2^R(R_1, E) = 0$, that $\mathrm{Kün}_0^R(\mathbf{U}_0, E)$ consists of a $\dim_k E/VE$-dimensional domino killed by p. If one admits, which is not hard to show (cf. [10]), that every domino killed by p is a direct sum of $\mathbf{U}_i$: s an argument similar to the one above shows that

$$\begin{aligned} &\mathrm{Kün}_0^R(\mathbf{U}_0, E) \xrightarrow{\sim} \mathbf{U}_0^{\dim_k E/VE} \\ (5.4)\qquad &\mathrm{Kün}_1^R(\mathbf{U}_0, E) \xrightarrow{\sim} \mathbf{U}_0(-1)^{\dim_k E/FE} \\ &\mathrm{Kün}_i^R(\mathbf{U}_0, E) = 0, \quad i \neq 0, 1. \end{aligned}$$

Consider, finally, $\mathbf{U}_0 \hat{*}_R^L k$. The same type of arguments show that

$$\begin{aligned} &\mathrm{Kün}_0^R(\mathbf{U}_0, k) \xrightarrow{\sim} \mathbf{U}_0 \\ (5.5)\qquad &\mathrm{Kün}_1^R(\mathbf{U}_0, k) \xrightarrow{\sim} \mathbf{U}_0 \oplus \mathbf{U}_0(-1) \\ &\mathrm{Kün}_2^R(\mathbf{U}_0, k) \xrightarrow{\sim} \mathbf{U}_0(-1) \\ &\mathrm{Kün}_i^R(\mathbf{U}_0, k) = 0, \quad i \neq 0, 1, 2. \end{aligned}$$

6. Let us now turn to $k \hat{*}_R^L k$. We have from (0: 9.1) that $D(k) = k[-2]\,(1)$. (1.5.4 ii) gives

$$\begin{aligned} &\mathrm{Hom}_R(k, k) = k \\ (6.1)\qquad &\mathrm{Ext}_R^1(k, k) = k^2 \\ &\mathrm{Ext}_R^2(k, k) = k. \end{aligned}$$

This gives that $\mathrm{Kün}_0^R(k, k)$, $\mathrm{Kün}_1^R(k, k)$ and $\mathrm{Kün}_2^R(k, k)$ contain a domino of dimension 1,2 resp. 1 and that the rest is of finite type. We see also that there is no semi-simple torsion.

From [16: I, 3.5] we find that, as $k[d]$-modules;

$$\begin{aligned} &R_1 \otimes_R k = k \\ (6.2)\qquad &\mathrm{Tor}_1^R(R_1, k) = (k \xrightarrow{\mathrm{id}} k(-1)) \\ &\mathrm{Tor}_2^R(R_1, k) = k(-1), \end{aligned}$$

which gives

$$\begin{aligned}
&\operatorname{Tor}_0^R(R_1, k\hat{*}_R^L k) = k\\
&\operatorname{Tor}_1^R(R_1, k\hat{*}_R^L k) = \big(k^2 \xrightarrow{\mathrm{id}} k^2(-1)\big)\\
(6.3)\qquad &\operatorname{Tor}_2^R(R_1, k\hat{*}_R^L k) = \big(k \xrightarrow{(\mathrm{id},-\mathrm{id})} k(-1)^2 \xrightarrow{\mathrm{id}+\mathrm{id}} k(-2)\big)\oplus k(-1)^2\\
&\operatorname{Tor}_3^R(R_1, k\hat{*}_R^L k) = \big(k^2(-1) \xrightarrow{\mathrm{id}} k^2(-2)\big)\\
&\operatorname{Tor}_4^R(R_1, k\hat{*}_R^L k) = k(-2).
\end{aligned}$$

It is clear from (0: 5.4.9) that if M is an R-module concentrated in non-negative degrees then $\operatorname{Tor}_2^R(R_1, M)^0=0$. (4.4) and the fact that $\text{Kün}_i^R(k, k)$ is concentrated in non-negative degrees show that

$$(6.4)\qquad R_1\otimes \text{Kün}_i^R(k, k)^0 \hookrightarrow \operatorname{Tor}_i^R(R_1, k\hat{*}_R^L k)^0.$$

If N is a domino then $\dim_k(R_1\otimes_R N)^0=\dim_k N^0/VN^0=\dim_k N$. The dominoes in $\text{Kün}_0^R(k, k)$, $\text{Kün}_1^R(k, k)$ and $\text{Kün}_2^R(k, k)$ thus give a contribution of 1,2 resp. 1 to the dimension of $R_1\otimes_R \text{Kün}_0^R(k, k)^0$, $R_1\otimes_R \text{Kün}_1^R(k, k)^0$ etc. (6.3) shows that $\dim_k \operatorname{Tor}_i^R(R_1, k\hat{*}_R^L k)^0$ is $1, 2, 1, 0, 0\ldots$ for $i=0, 1, 2, 3\ldots$ and (6.4) shows that $R_1\otimes_R \text{Kün}_*^R(k, k)^0=\operatorname{Tor}_*^R(R_1, k\hat{*}_R^L k)^0$ which, with the aid of (4.4), shows that $\operatorname{Tor}_1^R\big(R_1, \text{Kün}_*^R(k, k)\big)^0=0$. As $\text{Kün}_*^R(k, k)$ is zero in negative degrees (0: 5.4.9) shows that $\operatorname{Tor}_1^R\big(R_1, \text{Kün}_i^R(k, k)\big)^0={}_V\text{Kün}_i^R(k, k)^0$ and hence that V is injective in degree 0. This implies that there is no nilpotent torsion in degree 0 and by duality and (0: 9.1) we see that there is no nilpotent torsion whatsoever. As the $\text{Kün}_i^R(k, k)$ are torsion and, as we have seen, they contain neither semi-simple nor nilpotent torsion they have no hearts. The same argument as in the case of $E_{1/2}\hat{*}_R^L E_{1/2}$ shows that $\text{Kün}_0^R(k, k)\rightsquigarrow \mathbf{U}_1$. Duality and (0: 9.1) imply that $\text{Kün}_2^R(k, k)\rightsquigarrow \mathbf{U}_{-1}$. (4.1) degenerates in this case at E_2 for trivial reasons and, as we have seen, $E_\infty^{0,0}$ and $E_\infty^{1,-2}$ both have dimension 1 and therefore fill out the whole E_∞-term which implies that $H^*\big(\text{Kün}_1^R(k, k), d\big)=0$. As $\text{Kün}_1^R(k, k)$ is a domino killed by p we conclude:

$$\begin{aligned}
&\text{Kün}_0^R(k, k) \xrightarrow{\sim} \mathbf{U}_1\\
(6.5)\qquad &\text{Kün}_1^R(k, k) \xrightarrow{\sim} \mathbf{U}_0^2\\
&\text{Kün}_2^R(k, k) \xrightarrow{\sim} \mathbf{U}_{-1}\\
&\text{Kün}_i^R\, k, k) = 0, \quad i\neq 0, 1, 2.
\end{aligned}$$

Consider next $k\hat{*}_R^L E$ with E of positive slope concentrated in degree 0. By the now familiar arguments one shows easily that modulo finite type $\text{Kün}_0^R(k, E)$ and $\text{Kün}_1^R(k, E)$ contain dominoes and that the rest is 0 and that there is no semi-simple torsion. One sees by duality that only $\text{Kün}_i^R(k, E)$, $i=0, 1, 2$, could possibly contain nilpotent torsion. Hence $\text{Kün}_i^R(k, E)=0$ for $i>2$.

7. We can now use these calculations to obtain some general results.

Proposition 7.1. $(-)\hat{*}^L_R(-)$ *has amplitude* $[-2, 0]$ *on* $D^b_c(R)xD^b_c(R)$.

Indeed, we want to show that $\mathrm{Kün}^R_i(M, N)=0$ for $i>2$ and M, N coherent. Every coherent R-module is a successive extension of elementary ones (0: 8) so by half exactness of $\mathrm{Kün}^R_i(-, -)$ we may assume that M and N are elementary. (1.2 i) takes care of the case when either M or N are of slope zero or semisimple torsion and sections 4—6 take care of the remaining cases.

Proposition 7.2. i) *Let* $M, N\in D^b_c(R)$ *and assume that* $\mathrm{Kün}^R_*(M, N)$ *is of finite type as* W*-module. Then* $H^*(M)$ *and* $H^*(N)$ *are of finite type over* W *and one of them has* F *bijective.*

ii) *Let* X *and* Y *be smooth and proper varieties over* k. *If* $X\times Y$ *is Hodge—Witt then one of* X and Y *is Hodge—Witt and the other is ordinary.*

Proof: Assume i) false for some M and N and choose m and n such that $H^i(M)$ (resp. $H^j(N)$) has F bijective for $i>m$ $(j>n)$ and such that $H^m(M)$ $(H^n(N))$ does not, supposing this is possible. Using Props. 1.2 and 3.2 we see that $H^{m+n}(\tau_{\leqq m}M\hat{*}^L_R \tau_{\leqq n}N)$ is of finite type over W. By Prop. 3.2 again we see that $\mathrm{Kün}^R_0(H^m(M), H^n(N))$ is of finite type over W. It is easy to see that any coherent R-module with F not bijective has (a shifting of) k as quotient. By right exactness of $\mathrm{Kün}^R_0(-, -)$ we reach a contradiction using (6.5). We thus see that either $H^*(M)$ or $H^*(N)$ has F bijective and from (1.2) we see that the other one must be finitely generated over W. ii) is now clear from (II: 1.1).

Remark: (7.2 ii) has been shown, independently, by Katz in the case that $H^*_{\mathrm{crys}}(X\times Y/W)$ is without torsion.

8. Let $p=2$ and $k=\bar{k}$ and consider two elliptic curves E_1 and E_2 over W such that the special fibre of E_1 is ordinary and the one of E_2 is supersingular. Fix a W-point α of E_1 such that it and its reduction mod 2 have order 2. Let $\mathbf{C}_2=\langle\sigma\rangle$, the cyclic group of order 2, act on $E_1\times E_2$ by $\sigma(x, y):=(x+\alpha, -y)$ and put $I:= E_1\times E_2/\mathbf{C}_2$. We will want to calculate the cohomology of I and $I\times I$ and their special fibers and also to draw some consequences of these calculations.

To be able to do this we will need some preliminaries. For these it will not be necessary to assume that $p=2$.

Proposition 8.1. *Let* $f\colon X\to S$ *be smooth and proper with* S *p-local.*

i) *If* T *is a finite* S*-scheme then there are exact sequences*

$$0\to H^1_{fl}(T, R^{i-1}f_*\hat{G}_m)\to H^i_{fl}(X_T, \hat{G}_m)\to R^if_*\hat{G}_m(T)\to 0 \tag{8.1.1}$$

where the $R^if_*\hat{G}_m$ *are the flat higher direct images and* $\hat{(-)}$ *denotes the infinitesimal part i.e.* $H^i_{fl}(X_T, \hat{G}_m):=\mathrm{Ker}\,(H^i(X_T, G_m)\to H^i((X_T)_{\mathrm{red}}, G_m))$ *etc.* (*cf.* [20]).

ii) *We have V-tors* $(H^i(X, W\mathcal{O})) = {}^1TC((R^{i-1}f_*\hat{G}_m)_f)$ *and*

$$H^i(X, W\mathcal{O})/V\text{-tors} = TC((R^i f_* \hat{G}_m)_s).$$

Proof: We know from [20] that the $R^i f_* \hat{G}_m$ are pro-representable. The proof of the theorem of Grothendieck [13: Thm. 11.7] extends to show that if G is a smooth formal group over k then $H^i_{fl}(T, G) = H^i_{\acute{e}t}(T, G) = 0$ if $i > 0$. Any formal group is an extension of a smooth formal group by a finite group scheme which in turn is a successive extension of α_p: s and μ_p: s which embed in smooth formal groups. Hence $H^i_{fl}(T, G) = 0$ if $i > 1$ for any formal group G. This applies in particular to the $R^i f_* \hat{G}_m$. Leray's spectral sequence for the projection f: $X_{Tfl} \to X_{fl}$ takes the form

$$H^i_{fl}(T, R^j f_* \hat{G}_m) \Rightarrow H^{i+j}_{fl}(X_T, \hat{G}_m) \tag{8.1.2}$$

and by what has just been observed (8.1.2) degenerates to give (8.1.1). For ii) we start from (8.1.1) for $T_n = \operatorname{Spec} k[t]/(t^n)$, go to the limit and split off the non-typical part thus getting

$$0 \to {}^1TC(R^{i-1}f_* G_m) \to H^i(X, W\mathcal{O}) \to TC(R^i f_* G_m) \to 0. \tag{8.1.3}$$

By the theorem of Grothendieck ${}^1TC(G) = 0$ if G is smooth and as it is well-known V is injective on $TC(G)$. Furthermore, one proves easily that $\{H^i(T_n, G)\}$ is a Mittag—Leffler system for $i = 0, 1$ and G a formal group and hence that $TC(-)$ $({}^1TC(-))$ is left (right) exact. By dévissage and explicit computation for α_p and μ_p one then shows that if G is a finite group scheme then $TC(G) = 0$ and ${}^1TC(G)$ is killed by a power of V. Therefore, if G is a formal group, $TC(G) = TC(G_s)$ has V injective and ${}^1TC(G) = {}^1TC(G_f)$ has V nilpotent. This together with (8.1.3) gives ii).

Lemma 8.2. *Let f: $X \to V$ be flat and proper where V is the spectrum of a complete discrete valuation ring.*

i) *The dimension of $R^i f_* \hat{G}_m$ at the generic point is less than or equal to the dimension at the special point.*

ii) *If $R^i f_* \hat{G}_m$ and $R^{i+1} f_* \hat{G}_m$ are pro-representable then $R^i f_* \hat{G}_m$ is equidimensional.*

Indeed, recall [20: Thm. 2.3.7] that there is a complex $G^\cdot$ whose components are smooth formal groups and such that $R^i f_* \hat{G}_m = H^i(G^\cdot)$ where cohomology is taken in the flat topology. The cycles Z^i are formal groups and hence $\overline{\dim}\, Z^i \leqq \underline{\dim}\, Z^i$ where $\overline{\dim}$ and $\underline{\dim}$ denotes the dimension at the generic and special points respectively. The exact sequence $0 \to Z^i \to G^i \to B^{i+1} \to 0$ shows that $\overline{\dim}\, B^i \geqq \underline{\dim}\, B^i$ and the exact sequence $0 \to B^i \to Z^i \to H^i \to 0$ gives $\overline{\dim}\, H^i \geqq \underline{\dim}\, H^i$ which shows i). In case H^i is pro-representable then so is B^i being the kernel of $Z^i \to H^i$. Hence $\overline{\dim}\, B^i \leqq \underline{\dim}\, B^i$ which together with the inequality proved above implies that $\overline{\dim}\, B^i = \underline{\dim}\, B^i$. The exact sequence $0 \to Z^{i-1} \to G^{i-1} \to B^i \to 0$ now reveals that

$\overline{\dim} Z^{i-1} = \underline{\dim} Z^{i-1}$. If both H^i and H^{i+1} are pro-representable then $\overline{\dim} H^i = \overline{\dim} Z^i - \overline{\dim} B^i = \underline{\dim} Z^i - \underline{\dim} B^i = \underline{\dim} H^i$.

Put $\bar{E} := E_{1/2}/p$. From (1.5.1) we get an exact sequence $0 \to \hat{R} \xrightarrow{(p, F-V)} \hat{R} \oplus \hat{R} \xrightarrow{V-F+p} \hat{R} \to \bar{E} \to 0$ which gives

$$\begin{aligned} \operatorname{Hom}_R(\bar{E}, \bar{E}) &= (\mathbf{Z}/p)^2 \oplus k \\ \operatorname{Ext}^1_R(\bar{E}, \bar{E}) &= (\mathbf{Z}/p)^2 \oplus k^2 \\ \operatorname{Ext}^2_R(\bar{E}, \bar{E}) &= k. \end{aligned} \tag{8.3}$$

This gives us the number of dominoes appearing in $\text{Kün}^R_*(\bar{E}, \bar{E})$ but also that we will have semi-simple torsion as well as the exact amount appearing. Using arguments very similar to those used in the calculation of $\text{Kün}^R_*(k, k)$ one shows that

$$\begin{aligned} &\text{Kün}^R_0(\bar{E}, \bar{E}) \xrightarrow{\sim} \mathbf{U}_1 \\ &\text{Kün}^R_1(\bar{E}, \bar{E}) \xrightarrow{\sim} \mathbf{U}_{-1} \oplus \mathbf{U}_1 \oplus (W/p)^2(-1) \\ &\text{Kün}^R_2(\bar{E}, \bar{E}) \xrightarrow{\sim} \mathbf{U}_{-1} \oplus (W/p)^2(-1). \end{aligned} \tag{8.4}$$

(One also has to use that an extension of $\mathbf{U}_i$ by W/p killed by p splits but this follows by taking fixed points of F and using the corresponding result on extensions of G_a^{perf} by $\mathbf{Z}/p$.)

Let us now return to the calculation of the cohomology of I and $I \times I$ and let us begin with the Hodge—Witt cohomology of the special fiber $\bar{I}$. The Künneth formula gives us

$$\begin{aligned} &R^0\Gamma(W\Omega^{\cdot}_{\bar{E}_1 \times \bar{E}_2}) \xrightarrow{\sim} W \oplus W(-1) \\ &R^1\Gamma(W\Omega^{\cdot}_{\bar{E}_1 \times \bar{E}_2}) \xrightarrow{\sim} W \oplus E_{1/2}\zeta \oplus E_{1/2}(-1)\zeta \oplus W^2(-1) \oplus W(-2) \\ &R^2\Gamma(W\Omega^{\cdot}_{\bar{E}_1 \times \bar{E}_2}) \xrightarrow{\sim} E_{1/2}\zeta \oplus W(-1) \oplus E_{1/2}(-1)\zeta \oplus W(-2), \end{aligned} \tag{8.5}$$

where ζ denotes the non-trivial character of $\mathbf{C}_2$ thus giving us also the action of $\mathbf{C}_2$ on the Hodge—Witt cohomology.

We will now compute the cohomology of $\bar{I}$ using the spectral sequence

$$H^i(\mathbf{C}_2, R^j\Gamma(W\Omega^{\cdot}_{\bar{E}_1 \times \bar{E}_2})) \Rightarrow R^{i+j}\Gamma(W\Omega^{\cdot}_{\bar{I}}). \tag{8.6}$$

(Note that $\mathbf{C}_2$ acts freely on $\bar{E}_1 \times \bar{E}_2$.) This is a spectral sequence of R-modules. (Actually of coherent R-modules. As every coherent G-equivariant R-module, where G is a finite group, can be embedded in a $\Gamma(G, -)$-acyclic coherent R-module and coherent R-modules are closed under kernels and cokernels it is clear that $H^i(G, M)$ is coherent for M a G-equivariant coherent R-module.) If we consider first (8.6) in degree 2 we see that $H^0(W\Omega_{\bar{I}}^2)=0$, $H^1(W\Omega_{\bar{I}}^2)\xrightarrow{\sim} W$ and $H^2(W\Omega_{\bar{I}}^2)\xrightarrow{\sim} W$. This shows that $H^2(W\mathcal{O}_{\bar{I}})$ contains no semi-simple torsion as it would be dual to the semi-simple torsion of $H^1(W\Omega_{\bar{I}}^2)$. (8.6) in degree 0 now shows that $H^2(\mathbf{C}_2, H^0(W\mathcal{O}_{\bar{E}_1\times\bar{E}_2}))\xrightarrow{\sim} W/2$ must be killed and as $H^3(\mathbf{C}_2, H^0(W\mathcal{O}_{\bar{E}_1\times\bar{E}_2}))=0$, $H^1(\mathbf{C}_2, H^1(W\mathcal{O}_{\bar{E}_1\times\bar{E}_2}))\xrightarrow{\sim} \bar{E}$ survives and hence injects into $H^2(W\mathcal{O}_{\bar{I}})$. Finally $H^0(\mathbf{C}_2, H^2(W\mathcal{O}_{\bar{E}_1\times\bar{E}_2}))=0$ which implies that $H^2(W\mathcal{O}_{\bar{I}})\xrightarrow{\sim}\bar{E}$. In degree 1 we get from (8.6) that $H^0(W\Omega_{\bar{I}}^1)\xrightarrow{\sim} W$, $H^1(W\Omega_{\bar{I}}^1)\xrightarrow{\sim} W^2$ and $H^2(W\Omega_{\bar{I}}^1)/\text{torsion}\xrightarrow{\sim} W$. The nilpotent torsion in $H^2(W\Omega_{\bar{I}}^1)$ is dual to the one in $H^2(W\mathcal{O}_{\bar{I}})$ and is hence isomorphic to $\bar{E}$. The semi-simple torsion is dual to the semi-simple torsion in $H^1(W\Omega_{\bar{I}}^1)$ and is therefore 0. Splitting $H^2(W\Omega_{\bar{I}}^1)$ into its nilpotent and semi-simple parts we get:

$$
\begin{aligned}
&R^0\Gamma(W\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} W\oplus W(-1)\\
(8.7)\qquad &R^1\Gamma(W\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} W\oplus W^2(-1)\oplus W(-2)\\
&R^2\Gamma(W\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} \bar{E}\oplus W(-1)\oplus \bar{E}(-1)\oplus W(-2).
\end{aligned}
$$

Using very similar arguments one shows that

$$
\begin{aligned}
&R^0\Gamma(\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} W\oplus W(-1)\\
(8.8)\qquad &R^1\Gamma(\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} W\oplus W^2(-1)\oplus W/2(-1)\oplus W(-2)\oplus W/2(-2)\\
&R^2\Gamma(\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} W/2\oplus W(-1)\oplus W/2(-1)\oplus W(-2).
\end{aligned}
$$

We now want to use (8.7) and the Künneth formula to compute the Hodge—Witt cohomology of $\bar{I}\times\bar{I}$. In order to avoid fighting with more spectral sequences we note that

$$
(8.9)\qquad R\Gamma(W\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim} \bigoplus_i R^i(W\Omega_{\bar{I}}^{\cdot})[-i].
$$

This is seen as follows: The obstruction for splitting a complex $M\in D_c^b(R)$ into the sum $\tau_{\geq i}M\oplus\tau_{<i}M$ is the morphism $\tau_{\geq i}M\to\tau_{<i}M$ [1] which is part of the triangle $\tau_{<i}M\to M\to\tau_{\geq i}M\to\tau_{<i}M$ [1]. From Lemma 1.5 and dévissage it follows that every coherent R-module has projective dimension 2 over $\hat{R}$. Hence $\mathrm{Hom}_{D(R)}(\tau_{\geq i}M, \tau_{<i}M\,[1])=\mathrm{Ext}_R^2(H^i(M), H^{i-1}(M))^0$. Using again Lemma 1.5 one easily sees that for $M=R\Gamma(W\Omega_{\bar{I}}^{\cdot})$. $\mathrm{Ext}_R^2(H^i(M), H^{i-1}(M))=0$ for all i and we get the asserted decomposition. We therefore obtain $R\Gamma(W\Omega_{\bar{I}\times\bar{I}}^{\cdot})=R\Gamma(W\Omega_{\bar{I}}^{\cdot})\hat{*}_R^L R\Gamma(W\Omega_{\bar{I}}^{\cdot})\xrightarrow{\sim}\bigoplus_{i,j}R^i\Gamma(W\Omega_{\bar{I}}^{\cdot})\hat{*}_R^L R^j\Gamma(W\Omega_{\bar{I}}^{\cdot})\;[-i-j]$. Expanding this, using the additivity of $(-)\hat{*}_R^L(-)$, that W acts as a unit and (8.4) we get

$$
\begin{aligned}
&R^0\Gamma(W\Omega_{\bar{I}\times\bar{I}}^{\cdot})\xrightarrow{\sim} W\oplus W^2(-1)\oplus W(-2)\\
&R^1\Gamma(W\Omega_{\bar{I}\times\bar{I}}^{\cdot})\xrightarrow{\sim} W^2\oplus W^6(-1)\oplus W^6(-2)\oplus W^2(-3)
\end{aligned}
$$

$$R^2\Gamma(W\Omega^{\cdot}_{\bar I\times\bar I}) \xrightarrow{\sim} W\oplus \bar E^2\oplus \mathbf{U}_{-1}\oplus \bar E^4(-1)\oplus W^6(-1)\oplus \mathbf{U}^2_{-1}(-1)\oplus (W/2)^2(-1)$$
$$\oplus W^{10}(-2)\oplus \bar E^2(-2)\oplus (W/2)^4(-2)\oplus \mathbf{U}_{-1}(-2)\oplus W^6(-3)$$
$$\oplus (W/2)^4(-3),$$

(8.10)
$$R^3\Gamma(W\Omega^{\cdot}_{\bar I\times\bar I}) \xrightarrow{\sim} \bar E^2\oplus \mathbf{U}_1\oplus \mathbf{U}_{-1}\oplus \bar E^6(-1)\oplus W^2(-1)\oplus (W/2)^2(-1)$$
$$\oplus\, \mathbf{U}^2_1(-1)\oplus \mathbf{U}^2_{-1}(-1)\oplus W^6(-2)\oplus \bar E^6(-2)\oplus (W/2)^4(-2)$$
$$\oplus\, \mathbf{U}_1(-2)\oplus \mathbf{U}_{-1}(-2)\oplus \bar E^2(-3)\oplus W^6(-3)$$
$$\oplus (W/2)^2(-3)\oplus W^2(-4)$$

$$R^4\Gamma(W\Omega^{\cdot}_{\bar I\times\bar I}) \xrightarrow{\sim} \mathbf{U}_1\oplus \bar E^2(-1)\oplus \mathbf{U}^2_1(-1)\oplus W(-2)\oplus \bar E^4(-2)\oplus \mathbf{U}_1(-2)$$
$$\oplus W^2(-3)\oplus \bar E^2(-3)\oplus W(-4).$$

From (8.1 ii) we see that for f: $\bar I\times\bar I\to S$:

(8.11)
$$(R^2f_*\hat G_m)_s \xrightarrow{\sim} \hat G_m\oplus \hat G_a$$
$$(R^3f_*\hat G_m)_s \xrightarrow{\sim} \hat G_a^2$$
$$(R^4f_*\hat G_m)_s \xrightarrow{\sim} \hat G_a.$$

On the other hand, using the Künneth formula for the Hodge cohomology of the generic fiber I' of I and (8.8) we see that $h^{0,2}(I'\times I')=1$ and $h^{0,3}(I'\times I')=h^{0,4}(I'\times I')=0$. As in characteristic zero $h^{0,i}$ equals the dimension of $R^if_*\hat G_m$ we see that none of $R^if'_*G_m$, $i=2, 3, 4$, with f': $I\times I\to\operatorname{Spec} W$ are equi-dimensional. As $R^2f'_*G_m$ (cf. [20: Thm. 4.1.2]) and $R^5f'_*G_m$ are pro-representable we conclude from (8.2 ii) that none of $R^if'_*G_m$, $i=3, 4$, are pro-representable.

Let us also note that the Hodge-polygons of the generic fiber of $I\times I$ coincide with the Newton polygons of the special fiber yet $\bar I\times\bar I$ is not ordinary, in fact not even Hodge—Witt.

Remark: i) Similarly we can consider E_1 and E_2 over Spec $k[[t]]$ with E_1 ordinary at both the special and generic fiber and E_2 ordinary at the generic and supersingular at the special fiber. We will then get an example of a fourfold with $R^4f'_*\hat G_m$ and either $R^2f'_*\hat G_m$ or $R^3f'_*\hat G_m$ non-pro-representable.

ii) $p=2$ is not essential. In general we would consider $E_1\oplus E_2\otimes_{\mathbf{Z}}M$ where M is the augmentation ideal in $\mathbf{Z}[\mathbf{C}_p]$ and let $\mathbf{C}_p$ act by translation by an element of order p on E_1 and its natural action on the second factor. The quotient X will then have $H^2(W\mathcal{O}_X)\xrightarrow{\sim}\bar E$ and we would have phenomena similar to the ones just encountered. If we want a 2-dimensional example we will have to take successive hypersurface sections of sufficiently high degree as in [3] and use a weak Lefschetz theorem for the Hodge—Witt cohomology proved as in [3]. We will then have $H^2(W\mathcal{O}_X)\to H^2(W\mathcal{O}_Y)$ for such a 2-dimensional section Y. (The details will appear elsewhere.)

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