Spaces of Besov—Hardy—Sobolev type on complete Riemannian manifolds

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1. Introduction

Let $\mathbf{R}_n$ be the Euclidean n-space. The two scales of spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ with $-\infty<s<\infty$, $0<p\leqq\infty$ ($p<\infty$ in the case of the F-spaces), $0<q\leqq\infty$, cover many well-known classical spaces of functions and distributions on $\mathbf{R}_n$:

(i) the Besov—Lipschitz spaces $\Lambda^s_{p,q}(\mathbf{R}_n)=B^s_{p,q}(\mathbf{R}_n)$ if $s>0$, $1<p<\infty$, $1\leqq q\leqq\infty$;

(ii) the Bessel-potential spaces $H^s_p(\mathbf{R}_n)=F^s_{p,2}(\mathbf{R}_n)$ if $-\infty<s<\infty$, $1<p<\infty$, with the Sobolev spaces $W^m_p(\mathbf{R}_n)=H^m_p(\mathbf{R}_n)$ if $1<p<\infty$, m non-negative integer, as special cases;

(iii) the Hölder—Zygmund spaces $\mathscr{C}^s(\mathbf{R}_n)=B^s_{\infty,\infty}(\mathbf{R}_n)$ if $s>0$;

(iv) the (non-homogeneous) Hardy spaces $H_p(\mathbf{R}_n)=F^0_{p,2}(\mathbf{R}_n)$ if $0<p<\infty$.

After a modified extension of the definition of $F^0_{p,2}(\mathbf{R}_n)$ to $p=\infty$ one can even include the non-homogeneous version of the fashionable space BMO of functions of bounded mean oscillation.

The definition of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ is based on decompositions of the Fourier image of tempered distributions on $\mathbf{R}_n$. It is due to J. Peetre, P. I Lizorkin and the author, cf. [14—17, 25] (a more careful description of these historical aspects may be found in [29, 2.3.5]). Systematic treatments of these spaces have been given in [18] (mostly restricted to $B^s_{p,q}(\mathbf{R}_n)$ with $1\leqq p\leqq\infty$) and [29] (with [26, 27] as forerunners, cf. also [28]). The problem arises to introduce spaces of $B^s_{p,q}-F^s_{p,q}$ type on manifolds, where (complete) Riemannian manifolds and Lie groups seem to be of peculiar interest. As far as special cases of the above spaces are concerned something has been done in this direction. T. Aubin [1, and Chapter 2 in 2] studied Sobolev spaces on (complete) Riemannian manifolds, which are defined via covariant derivatives. Weighted Sobolev spaces on Riemannian manifolds may be found in [4]. Based on a study of the Laplace—Beltrami operator R. S. Strichartz [24] introduced Bessel potential spaces on complete Riemannian manifolds. Lipschitz—Besov—Hardy spaces on Lie groups attracted even more attention.

We refer to [6, 7, 8, 13, 22] and in particular to [5, 9, 10, 20, 21] where the latter papers may be of interest in connection with the approach presented in this paper and in a planned subsequent paper [32]. Finally we mention [19, 23] where local versions of Hardy spaces on manifolds have been considered.

The present paper deals with the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ on a complete Riemannian manifold M for the full range of the parameters s, p, q under some restrictions on M. The investigations are based on some recent developments in the theory of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ described in [30, 31]. In particular the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ can be treated for all admissible parameters s, p, q in a strictly local way.

The plan of the paper is the following. In Section 2 we give a brief description of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$, in particular of those properties which we need in the sequel and which anticipate the typical constructions for corresponding spaces on Riemannian manifolds. Section 3 deals with the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ on a complete Riemannian manifold M: definitions, results, comments. In 3.1 we collect the required properties of the underlying manifold M, whereas the definitions of the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ and of desirable equivalent quasi-norms are given in 3.2. In 3.3—3.5 we describe our main results: basic properties, equivalent quasi-norms, special cases (Bessel-potential and Sobolev spaces in comparison with the spaces introduced in [1, 2, 24]), embeddings, lifts, interpolation. Finally we formulate in 3.6 further assertions which can be expected: Equivalent quasi-norms in $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ which are defined via differences of functions, where these differences are now taken along geodesics. However a more detailed (and also more careful) treatment of this (appearently somewhat delicate) subject is postponed to a second paper [32]. We wish to emphasize the following surprising effect: On a complete Riemannian manifold M the spaces $F^s_{p,q}(M)$ are more natural and easier to handle than the spaces $B^s_{p,q}(M)$ (after a corresponding theory of these spaces on $\mathbf{R}_n$ has been established!). From a technical point of view we refer to (29) which has no counterpart for the spaces $B^s_{p,q}(M)$ (with $p \neq q$). But the deeper reason is clearly exhibited by the assertions formulated in 3.6. In (54) the inner L_q-quasi-norm with its integration over $B_P(r)=\{X|X\in T_PM,\ \|X\|<r\}$ makes sense for every fixed $P\in M$. Afterwards the quasi-norm $\|\cdot|L_p(M)\|$ is taken. In the Euclidean case (i.e. a complete Riemannian manifold with distant parallelism!) these two quasi-norms are interchangeable and one obtains an equivalent quasi-norm in $B^s_{p,q}(\mathbf{R}_n)$, cf. (57) and [29, 2.5.12]. But in general this exchange is not possible. (In the case of a Lie group the situation is different because all tangent spaces are linked in a natural way with the distinguished tangent space of the identity element and there are counterparts of (54), cf. [9, 10, 20, 21].) Proofs of the theorems and of the other assertions formulated in Section 3 (with expection of that ones of 3.6) are given in Section 4.

2. Spaces on R_n

2.1. Definitions. We follow essentially [29, in particular 2.3.1]. Let $\mathbf{R}_n$ be the Euclidean n-space. $S(\mathbf{R}_n)$ and $S'(\mathbf{R}_n)$ stand for the Schwartz space of all infinitely differentiable rapidly decreasing complex-valued functions on $\mathbf{R}_n$ and the collection of all complex-valued tempered distributions on $\mathbf{R}_n$, respectively. F and F^{-1} denote the Fourier transform and its inverse on $S'(\mathbf{R}_n)$, respectively. Let $0<p\leqq\infty$, then

$$\|f|L_p(\mathbf{R}_n)\| = \left(\int_{\mathbf{R}_n} |f(x)|^p dx\right)^{1/p}$$

(modification if $p=\infty$) has the usual meaning. Let $\Phi(\mathbf{R}_n)$ be the collection of all systems $\{\varphi_j(x)\}_{j=0}^{\infty}\subset S(\mathbf{R}_n)$ with the following properties:

(i) $$\varphi_j(x) = \varphi(2^{-j}x) \quad \text{if} \quad j = 1, 2, 3, \ldots,$$

(ii) $$\operatorname{supp} \varphi_0 \subset \{x \,|\, |x| \leqq 2\},$$

$$\operatorname{supp} \varphi \subset \left\{x \,\middle|\, \tfrac{1}{2} \leqq |x| \leqq 2\right\},$$

(iii) $$\sum_{j=0}^{\infty} \varphi_j(x) \equiv 1 \quad \text{for every } x\in\mathbf{R}_n.$$

Definition 1. Let $\{\varphi_j(x)\}_{j=0}^{\infty}\in\Phi(\mathbf{R}_n)$. Let $-\infty<s<\infty$ and $0<q\leqq\infty$.

(i) Let $0<p\leqq\infty$. Then

$$B^s_{p,q}(\mathbf{R}_n) = \Big\{f \,|\, f\in S'(\mathbf{R}_n),\ \ \|f|B^s_{p,q}(\mathbf{R}_n)\|^{\{\varphi_j\}} = \Big(\sum_{j=0}^{\infty} 2^{sjq}\|F^{-1}[\varphi_j Ff]|L_p(\mathbf{R}_n)\|^q\Big)^{1/q} < \infty\Big\} \tag{1}$$

(usual modification if $q=\infty$).

(ii) Let $0<p<\infty$. Then

$$F^s_{p,q}(\mathbf{R}_n) = \Big\{f \,|\, f\in S'(\mathbf{R}_n),\ \|f|F^s_{p,q}(\mathbf{R}_n)\|^{\{\varphi_j\}} = \Big\|\Big(\sum_{j=0}^{\infty} 2^{sjq}|(F^{-1}[\varphi_j Ff])(\cdot)|^q\Big)^{1/q}\Big|L_p(\mathbf{R}_n)\Big\| < \infty\Big\} \tag{2}$$

(usual modification if $q=\infty$).

Remark 1. The theory of these spaces has been developed in [29], cf. also the above introduction as far as the historical roots are concerned. The definitions make sense because $F^{-1}[\varphi_j Ff]$ is an analytic function for any $f\in S'(\mathbf{R}_n)$. All spaces are quasi-Banach spaces (Banach spaces if $p\geqq 1$ and $q\geqq 1$) and they cover the classical spaces mentioned in the introduction. Different choices of $\{\varphi_j(x)\}_{j=0}^{\infty}\in\Phi(\mathbf{R}_n)$ yield equivalent quasi-norms in the respective spaces. We shall not distinguish between equivalent quasi-norms of a given space. In this sense we write in the sequel $\|f|B^s_{p,q}(\mathbf{R}_n)\|$ instead of $\|f|B^s_{p,q}(\mathbf{R}_n)\|^{\{\varphi_j\}}$ with $\{\varphi_j\}\in\Phi(\mathbf{R}_n)$ or of any of the equivalent quasi-norms described in the next subsections. Similarly for the spaces $F^s_{p,q}(\mathbf{R}_n)$.

2.2. Equivalent characterizations. In order to calculate

$$(F^{-1}[\varphi_j Ff])(x) = c\int_{\mathbf{R}_n} (F^{-1}\varphi_j)(y)f(x-y)\,dy$$

at a given point $x\in\mathbf{R}_n$, one needs the knowledge of $f(y)$ for all $y\in\mathbf{R}_n$. This shows that the quasi-norms in (1) and (2) do not reflect the local nature of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ which is well-known at least for some classical spaces mentioned in the introduction and which is true for all spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$. Furthermore quasi-norms which exhibit the local nature of all of these spaces are highly desirable if one wishes to transfer spaces of $B^s_{p,q}-F^s_{p,q}$ type from $\mathbf{R}_n$ to manifolds. We formulate a theorem which is proved in [30, 31] and which is the basis for our study of corresponding spaces on complete Riemannian manifolds.

Let $B=\{y\ |y|<1\}$ be the unit ball in $\mathbf{R}_n$. Let $k_0\in S(\mathbf{R}_n)$ and $k\in S(\mathbf{R}_n)$ with

$$\text{supp}\, k_0\subset B,\quad \text{supp}\, k\subset B, \tag{3}$$

$$(Fk)(0)\neq 0 \tag{4}$$

and

$$(Fk_0)(y)\neq 0 \quad \text{for all} \quad y\in\mathbf{R}_n. \tag{5}$$

Functions k_0 with (3) and (5) exist. For example, if $\varkappa\in S(\mathbf{R}_n)$ with $\varkappa(x)=\varkappa(-x)$ is an appropriate real function with a compact support near the origin then $k_0(x)=e^{-|x|^2}(\varkappa*\varkappa)(x)$ has the desired properties. Let $k_N=\left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\right)^N k$ where N is a natural number. We introduce the means

$$[K^e(k_N,t)f](x) = \int_{\mathbf{R}_n} k_N(y)f(x+ty)\,dy,\quad x\in\mathbf{R}_n,\quad t>0, \tag{6}$$

and $N=0, 1, 2, \ldots$, where "e" stands for "Euclidean". This makes sense for any $f\in S'(\mathbf{R}_n)$ (usual interpretation). As in [29] or in [30, (12)] we use the abbreviation $\tilde{\sigma}_p=0$ if $1\leqq p\leqq\infty$ and $\tilde{\sigma}_p=n\left(\frac{1}{p}-1\right)$ if $0<p<1$.

Theorem A. *Let $-\infty<s<\infty$ and $0<q\leqq\infty$. Let $0<\varepsilon<\infty$ and $0<r<\infty$. Let k_N with $N=0, 1, 2, \ldots$ be the above functions, cf.* (3)—(5). *Let $K^e(k_N,t)$ be the above means.*

(i) *Let $0<p\leqq\infty$ and $2N>\max(s,\tilde{\sigma}_p)$. Then*

$$\|K^e(k_0,\varepsilon)f|L_p(\mathbf{R}_n)\| + \left(\int_0^r t^{-sq}\|K^e(k_N,t)f|L_p(\mathbf{R}_n)\|^q \frac{dt}{t}\right)^{1/q} \tag{7}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $B^s_{p,q}(\mathbf{R}_n)$.

(ii) *Let* $0<p<\infty$ *and* $2N>\max(s, \tilde{\sigma}_p)$. *Then*

$$\text{(8)} \qquad \|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\| + \left\| \left(\int_0^r t^{-sq} |(K^e(k_N, t)f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} \Big| L_p(\mathbf{R}_n) \right\|$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $F^s_{p,q}(\mathbf{R}_n)$.

Remark 2. The theorem is a slight modification of Proposition 1 in [30], cf. also Remark 3 in [30]. One can replace (5) by the weaker assumption $(Fk_0)(0)\neq 0$, as we did in [30, 31]. But (7) and (8) are just those quasi-norms which can be transferred to complete Riemannian manifolds and then the stronger assumption (5) will be of great service for us.

Proposition A. *Let the kernels* k_N, *the means* $K^e(k_N, t)$ *and the numbers* s, q, ε *be the same as in Theorem A. Let L be an integer.*

(i) *Let* $0<p\leq\infty$ *and* $2N>\max(s, \tilde{\sigma}_p)$. *Then*

$$\text{(9)} \qquad \|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\| + \left(\sum_{j=L}^{\infty} 2^{jsq} \|K^e(k_N, 2^{-j})f|L_p(\mathbf{R}_n)\|^q\right)^{1/q}$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $B^s_{p,q}(\mathbf{R}_n)$.

(ii) *Let* $0<p<\infty$ *and* $2N>\max(s, \tilde{\sigma}_p)$. *Then*

$$\text{(10)} \qquad \|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\| + \left\|\left(\sum_{j=L}^{\infty} 2^{jsq} |(K^e(k_N, 2^{-j})f)(\cdot)|^q\right)^{1/q} \Big| L_p(\mathbf{R}_n)\right\|$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $F^s_{p,q}(\mathbf{R}_n)$.

Remark 3. The above proposition is simply the discrete version of the continuous version described in Theorem A. There is no big difference between discrete and continuous versions of equivalent quasi-norms in spaces of $B^s_{p,q}-F^s_{p,q}$ type. E.g. in Theorem 1 in [30] we began with a discrete version and explained in the proof of Theorem 2 in [30] how to switch over to a corresponding continuous version. The philosophy is the following: continuous versions look more handsome, but discrete versions are at least occasionally more effective. In this paper we prefer continuous versions. But we formulate the counterpart of the above proposition on Riemannian manifolds for sake of completeness, in order to prepare later applications and to make some proofs in this paper more transparent.

2.3. Further properties. We describe few properties of the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ which will be useful later on.

Proposition B. *Let* $0<q\leq\infty$.

(i) *Let* $0<p\leq\infty$ *and* $s>\tilde{\sigma}_p$. *Then* (7) *and* (9) *with* $\|f|L_p(\mathbf{R}_n)\|$ *instead of* $\|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\|$ *are equivalent quasi-norms in* $B^s_{p,q}(\mathbf{R}_n)$.

(ii) *Let* $0<p<\infty$ *and* $s>\tilde{\sigma}_p$. *Then* (8) *and* (10) *with* $\|f|L_p(\mathbf{R}_n)\|$ *instead of* $\|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\|$ *are equivalent quasi-norms in* $F^s_{p,q}(\mathbf{R}_n)$.

(iii) *Let* $0<p<\infty$ *and* $s>n/p$. *Let* $\delta>0$ *and let* $B_x(\delta)$ *be the ball centered at* $x\in\mathbf{R}_n$ *with radius* δ. *Then* (8) *and* (10) *with* $\|\sup_{t\in B_x(\delta)}|f(z)|\,|L_p(\mathbf{R}_n)\|$ *instead of* $\|K(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\|$ *are equivalent quasi-norms in* $F^s_{p,q}(\mathbf{R}_n)$.

Remark 4. Part (i) of the proposition is covered by Theorem 3 in [30]. Part (ii) is a consequence of part (i) and elementary embeddings. Finally if $s>\sigma>n/p$ then we have

$$\Big\|\sup_{z\in B_x(\delta)}|f(z)|\,|L_p(\mathbf{R}_n)\Big\| \leqq c\|f|F^{\sigma}_{p,p}(\mathbf{R}_n)\| \leqq c'\Big\|f|F^s_{p,q}(\mathbf{R}_n)\Big\|, \tag{11}$$

cf. [29, p. 100, the end of the proof of Corollary 1 in 2.5.9]. This proves (iii). Obviously, $\sup_{t\in B_x(\delta)}|f(z)|$ in (11) and in (iii) must be understood as a function of $x\in\mathbf{R}_n$. Of course, part (iii) has an immediate counterpart for the spaces $B^s_{p,q}(\mathbf{R}_n)$. But we need only (11).

3. Spaces on manifolds: definitions, results, comments

3.1. Complete Riemannian manifolds. We collect those basic facts about Riemannian manifolds which we need in the sequel. We follow essentially [2], cf. also [12].

We assume in this paper that M stands for a connected C^∞-manifold of dimension n furnished with a smooth Riemannian metric g. Recall that a smooth Riemannian metric is given by a real twice-covariant C^∞ tensor field g such that g_P at each point $P\in M$ is a positive definite bilinear symmetric form

$$g_P(X,Y) = g_P(Y,X) \quad \text{and} \quad g_P(X,X)>0 \quad \text{if} \quad T_P\ni X\neq 0, \qquad Y\in T_PM.$$

Here T_PM stands for the tangent space at $P\in M$. Covariant (or Levi—Civita) derivatives are taken with respect to the (unique) Riemannian connection. In particular, in a local chart (Ω, φ) where Ω is an open set of M and φ is a homeomorphism from Ω onto an open set U in $\mathbf{R}_n$, the Christoffel symbols are given by

$$\Gamma^l_{ij} = \tfrac{1}{2}\, g^{kl}[\partial_i g_{kj}+\partial_j g_{ki}-\partial_k g_{ij}]$$

(summation convention). Let $c(t)=c(P, X, t)$ be the geodesic with $c(P, X, 0)= P\in M$ and $\left.\dfrac{dc}{dt}\right|_{t=0} = X\in T_PM$, where $X\neq 0$ and $-\infty<t<\infty$ (usually only $t\geqq 0$ is of interest for us). We assume always that the Riemannian manifold M is complete, i.e. all geodesics are infinitely extendable with respect to the arc length (by the Hopf—Rinow theorem this is equivalent to the assumption that M is complete as a metric space). Let $C(t)=\varphi(c(t))$ in the above local chart. Then we have

(12) $$\frac{d^2C^j(t)}{dt^2}+\Gamma^j_{ik}(C(t))\frac{dC^i(t)}{dt}\frac{dC^k(t)}{dt}=0$$

with $C(0)=\varphi(P)$ and $\left.\frac{dC}{dt}\right|_{t=0}=\varphi_* X\in T_{\varphi(P)}M$, where $C^j(t)$ are the components of $C(t)$. Recall that

$$g_{ij}(C(t))\frac{dC^i(t)}{dt}\frac{dC^j(t)}{dt}$$

is constant along the geodesic $C(t)$. In particular $\sigma=\|X\|\,t$ where σ stands for the arc length ($\sigma=0$ corresponds to $\varphi(P)$) and $\|X\|=\sqrt{g(X,X)}$.

Of special interest for us is the exponential map $\exp_P$ where $P\in M$, which is given by

$$\exp_P(X)=c(P,X,1),\quad X\in T_PM,$$

where $c(P,X,t)$ is the above geodesic. We put $\exp_P(0)=P$. It is convenient and usual (although a slight abuse of notations) to identify T_PM with $\mathbf{R}_n$. This can be done if one identifies T_PM with the above space $T_{\varphi(P)}U$ where the latter has a natural $\mathbf{R}_n$-structure via $\varphi_* X=a^i\partial_i$. If $r>0$ is small then $\exp_P$ is a diffeomorphism from

(13) $$B(r)=\{X\,|\,X\in\mathbf{R}_n,\ \|X\|<r\}\quad\text{onto}\quad \Omega_P(r)=\exp_P B(r).$$

In particular, $(\Omega_P(r),\exp_P^{-1})$ is a local chart (where we used the above identification of T_PM with $\mathbf{R}_n$). The corresponding local coordinates are denoted as normal geodesic coordinates, where "normal" refers to

(14) $$g_{ij}(0)=\delta_{ij}\quad\text{and}\quad \partial_k g_{ij}(0)=0,$$

where i, j, k are natural numbers between 1 and n. In particular $\Gamma^l_{ij}(0)=0$. Let r_P be the supremum of all numbers r such that $\exp_P$ yields a diffeomorphism in the sense of (13). Then $r_0=\inf r_P$ is called the injectivity radius of M, where the infimum is taken over all $P\in M$. In this paper we assume $r_0>0$ what need not be satisfied in general, cf. [12, p. 131] for a counter-example. We discuss this point in Remark 5 below. Furthermore we assume that the manifold M *is uniform in the following* sense. Let $0<r<r_0$, where r_0 is the injectivity radius. There exist a positive number c and, for every multi-index α, positive numbers c_α with

(15) $$\det(g_{ij})\geqq c,\quad |D^\alpha g_{jk}|\leqq c_\alpha,$$

in the normal geodesic coordinates of every local chart $(\Omega_P(r),\exp_P^{-1})$ with $P\in M$

in the sense of (13). We assume that there exist two positive numbers c and ν such that

$$\text{vol}\,\Omega_P(\varrho) \leqq c\varrho^{\nu}\,^{*)} \tag{15'}$$

for all $P\in M$, $1<\varrho<\infty$, and $\Omega_P(\varrho)=\{Q|Q\in M,\ \text{dist}\,(P, Q)<\varrho\}$.

Hypotheses (about the manifold). M is a connected n-dimensional complete Riemannian manifold with the positive injectivity radius r_0. If r with $r<r_0$ is given then (15) holds in the distinguished local charts $(\Omega_P(r), \exp_P^{-1})$ of normal geodesic coordinates for every $P\in M$. Furthermore vol $\Omega_P(\varrho)$ can be estimated by (15′) *).

Remark 5. Compact manifolds satisfy these hypotheses, cf. [12, p. 131], but also large classes of non-compact manifolds. Of peculiar interest are manifolds with negative curvature (more precisely: non-positive sectional curvature, cf. [11, p. 72]). In this case we have $r_0=\infty$. In particular if M is a simply connected complete Riemannian manifold with negative curvature the $\exp_P$ is a diffeomorphism from $T_PM\sim\mathbf{R}_n$ onto M, cf. [11, p. 74]. As a special case, $\exp_P$ is a diffeomorphism from $\mathbf{R}_n$ onto M if M is a symmetric Riemannian manifold of non-compact type, cf. [12, p. 152].

Proposition C. *Let M be a manifold which satisfies the above Hypotheses. If $\delta>0$ is small then there exist a uniformly locally finite covering of M by a sequence of open balls $\Omega_{P_j}(\delta)$ and a corresponding smooth resolution of unity $\{\psi_j\}$ where the ψ_j's are C^∞-functions on M with* $\text{supp}\,\psi_j\subset\Omega_{P_j}(\delta)$.

Remark 6. $\Omega_{P_j}(\delta)$ stands for the ball centered at $P_j\in M$ and of radius δ, cf. (13). "Uniformly locally finite" means that there exist a natural number L such that any fixed ball $\Omega_{P_j}(\delta)$ has a non-empty intersection with at most L of the remaining other balls. This part of the proposition follows essentially from the above Hypotheses and the Lemmas 2 25 and 2.26 in [2], which are due to Calabi and Aubin. We assume that $\delta>0$ is small. Then the construction of a smooth resolution of unity can be done in normal geodesic coordinates, cf. (13) with P_j instead of P and $0<\delta<r<r_0$. The proof of Lemma 2.26 in [2] shows that one can proceed similar as in the Euclidean case. In other words,

$$\psi_j\in C^\infty(M),\quad 0\leqq\psi_j\leqq 1,\quad \textstyle\sum_j\psi_j\equiv 1 \quad\text{on}\quad M, \tag{16}$$

$$\text{supp}\,\psi_j\subset\Omega_{P_j}(\delta), \tag{17}$$

*) We need this assumption only in Step 2 of the proof of Proposition 2 in 4.4, cf. also Remark 23. In particular the theory for the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ developed in this paper remains valid without assumption (15′) provided that $1\leqq p\leqq\infty$. If $0<p<1$ and s large (more precisely $s>5+2n/p$) then only Theorem 6 (lift property) depends on (15′), all the other assertions are valid without the assumption (15′). In other words: We need (15′) only to prove Theorem 6 for the p's with $0<p<1$ and in order to develop the theory of the space $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ with $0<p<1$ and $s<5+2n/p$.

for every multi-index α there exists a positive number b_α with

(18) $$|D^\alpha(\psi_j \circ \exp_{P_j})(x)| \leqq b_\alpha, \quad x \in B(r), \quad j = 1, 2, \ldots,$$

cf. (13). Without restriction of generality we assume $\{\psi_j\} = \{\psi_j\}_{j=1}^\infty$ in the sequel. But, of course, our approach covers also the case of a finite resolution of unity $\{\psi_j\} = \{\psi_j\}_{j=1}^J$ which refers to a compact manifold. Furthermore we tacitly assume in the sequel that $\delta > 0$ is smal enough, in any case smaller than say, $\frac{r_0}{8}$. This gives the possibility to handle for every fixed j all functions ψ_k with $\operatorname{dist}(\operatorname{supp}\psi_j, \operatorname{supp}\psi_k) \leqq \delta$ within the same local chart, e.g. $(\Omega_{P_j}(r), \exp_{P_j}^{-1})$ with r near r_0.

3.2. Definitions. Let M be the above manifold, in particular, the Hypotheses from 3.1 are satisfied. Let $D'(M)$ be the collection of all complex distributions on M, the dual of $D(M) = C_0^\infty(M)$. Maybe the most convincing way to introduce the spaces $F_{p,q}^s(M)$ is to find an appropriate counterpart of the quasi-norm (8) and to define $F_{p,q}^s(M)$ as the collection of all $f \in D'(M)$ such that this modified quasi-norm is finite. Similarly for the spaces $B_{p,q}^s(M)$. But this causes some technical problems which we wish to avoid. For this reason we give a definition of $F_{p,q}^s(M)$ which reduces these spaces from the very beginning to the corresponding spaces on $\mathbf{R}_n$, and we incorporate $B_{p,q}^s(M)$ afterwards via real interpolation. On the other hand it is the main goal of this paper to give intrinsic characterizations (equivalent quasi-norms) of these spaces. For this reason we first try to find a substitute of the means from (6). Let k_0, k and $k_N = \left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\right)^N k$ be the same functions as in 2.2, where we now assume, *in addition*, that both k_0 and k, and hence also k_N, are rotation-invariant, i.e.

(19) $$k_0(x) = k_0'(|x|), \quad k(x) = k'(|x|)$$

and hence $k_N(x) = k_N'(|x|)$ with

(20) $$k_N'(\varrho) = \left(\frac{d^2}{d\varrho^2} + \frac{n-1}{\varrho}\frac{d}{d\varrho}\right)^N k'(\varrho).$$

If (Ω, φ) is a local chart, $P \in \Omega$ and $U = \varphi(\Omega)$ then $T_{\varphi(P)}U$ has a natural $\mathbf{R}_n$-structure via the representation $\varphi_* X = a^i \partial_i$, where $X \in T_P M$. Let $d\varphi_* X$ be the usual Euclidean volume element (taken with respect to the components a^i interpreted as Cartesian coordinates). As in the case of the standard Riemannian volume element, cf. e.g. [2, 1.74] it is not hard to see that $\sqrt{|\det g_{\varphi(P)}|}\, d\varphi_* X$ has an invariant meaning which allows us to introduce an invariant volume element dX on $T_P M$. The geodesic $c(P, X, t)$ has the same meaning as in 3.1. The counterpart of (6) reads as follows,

$$[K(k_N, t)f](P) = \int_{T_P M} k'_N(\|X\|) f(c(P, X, t))\, dX \tag{21}$$

$$= \int_{T_{\varphi(P)}U} k'_N(\|\varphi_* X\|)(f\circ\varphi^{-1})\big(C(\varphi(P), \varphi_* X, t)\big) \sqrt{|\det g_{\varphi(P)}|}\, d\varphi_* X$$

where the latter expression is the definition of the former one, $t>0$ small, and the integrand is extended outside of the unit ball in $T_{\varphi(P)}U$ by zero. We have to check that these expressions have an invariant meaning and that they make sense for any $f\in D'(M)$. Let $(\tilde{\Omega}, \tilde{\varphi})$ be a second local chart with $P\in\Omega\cap\tilde{\Omega}$. Recall $\|X\|=\|\varphi_* X\|=\|\tilde{\varphi}_* X\|$ if $X\in T_P M$ and $\sigma=t\|X\|$, where σ is the arc length of the geodesic $c(P, X, t)$. Then we have

$$(f\circ\varphi^{-1})\big(C(\varphi(P), \varphi_* X, t)\big) = (f\circ\tilde{\varphi}^{-1})\big(C(\tilde{\varphi}(P), \tilde{\varphi}_* X, t)\big) \tag{22}$$

with $t>0$ small. Together with the above remarks about the invariant volume element dX it follows that the last expression in (21) is independent of the chosen local chart (Ω, φ). In particular in normal geodesic coordinates with the local chart $(\Omega_P(r), \exp_P^{-1})$ we have

$$[K(k_N, t)f](P) = \int_{T_0 U} k'_N(\|\exp_{P*}^{-1} X\|)(f\circ\exp_P)(t \exp_{P*}^{-1} X)\, d \exp_{P*}^{-1} X. \tag{23}$$

In particular, $K(k_N, t)f$ makes sense for every t with $0<t<r_0$, where the latter is the injectivity radius of M (as above we may assume that $f\circ\exp_P$ is extended outside of the unit ball in $T_0 U$ by zero). As we said we identify $T_0 U$ with R_n and put $\exp_{P*}^{-1} X = Y$. Then we have

$$[K(k_N, t)f](P) = \int_{\mathbf{R}_n} k'_N(|Y|)(f\circ\exp_P)(tY)\, dY. \tag{24}$$

This shows that (24) makes sense for every $f\in D'(M)$ (i.e. $f\circ\exp_P\in D'(\mathbf{R}_n)$ after $f\circ\exp_P$ has been extended outside of the unit ball by zero). Hence, (21) can be defined for every $f\in D'(M)$. If $0<p\leqq\infty$ then $L_p(M)$ has the usual meaning with respect to the invariant Riemannian volume element (which in local coordinates x^j is given by $\sqrt{|\det g|}\, dx$, cf. [2, 1.74]).

Definition 2. Let M be the complete Riemannian manifold described in 3.1 with the injectivity radius $r_0>0$. Let k_0, k and k_N be the functions from (3)—(5) and (19), (20). Let $K(k_0, t)f$ and $K(k_N, t)f$ be given by (21). Let $0<r<r_0$ and $-\infty<s<\infty$.

(i) Let either $0<p<\infty$, $0<q\leqq\infty$ or $p=q=\infty$. Let $N>\max(s, 5+2n/p)+\tilde{\sigma}_p$ be a natural number and $0<\varepsilon<\varepsilon_0<r_0$. Then we put

$$\|f|F^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r} = \|K(k_0,\varepsilon)f|L_p(M)\| + \left\|\left(\int_0^r t^{-sq}|(K(k_N,t)f)(\cdot)|^q \frac{dt}{t}\right)^{1/q} |L_p(M)\right\| \tag{25}$$

(modification if $q=\infty$).

(ii) Let $0<p\leqq\infty$, $0<q\leqq\infty$. Let $N>\max(s, 5+2n/p)+\tilde{\sigma}_p$ be a natural number and $0<\varepsilon<\varepsilon_0<r_0$. Then we put

$$\|f|B^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r} = \|K(k_0,\varepsilon)f|L_p(M)\| + \left(\int_0^r t^{-sq}\|K(k_N,t)f|L_p(M)\|^q \frac{dt}{t}\right)^{1/q} \tag{26}$$

(modification if $q=\infty$).

Remark 7. Both (25) and (26) make sense for any $f\in D'(M)$ (with ∞ as an admissible value). It is the main goal of this paper to prove that these expressions are equivalent quasi-norms in the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ which will be defined below provided that ε_0 in dependence on s, p, q and on the chosen means is sufficiently small and N has the above meaning. $\tilde{\sigma}_p$ has been defined in front of Theorem A. Of course such an assertion is an extension of Theorem A from $\mathbf{R}_n$ to M. The assumption for N is rough and not natural, in contrast to the corresponding assumption for N in Theorem A. In the Remarks 20 and 22 below we give a discussion about the numbers N and ε_0. The spaces $F^s_{p,q}(M)$ on Riemannian manifolds are more natural than the spaces $B^s_{p,q}(M)$. For this reason we include now the spaces with $p=q=\infty$ in the F-scale, in contrast to our habit for the corresponding spaces on $\mathbf{R}_n$.

Now we define the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$. Recall that $(A_0, A_1)_{\Theta,q}$ with $0<\Theta<1$ and $0<q\leqq\infty$ stands for the real interpolation method, where $\{A_0, A_1\}$ is an interpolation pair of quasi-Banach spaces. A description of the real interpolation method in the case of Banach spaces has been given in [28, 1.3]. There are no problems to extend these considerations to quasi-Banach spaces, cf. [3, Chapter 3] or [29, 2.4.1]. Recall that $D'(M)$ stands for the collection of all complex distributions on M. Furthermore we put $F^s_{\infty,\infty}(\mathbf{R}_n)=B^s_{\infty,\infty}(\mathbf{R}_n)$, $-\infty<s<\infty$, in the sense of the last remark.

Definition 3. Let M be the complete Riemannian manifold described in 3.1. Let $\psi=\{\psi_j\}_{j=1}^\infty$ be a resolution of unity in the sense of Proposition C and Remark 6. Let $-\infty<s<\infty$.

(i) Let either $0<p<\infty$, $0<q\leqq\infty$ or $p=q=\infty$. Then

(27) $$F^s_{p,q}(M)=\{f\,|\,f\in D'(M),\ \|f\,|F^s_{p,q}(M)\|^{\psi}=(\sum_{j=1}^{\infty}\|\psi_j f\circ\exp_{P_j}|F^s_{p,q}(\mathbf{R}_n)\|^p)^{1/p}<\infty\}$$

(modification if $p=\infty$).

(ii) Let $0<p\leqq\infty$, $0<q\leqq\infty$ and $-\infty<s_0<s<s_1<\infty$. Then

(28) $$B^s_{p,q}(M)=(F^{s_0}_{p,p}(M),\ F^{s_1}_{p,p}(M))_{\Theta,q}$$

with $s=(1-\Theta)s_0+\Theta s_1$.

Remark 8. If $M=\mathbf{R}_n$ then $F^s_{p,q}(M)$ coincides with $F^s_{p,q}(\mathbf{R}_n)$. In this case one can assume that $\mathbf{R}_n$ is covered by congruent balls $\Omega_{P_j}(\delta)$ in the previous notation and that the ψ_j's are mutually connected via translations. But we shall not need this known fact, we shall obtain it as a by-product of our more general considerations. Of course, it is assumed that $\psi_j f\circ\exp_{P_j}$ in (27) is extended outside of $B(r)$ by zero, cf. (13). However this extension is out of interest if one uses appropriate local quasi-norms for $F^s_{p,q}(\mathbf{R}_n)$, e.g. that ones from Theorem A. In order to justify (27) we must prove that the definition of $F^s_{p,q}(M)$ is independent of the chosen resolution of unity. Furthermore, the definition (28) is justified if we know that it is independent of s_0 and s_1.

3.3. Basic theorems. All notations have the same meaning as in the two preceding subsections.

Theorem 1. *Let* $-\infty<s<\infty$.

(i) *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Then* $F^s_{p,q}(M)$ *from Definition* 3(i) *is a quasi-Banach space (Banach space if* $p\geqq 1$ *and* $q\geqq 1$*). It is independent of the chosen local charts* $\{(\Omega_{P_j}(r),\exp^{-1}_{P_j})\}$ *and the corresponding resolution of unity* $\psi=\{\psi_j\}$.

(ii) *Let* $0<p\leqq\infty$, $0<q\leqq\infty$. *Then* $B^s_{p,q}(M)$ *from Definition* 3(ii) *is a quasi-Banach space (Banach space if* $p\geqq 1$ *and* $q\geqq 1$*). It is independent of the numbers* s_0 *and* s_1.

Our next theorem is the main goal of this paper. Let again $\psi=\{\psi_j\}_{j=1}^{\infty}$ be a resolution of unity in the sense of Proposition C and Remark 6. All the other notations have the same meaning as in Definition 2. In particular, $\varepsilon_0>0$ is assumed to be sufficiently small, where a discussion of this point will be given below in the Remarks 20 and 22.

Theorem 2. *Let the hypotheses of the Definitions* 2 *and* 3 *be satisfied. Let* $-\infty<s<\infty$.

(i) *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Then* $\|f\,|F^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r}$, *cf.*

(25), *is an equivalent quasi-norm in* $F^s_{p,q}(M)$. *Furthermore*

$$\|f|F^s_{p,q}(M)\|^p \sim \sum_{j=1}^{\infty} \|\psi_j f|F^s_{p,q}(M)\|^p \quad \text{if} \quad p<\infty \tag{29}$$

and

$$\|f|F^s_{\infty,\infty}(M)\| \sim \sup_j \|\psi_j f|F^s_{\infty,\infty}(M)\| \tag{30}$$

(equivalent quasi-norms).

(ii) *Let* $0<p\leqq\infty$, $0<q\leqq\infty$. *Then* $\|f|B^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r}$, *cf.* (26), *is an equivalent quasi-norm in* $B^s_{p,q}(M)$.

(iii) *Let* $0<p\leqq\infty$. *Then holds* $F^s_{p,p}(M)=B^s_{p,p}(M)$.

Remark 9. We shall not distinguish between equivalent quasi-norms of a given space, $\|f|F^s_{p,q}(M)\|$ stands for an arbitrary equivalent quasi-norm in $F^s_{p,q}(M)$, and (29), (30) must be understood in this sense. A representation of type (29), (30) for the spaces $B^s_{p,q}(M)$ with $p\neq q$ cannot be expected.

Recall that $\tilde{\sigma}_p=0$ if $1\leqq p\leqq\infty$ and $\tilde{\sigma}_p=n\left(\frac{1}{p}-1\right)$ if $0<p<1$.

Theorem 3. *Let the hypotheses of the Definitions* 2 *and* 3 *be satisfied.*

(i) *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Let* $s>\tilde{\sigma}_p$. *Then*

$$\|f|F^s_{p,q}(M)\|^{k_N}_r = \|f|L_p(M)\| + \left\| \left(\int_0^r t^{-sq} |(K(k_N, t)f)(\cdot)|^q \frac{dt}{t} \right)^{1/q} |L_p(M) \right\| \tag{31}$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $F^s_{p,q}(M)$.

(ii) *Let* $0<p\leqq\infty$, $0<q\leqq\infty$ *and* $s>\tilde{\sigma}_p$. *Then*

$$\|f|B^s_{p,q}(M)\|^{k_N}_r = \|f|L_p(M)\| + \left(\int_0^r t^{-sq} \|K(k_N, t)f|L_p(M)\|^q \frac{dt}{t} \right)^{1/q} \tag{32}$$

is an equivalent quasi-norm in $B^s_{p,q}(M)$.

Remark 10. The Theorems 2 and 3 are the counterparts of Theorem A and Proposition B(i, ii).

3.4. Special cases. Let M be again the complete Riemannian manifold from 3.1. Three types of spaces on M are of interest: (i) Hölder—Zygmund spaces $\mathscr{C}^s(M)$ with $s>0$, (ii) the Sobolev spaces $W^k_p(M)$ with $k=0, 1, 2, \ldots$, and $1<p<\infty$ in the sense of T. Aubin, cf. [1, 2] and (iii) the Bessel-potential spaces $H^s_p(M)$ with $-\infty<s<\infty$ and $1<p<\infty$ in the sense of *R.* S. Strichartz [24]. Recall that the Hölder—Zygmund spaces $\mathscr{C}^s(\mathbf{R}_n)$ are defined via differences of functions and sup-norms. One can imitate this procedure where differences are now taken along geodesics. But we postpone this natural way to introduce the spaces $\mathscr{C}^s(M)$ to the announced second paper [32] and restrict ourselves at the moment to the formal definition

$$\mathscr{C}^s(M) = F^s_{\infty,\infty}(M), \quad s > 0. \tag{33}$$

We refer to 3.6, in particular Remark 17, where we add some remarks about expected equivalent norms in $\mathscr{C}^s(M)$. In order to define the Sobolev spaces $W_p^k(M)$ we begin with some preliminaries. The covariant derivatives with respect to a given local chart, are denoted by ∇_j. If f is a complex C^∞-function on M and $k=0, 1, 2, \ldots$ then we put

$$|\nabla^k f|^2 = g^{\alpha_1\beta_1} \ldots g^{\alpha_k\beta_k} \nabla_{\alpha_1} \ldots \nabla_{\alpha_k} f \cdot \nabla_{\beta_1} \ldots \nabla_{\beta_k} \bar{f}. \tag{34}$$

Of course, (34) is invariant. With respect to normal coordinates, cf. (14), we have

$$|\nabla^k f|^2 = \sum_{\alpha_l=1}^n |\nabla_{\alpha_1} \ldots \nabla_{\alpha_k} f|^2. \tag{35}$$

The Laplace—Beltrami operator Δ in local coordinates is given by

$$\Delta f = \frac{1}{\sqrt{|\det g|}} \partial_j \left(\sqrt{|\det g|}\, g^{jk} \partial_k f\right), \tag{36}$$

where $\partial_k = \dfrac{\partial}{\partial x_k}$. On complete Riemannian manifolds this operator has been studied in detail in [24]. Recall that $D(M) = C_0^\infty(M)$ is the collection of all complex C^∞-functions on M with compact support. With $D(M)$ as its domain of definition it comes out that $E-\Delta$ is a positive-definite and essentially self-adjoint operator with respect to the Hilbert space $L_2(M)$, where E is the identity. The heat semigroup $\{e^{t\Delta}\}_{t\geq 0}$ and the Bessel-potentials $(E-\Delta)^{-s/2}$ with $s>0$ can be defined in $L_2(M)$ via the spectral theorem. They can be extended afterwards from $L_2(M)$ to $L_p(M)$ with $1<p<\infty$, cf. [24, Theorem 3.5 and Section 4] for details.

Definition 4. (i) Let $1<p<\infty$ and let k be a natural number. Let

$$\|f|W_p^k(M)\| = \sum_{l=0}^k \|\nabla^l f|L_p(M)\|. \tag{37}$$

Then $W_p^k(M)$ is the completion of $\{h|h\in C^\infty(M), \|f|W_p^k(M)\|<\infty\}$ in the norm (37).

(ii) Let $1<p<\infty$. If $s>0$ then $H_p^s(M)$ is the collection of all $f\in L_p(M)$ such that $f=(E-\Delta)^{-s/2}h$ for some $h\in L_p(M)$, with the norm $\|f|H_p^s(M)\| = \|h|L_p(M)\|$. If $s<0$ then $H_p^s(M)$ is the collection of all $f\in D'(M)$ of the form $f=(E-\Delta)^k h$ with $h\in H_p^{2k+s}(M)$, where k is a natural number such that $2k+s>0$, and $\|f|H_p^s(M)\| = \|h|H_p^{2k+s}(M)\|$. If $s=0$ then $H_p^0(M) = L_p(M)$.

Remark 11. Part (i) of the definition coincides essentially with *T.* Aubin's definition, cf. [1, or 2, p. 32]. As we said part (ii) of the definition goes back to R. S. Strichartz, cf. [24, in particular Definition 4.1]. The spaces $H_p^s(M)$ with $s<0$ are independent of k (equivalent norms).

Theorem 4. (i) *(Paley—Littlewood theorem). Let* $1<p<\infty$ *and* $-\infty<s<\infty$. *Then*

$$H_p^s(M) = F_{p,2}^s(M). \tag{38}$$

(ii) *Let* $1<p<\infty$ *and* $k=0, 1, 2, \ldots$. *Then*

$$W_p^k(M) = H_p^k(M) = F_{p,2}^k(M). \tag{39}$$

3.5. Further properties. By Definition 3(i) it is not surprising that many properties of the $F_{p,q}^s$-spaces, and, via elementary embeddings, also of the $B_{p,q}^s$-spaces, can be transferred from $\mathbf{R}_n$ to the above Riemannian manifold M (it is always assumed that M satisfies the hypotheses from 3.1). We describe some of these properties, in particular those ones which are useful in this paper and in [32]. As above $(A_0, A_1)_{\Theta,q}$ stands for the real interpolation method, where $\{A_0, A_1\}$ is an interpolation pair of two quasi-Banach spaces, $0<\Theta<1$, $0<q\leqq\infty$. Recall that $D(M)$ is the collection of all complex C^∞-functions on M with compact support. Finally, "$\subset$" indicates continuous embedding.

Theorem 5. (i) *(Density). Let* $0<p<\infty$, $0<q<\infty$ *and* $-\infty<s<\infty$. *Then* $D(M)$ *is dense both in* $F_{p,q}^s(M)$ *and* $B_{p,q}^s(M)$.

(ii) *(Elementary embeddings.) Let* $0<p<\infty$, $0<q\leqq\infty$ *and* $-\infty<s<\infty$. *Then*

$$B_{p,\min(p,q)}^s(M) \subset F_{p,q}^s(M) \subset B_{p,\max(p,q)}^s(M). \tag{40}$$

(iii) *(Embeddings with different metrics.) If* $0<p\leqq\infty$, $0<q\leqq\infty$ *and* $0<\sigma\leqq s-n/p$ *then*

$$B_{p,q}^s(M) \subset \mathscr{C}^\sigma(M). \tag{41}$$

If $0<p\leqq 1$, $0<q\leqq\infty$ *and* $s>n\left(\frac{1}{p}-1\right)$ *then*

$$B_{p,q}^s(M) \subset L_1(M). \tag{42}$$

(iv) *(Real interpolation.) Let* $0<p<\infty$, $0<q_0\leqq\infty$, $0<q_1\leqq\infty$ *and* $-\infty<s_0<s_1<\infty$. *Let* $0<q\leqq\infty$, $0<\Theta<1$, *and*

$$s = (1-\Theta)s_0+\Theta s_1. \tag{43}$$

Then

$$\left(B_{p,q_0}^{s_0}(M), B_{p,q_1}^{s_1}(M)\right)_{\Theta,q} = \left(F_{p,q_0}^{s_0}(M), F_{p,q_1}^{s_1}(M)\right)_{\Theta,q} = B_{p,q}^s(M). \tag{44}$$

Remark 12. These assertions are well-known if one replaces M by $\mathbf{R}_n$. The extension from $\mathbf{R}_n$ to M is mainly based on Definition 3. In that way the full range of embedding theorems with different metrics proved in [29, 2.7] can be extended from $\mathbf{R}_n$ to M, in this sense (41) and (42) are examples which we need in [32]. Furthermore, (40) can be complemented by the monotonicity assertions (i)

$$B_{p,q_0}^s(M) \subset B_{p,q_1}^s(M) \quad \text{and} \quad F_{p,q_0}^s(M) \subset F_{p,q_1}^s(M) \tag{45}$$

where $0<p\leqq\infty$ ($p<\infty$ in the case of the F-spaces), $-\infty<s<\infty$, $0<q_0\leqq q_1\leqq\infty$, and (ii)

$$B^{s_0}_{p,q_0}(M)\subset B^{s_1}_{p,q_1}(M) \quad \text{and} \quad F^{s_0}_{p,q_0}(M)\subset F^{s_1}_{p,q_1}(M) \tag{46}$$

with $0<p\leqq\infty$ ($p<\infty$ in the case of the F-spaces), $0<q_0\leqq\infty$, $0<q_1\leqq\infty$ and $-\infty<s_1<s_0<\infty$.

Remark 13. The problem of interpolation (real and complex) is more delicate, in particular if the spaces $B^s_{p,q}(M)$ are involved, and the reduction of the spaces on M to those ones on $\mathbf{R}_n$ is only of restricted use. On the other hand in some cases interpolation results can be simply transferred from $\mathbf{R}_n$ to M. We describe an example. Let $[A_0, A_1]_\Theta$ with $0<\Theta<1$ be the classical complex interpolation method, cf. [28, 1.9]. We have

$$[F^{s_0}_{p_0,q_0}(M), F^{s_1}_{p_1,q_1}(M)]_\Theta = F^s_{p,q}(M) \tag{47}$$

where $-\infty<s_0<\infty$, $-\infty<s_1<\infty$, $1<p_0<\infty$, $1<p_1<\infty$, $1<q_0<\infty$, $1<q_1<\infty$, $0<\Theta<1$, and

$$s=(1-\Theta)s_0+\Theta s_1,\ \frac{1}{p}=\frac{1-\Theta}{p_0}+\frac{\Theta}{p_1},\ \frac{1}{q}=\frac{1-\Theta}{q_0}+\frac{\Theta}{q_1}. \tag{48}$$

In particular, by (38), we obtain

$$[H^{s_0}_p(M), H^{s_1}_p(M)]_\Theta = H^s_p(M) \tag{49}$$

with (48). This coincides with [24, Corollary 4.6] proved by other means $\big($: $(E-\Delta)^{it}$, t real, is bounded on $L_p(M)$, $1<p<\infty\big)$ and under weaker assumptions for M.

The next theorem is crucial for our approach. Recall that the Laplace—Beltrami operator Δ is given by (36).

Theorem 6. *(Lift property.) Let* $0<p\leqq\infty$ *(with* $p<\infty$ *in the case of the spaces* $F^s_{p,q}(M)$*),* $0<q\leqq\infty$ *and* $-\infty<s<\infty$*. Then* $f\to f-\Delta f$ *yields an isomorphic map from* $F^s_{p,q}(M)$ *onto* $F^{s-2}_{p,q}(M)$ *and from* $B^s_{p,q}(M)$ *onto* $B^{s-2}_{p,q}(M)$*.*

Remark 14. Recall that fractional powers of $E-\Delta$ had been used in Definition 4(ii) in order to introduce the Bessel potential spaces $H^s_p(M)$, cf. [24].

In Proposition A we formulated the discrete versions (9) and (10) of (7) and (8), respectively, for the respective quasi-norms in the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$. Maybe the continuous versions look more handsome but the discrete versions are often more useful.

Theorem 7. *Let the hypotheses of the Definitions* 2 *and* 3 *be satisfied. Let* $-\infty<s<\infty$ *and let* L *be a natural number with* $2^{-L}\leqq r$*, cf. Definition* 2.

(i) *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Let* N *and* ε *be the same numbers as in Definition* 2(i). *Then*

$$\|K(k_0,\varepsilon)f|L_p(M)\|+\left\|\left(\sum_{j=L}^{\infty}2^{jsq}|(K(k_N,2^{-j})f)(\cdot)|^q\right)^{1/q}|L_p(M)\right\| \tag{50}$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $F^s_{p,q}(M)$. *If, in addition,* $s>\tilde{\sigma}_p$ *then* $\|K(k_0,\varepsilon)f|L_p(M)\|$ *in* (50) *can be replaced by* $\|f|L_p(M)\|$.

(ii) *Let* $0<p\leqq\infty$, $0<q\leqq\infty$. *Let* N *and* ε *be the same numbers as in Definition* 2(ii). *Then*

$$\|K(k_0,\varepsilon)f|L_p(M)\|+\left(\sum_{j=L}^{\infty}2^{jsq}\|K(k_N,2^{-j})f|L_p(M)\|^q\right)^{1/q} \tag{51}$$

(modification if $q=\infty$*) is an equivalent quasi-norm in* $F^s_{p,q}(M)$. *If, in addition,* $s>\tilde{\sigma}_p$ *then* $\|K(k_0,\varepsilon)f|L_p(M)\|$ *in* (51) *can be replaced by* $\|f|L_p(M)\|$.

Remark 15. All numbers have the same meaning as in Definition 2. The above theorem is the counterpart of Theorem 2(i, ii) and Theorem 3.

3.6. Characterizations via differences. It is a known fact that the spaces $B^s_{p,q}(\mathbf{R}_n)$ and $F^s_{p,q}(\mathbf{R}_n)$ for large values of s can be characterized via derivatives and differences, cf. [29, 2.5.9—2.5.12] and [30]. The question arises whether corresponding characterizations for the spaces $F^s_{p,q}(M)$ and $B^s_{p,q}(M)$ exist. A detailed study of this problem will be given in [32]. However in order to provide an impression we formulate here some assertions which can be expected.

Let $P\in M$ and $X\in T_PM$. Let m be a natural number. Then

$$(\Delta^m_X f)(P)=\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f\left(c\left(P,X,\frac{j}{m}\right)\right) \tag{52}$$

are the m-th differences along the geodesic $c(P,X,t)$. Similar as in [30] we introduce means of differences. Let $G\in S(\mathbf{R}_n)$ be non-negative and rotation-invariant (i.e. $G(x)$ depends only on $|x|$) with $G(0)>0$ and $\operatorname{supp} G\subset\{y||y|\leqq 1\}$. Then we put

$$[D^m(G,t)f](P)=\int_{T_PM}G(X)(\Delta^m_{tX}f)(P)\,dX,\quad P\in M,\quad 0<t<r<r_0, \tag{53}$$

where m is a natural number and r_0 stands for the injectivity radius of M. Recall that we introduced in 3.2 an invariant volume element dX on T_PM. The integration in (53) must be understood in this sense. Recall $\tilde{\sigma}_p=n\left(\frac{1}{\min(p,1)}-1\right)$ and $\tilde{\sigma}_{p,q}=n\left(\frac{1}{\min(p,q,1)}-1\right)$. Let $B_P(r)=\{X|X\in T_PM,\ \|X\|<r\}$ be a ball of radius r in the tangent space T_PM, where $P\in M$.

Conjecture. Let $0<q\leqq\infty$ and $0<r<r_0$ where the latter stands for the injectivity radius.

(i) Let either $0<p<\infty$, $0<q\leqq\infty$ or $p=q=\infty$. Let $\frac{n}{\min(p\,q)}<s<m$ where m is a natural number. Then

$$\|f|L_p(M)\|+\left\|\left(\int_{B_P(r)}\|X\|^{-sq}|(\Delta_X^m f)(\cdot)|^q\frac{dX}{\|X\|^n}\right)^{1/q}|L_p(M)\right\| \tag{54}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $F^s_{p,q}(M)$.

(ii) Let $0<p<\infty$ and let $\tilde{\sigma}_{p,q}<s<m$, where m is a natural number. Then

$$\|f|L_p(M)\|+\left\|\left(\int_0^r t^{-sq}|(D^m(G,t)f)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(M)\right\| \tag{55}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $F^s_{p,q}(M)$.

(iii) Let $0<p\leqq\infty$ and let $\tilde{\sigma}_p<s<m$, where m is a natural number. Then

$$\|f|L_p(M)\|+\left(\int_0^r t^{-sq}\|D^m(G,t)f|L_p(M)\|^q\frac{dt}{t}\right)^{1/q} \tag{56}$$

(modification if $q=\infty$) is an equivalent quasi-norm in $B^s_{p,q}(M)$.

Remark 16. If $M=\mathbf{R}_n$ then the above conjecture is valid, cf. [30] (some assertions may also be found in [29]). It is one of the main aims of the announced paper [32] to prove this conjecture, at least partly, (maybe under some additional restrictions for s and m). Maybe the most striking feature of this conjecture is formula (54). If $M=\mathbf{R}_n$, then one has a well-known counterpart for the spaces $B^s_{p,q}(\mathbf{R}_n)$:

$$\|f|L_p(\mathbf{R}_n)\|+\left(\int_{|h|\leqq r}|h|^{-sq}\|\Delta_h^m f|L_p(\mathbf{R}_n)\|^q\frac{dh}{|h|^n}\right)^{1/q} \tag{57}$$

is an equivalent quasi-norm on $B^s_{p,q}(\mathbf{R}_n)$, provided that $0<p\leqq\infty$, $0<q\leqq\infty$ and $s>\tilde{\sigma}_p$. However for general Riemannian manifolds of the above type one has no counterpart of (57). One would need a natural one-to-one map between the vectors of different tangent spaces. If M is a Riemannian manifold with distant parallelism then one has such a relation, but this is essentially the Euclidean case.

Remark 17. Recall $\mathscr{C}^s(M)=F^s_{\infty,\infty}(M)$, $s>0$, cf. (33). In this case a modified version of (54) reads as follows,

$$\|f|\mathscr{C}^s(M)\|\sim\sup_{P\in M}|f(P)|+\sup_{P\in M}\sup_{X\in T_PM}\|X\|^{-s}|(\Delta_X^m f)(P)|, \tag{58}$$

$s>0$ and $m>s$, where m is a natural number. We hope to return to this characterization in [32].

4. Spaces on manifolds: Proofs

4.1. Proof of Theorem 1 (i). *Step 1.* (Independence.) We use in this step the pointwise multiplier property and the diffeomorphism property for the spaces $F^s_{p,q}(\mathbf{R}_n)$. We refer to [31] where one can find formulations which are especially well adapted to our purposes in this step. However these properties are known, cf. [29, 2.8.2 and 2.10.2]. Let $\psi=\{\psi_j\}_{j=1}^{\infty}$ and $\psi'=\{\psi'_k\}_{k=1}^{\infty}$ be two admissible resolutions of unity as they have been described in Proposition C and Remark 6. Let $(\Omega_{P_j}(r), \exp_{P_j}^{-1})$ and $(\Omega_{P'_k}(r'), \exp_{P'_k}^{-1})$ be the respective local charts. If j is given then J' collects all k's for which $\operatorname{supp}\psi_j$ and $\operatorname{supp}\psi'_k$ have a non-empty intersection. Here card J' can be estimated from above by a natural number which is independent of j. By our assumptions ψ_j and ψ'_k with $k\in J'$ can be treated within the same local chart, e.g. $(\Omega_{P_j}(r), \exp_{P_j}^{-1})$. By the pointwise multiplier property of the spaces $F^s_{p,q}(\mathbf{R}_n)$ we have

$$\|\psi_j f\circ \exp_{P_j}|F^s_{p,q}(\mathbf{R}_n)\| \leqq c\sum_{k\in J'}\|\psi'_k f\circ \exp_{P_j}|F^s_{p,q}(\mathbf{R}_n)\|,\quad f\in F^s_{p,q}(\mathbf{R}_n), \tag{59}$$

where c is independent of j. By the diffeomorphism property of the spaces $F^s_{p,q}(\mathbf{R}_n)$ we have

$$\|\psi'_k f\circ \exp_{P_j}|F^s_{p,q}(\mathbf{R}_n)\| \leqq c\|\psi'_k f\circ \exp_{P'_k}|F^s_{p,q}(R_n)\|,\quad k\in J'. \tag{60}$$

By our assumptions about the resolutions of unity $\{\psi_j\}$ and $\{\psi'_k\}$, and of the normal geodesic coordinates, cf. in particular (12) and (15), it follows that c in (60) can be chosen independently of j and $k\in J'$ (the diffeomorphic map in question is given by $\exp_{P'_k}^{-1}\circ\exp_{P_j}$ in a small neighbourhood of the origin). Now (59) and (60) prove the desired independence.

Step 2. (Completeness.) Let $\{\psi_j\}$ be a given resolution of unity on M in the above sense. Let $\lambda_j=(\sum_{k\in J}\psi_k)\circ\exp_{P_j}$ where J is the collection of all k such that $\operatorname{supp}\psi_k$ and $\operatorname{supp}\psi_j$ have a non-empty intersection. Furthermore $l_p(F^s_{p,q}(\mathbf{R}_n))$ has the usual meaning, in particular it is a (complete) quasi-Banach space with respect to the quasi-norm

$$\left\|\{f_l\}|l_p(F^s_{p,q}(\mathbf{R}_n))\right\| = \left(\sum_{l=1}^{\infty}\|f_l|F^s_{p,q}(\mathbf{R}_n)\|^p\right)^{1/p} \tag{61}$$

(modification if $p=\infty$). By the pointwise multiplier property it follows that the operator Λ, given by

$$\Lambda\{f_l\} = \sum_{j=1}^{\infty}\lambda_j f_j\circ\exp_{P_j}^{-1} \tag{62}$$

is a linear and bounded operator from $l_p(F^s_{p,q}(\mathbf{R}_n))$ into $F^s_{p,q}(M)$. (Of course, $\lambda_j f_j\circ\exp_{P_j}^{-1}$ is extended outside of $\Omega_{P_j}(r)$ by zero.) Furthermore, by definition,

Ψ, given by

(63) $$\Psi f=\{\psi_j f\circ \exp_{P_j}\}_{j=1}^{\infty},$$

is a linear and bounded operator from $F^s_{p,q}(M)$ into $l_p(F^s_{p,q}(\mathbf{R}_n))$. We have $\Lambda\Psi=\mathrm{id}$ (identity in $F^s_{p,q}(M)$). Let $\{f^k\}$ be a fundamental sequence in $F^s_{p,q}(M)$. Then $\{\Psi f^k\}$ is a fundamental sequence in $l_p(F^s_{p,q}(\mathbf{R}_n))$. Let f be its limit element. Then we have $\Lambda f\in F^s_{p,q}(M)$ and

$$\|\Lambda f-f^k|F^s_{p,q}(M)\| = \|\Lambda f-\Lambda\Psi f^k|F^s_{p,q}(M)\| \leqq c\big\|f-\Psi f^k|l_p\big(F^s_{p,q}(\mathbf{R}_n)\big)\big\|.$$

Hence, Λf is the desired limit element and $F^s_{p,q}(M)$ is complete.

4.2. Proof of Theorem 2(i): the case of large values of s. We prove Theorem 2(i) and (29) for $0<p<\infty$, $0<q\leqq\infty$ and large values of s. In the same way one proves Theorem 2(i) and (30) for $p=q=\infty$ and large values of s.

Step 1. The resolution of unity $\{\psi_j\}_{j=1}^{\infty}$ has the same meaning as in Proposition C and Remark 6, in particular δ in (17) is assumed to be small. Let the numbers ε and r, and the kernels k_0 and k_N be fixed, cf. Definition 2. For sake of brevity we write $\|\cdot|F^s_{p,q}(M)\|$ instead of $\|\cdot|F^s_{p,q}(M)\|^{k_0 k_N}_{\varepsilon,r}$, cf. (25). But $\|\cdot|F^s_{p,q}(M)\|$ should not be mixed with $\|\cdot|F^s_{p,q}(M)\|^{\psi}$ from (27): We have to prove that $\|\cdot|F^s_{p,q}(M)\|$ is an equivalent quasi-norm in $F^s_{p,q}(M)$. By the local character of the means $K(k_N,t)f$, cf. (21), and the properties of the functions ψ_j we have

(64) $$\|f|F^s_{p,q}(M)\|^p\leqq c\sum_{j=1}^{\infty}\|\psi_j f|F^s_{p,q}(M)\|^p.$$

Step 2. We prove

(65) $$\|\psi_j f|F^s_{p,q}(M)\|\sim\|(\psi_j f)\circ\exp_{P_j}|F^s_{p,q}(\mathbf{R}_n)\|,\quad f\in F^s_{p,q}(M)$$

(equivalent quasi-norms, where the equivalence-constants may be chosen independently of j). Recall that $\|\cdot|F^s_{p,q}(M)\|$ is the abbreviation from Step 1, cf. also Definition 3. Let $h=\psi_j f\circ\exp_{P_j}$ and $U=B(r')$, cf. (13), with $\delta+r<r'<r_0$, where r_0 is the injectivity radius of M and r has the same meaning as in Definition 2. In order to calculate $[K(k_0,\varepsilon)\psi_j f](P)$ and $[K(k_N,t)\psi_j f](P)$ with $\varepsilon<r$ and $t<r$ we may assume $P\in\Omega_{P_j}(r')=\exp_{P_j}U$ because for other points $P\in M$ these means are zero. Furthermore, if $P\in\Omega_{P_j}(r')$ then only geodesics $c(P,X,t)$ with $0<t<r$, $X\in T_PM$, $\|X\|\leqq 1$, are of interest which have a non-empty intersection with $\operatorname{supp}\psi_j$ and for which, consequently, $c(P,X,t)\subset\Omega_{P_j}(r')$ holds. Hence the proof of (65) can be reduced to the proof of

(66) $$\|h|F^s_{p,q}(\mathbf{R}_n)\|\sim\|K(k_0,\varepsilon)h|L_p(U)\| +\left\|\left(\int_0^r t^{-sq}|(K(k_N,t)h)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(U)\right\|$$

with $h(x)=0$ if $|x|>r'$,

$$[K(k_N, t)h](x) = \int_{\mathbf{R}_n} k_N'(|X|)h(C(x, X, t))dX \quad \text{if} \quad x\in U, \tag{67}$$

$|X|\leqq 1$, $t<r$, and $C(x, X, t)\subset U$ (similarly for k_0 instead of k_N). For the components C^a of C we have

$$C^a(x, X, t) = x^a+tX^a+\sum_{l=2}^{2L-1} \frac{t^l}{l!}\frac{d^l C^a}{dt^l}(x, X, 0)+\frac{t^{2L}}{(2L)!}\frac{d^{2L}C^a}{dt^{2L}}(x, X, \vartheta^a t), \tag{68}$$

where $0\leqq\vartheta^a\leqq 1$. The natural number L will be chosen later on. By (12) and (15) we have

$$C^a(x, X, t) = x^a+tX^a+\sum_{2\leqq|\alpha|\leqq 2L-1} t^{|\alpha|}b_\alpha^a(x)X^\alpha+t^{2L}R_{2L}^a(x, X, t), \tag{69}$$

$a=1, ..., n$ and $0<t<r$, the $b_\alpha^a(x)$'s are uniformly bounded C^∞-functions on U (with respect to j) and $|R_{2L}^a(x, X, t)|\leqq c$ for all $x\in U$, $0<t<r$ and $|X|\leqq 1$ (we used again (12) and (15) and the fact that $g_{ij}\frac{dC^i}{dt}\frac{dC^j}{dt}=|X|^2$ is constant along the geodesic). We have

$$|h(C(x, X, t))-h(x+tX+\sum_{2\leqq|\alpha|\leqq 2L-1} t^{|\alpha|}b_\alpha(x)X^\alpha)| \leqq ct^{2L}\sum_{a=1}^n \left|\frac{\partial h}{\partial x^a}(...)\right| \tag{70}$$

with $b_\alpha(x)=(b_\alpha^a(x))_{a=1}^n$. We put (69) in (67) and use (70). The term which comes from the right-hand side of (70) can be estimated from above by

$$ct^{2L}\sum_{a=1}^n \sup_{|x-y|\leqq C'}\left|\frac{\partial h}{\partial y^a}(y)\right|, \tag{71}$$

where c and c' are independent of t, X and x. Let $2L>s$. We put (67) in the second term of the right-hand side in (66) and use the above considerations. The term which comes from (71) can be estimated from above by

$$c\left\|\sup_{|x-y|\leqq C'}\left|\frac{\partial h}{\partial y^a}(y)\right| |L_p(U)\right\| \leqq c''\|h|F_{p,q}^\sigma(\mathbf{R}_n)\| \tag{72}$$

with $\sigma>1+n/p$, cf. (11). Now we fix our assumptions about s by

$$L+n/p < s < 2L. \tag{73}$$

If p and q are given, then (73) covers all sufficiently large numbers s. It is convenient for us to assume that not only s but all numbers s' with $s-1<s'<\infty$ are covered by (73) (for suitable natural numbers L). We have

$$(74)\qquad h\left(x+tX+\sum_{2\leq|\alpha|\leq 2L-1} t^{|\alpha|} b_\alpha(x) X^\alpha\right)$$
$$= \sum_{0\leq|\beta|\leq L-1} D^\beta h(x+tX)\left(\sum_{2\leq|\alpha|\leq 2L-1} t^{|\alpha|} b_\alpha(x) X^\alpha\right)^\beta \frac{1}{\beta!}\cdot$$
$$+\sum_{|\beta|=L} D^\beta h(\ldots)\left(\sum_{2\leq|\alpha|\leq 2L-1}\ldots\right)^\beta \frac{1}{\beta!}.$$

By (73) we have $F^s_{p,q}(\mathbf{R}_n)\subset C^L(\mathbf{R}_n)$, cf. [29, 2.7.1]. Hence (74) makes sense. We use (67) with the left-hand side of (74) instead of $h(C(x, X, t))$ and substitute (74). Afterwards we put the result in the second term on the right-hand side of (66). The term which comes from the last expression in (74) can be estimated from above by

$$(75)\qquad c\sum_{|\beta|=L}\left\|\sup_{|x-y|\leq c'} |(D^\beta h)(y)|\,|L_p(U)\right\| \leq c''\|h|F^\varkappa_{p,q}(\mathbf{R}_n)\|$$

with $\varkappa>L+n/p$, where we used $2L>s$ and (11). The term in (66) which comes from the expression with $|\beta|=0$ on the right-hand side of (74) is just what we want, cf. (8) and (6). Then there remain the expressions on the right-hand side of (74) with $0<|\beta|\leq L-1$. These expressions yield means of the type

(76)
$$\int_{\mathbf{R}_n} k'_N(|X|)\,t^{|\alpha|} X^\alpha (D^\beta h)(x+tX)\,dX,\quad 0<|\beta|\leq L-1,\quad 2|\beta|\leq|\alpha|\leq(2L-1)|\beta|,$$

multiplied with smooth bounded functions depending only on x. If $|\alpha|>s$ in (76) then the corresponding expressions in the second term on the right-hand side of (66) can be estimated in the same way as in (75), now with $|\beta|<L$, but with the same $\varkappa$. Let $|\alpha|\leq s$. Recall $k'_N(|X|)=k_N(X)$, $k_N=\left(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}\right)^N k$ and $N>s+\tilde{\sigma}_p$, cf. (19), (20) and Definition 2 where we assume $s>5+2n/p$. The kernels $k_N(X)X^\alpha=\tilde{k}_{N-|\alpha|}(X)$ are of the type needed in Theorem A with a function $\tilde{k}$ instead of k, $N-|\alpha|$ instead of N, and $s-|\alpha|$ instead of s: Maybe the counterpart of (4) (with $\tilde{k}$ instead of k) is not satisfied, but this is not necessary, because we are only interested in estimates from above, cf. Remark 3 in [31]. By these remarks, $N-|\alpha|>s-|\alpha|+\tilde{\sigma}_p$ and Theorem A it follows that the remaining terms in (76), incorporated in the second term on the right-hand side of (66), can be estimated from above by

$$(77)\qquad c\|D^\beta h|F^{s-|\alpha|}_{p,q}(\mathbf{R}_n)\| \leq c'\|h|F^{s-1}_{p,q}(\mathbf{R}_n)\|.$$

By (70), (72), (74), (75) (also with $|\beta|<L$) and (77) we have

$$(78)\qquad \left\|\left(\int_0^r t^{-sq}|(K(k_N, t)h-K^e(k_N, t)h)(\cdot)|^q\frac{dt}{t}\right)^{1/q} |L_p(U)\right\| \leq c\|h|F^{s-1}_{p,q}(\mathbf{R}_n)\|.$$

(Here we used that not only s, but also $s-1$ satisfies (73).) Recall that K^e stands for the Euclidean means defined in (6). The number c in (78) can be estimated by $c \leqq c'L$, where c' is independent of L and s. In the same way one obtains

$$(79) \qquad \|K(k_0, \varepsilon)h - K^e(k_0, \varepsilon)h|L_p(U)\| \leqq c\varepsilon\|h|F^\sigma_{p,q}(\mathbf{R}_n)\|;$$

where c is independent of ε with $0<\varepsilon<1$ and of $\sigma>1+n/p$: It is sufficient to use (70) with $L=1$ and (72). By (78) and (79) it follows

$$(80) \qquad \|\psi_j f|F^s_{p,q}(M)\| \leqq c\|h|F^s_{p,q}(\mathbf{R}_n)\|.$$

Furthermore, by (78) and (79) we have

$$(81) \qquad \|h|F^s_{p,q}(\mathbf{R}_n)\| \leqq c\|\psi_j f|F^s_{p,q}(M)\| + c'\|h|F^{s-1}_{p,q}(\mathbf{R}_n)\|$$
$$\leqq c\|\psi_j f|F^s_{p,q}(M)\| + \tfrac{1}{2}\|h|F^s_{p,q}(\mathbf{R}_n)\| + c''\sum_{b=0}^{B} 2^{bs}\|F^{-1}[\varphi_b Fh]|L_p(\mathbf{R}_n)\|,$$

cf. Definition 1, where B is an appropriate natural number. Similar as in (78) the number c'' in (81) can be estimated from above by $c'''L$, and hence by $\tilde{c}s$. Furthermore $2^B \sim s$. We use

$$(82) \qquad \|F^{-1}[\varphi_b Fh]|L_p(\mathbf{R}_n)\| \leqq \sigma\|h|F^s_{p,q}(\mathbf{R}_n)\| + c_\sigma\|K^e(k_0, \varepsilon)h|L_p(\mathbf{R}_n)\|,$$

where $\sigma>0$ is at our disposal and c_σ is independent of ε and s. We outline a proof of (82) in Remark 19 below. Now (81), (82) and (79) prove

$$(83) \qquad \|h|F^s_{p,q}(\mathbf{R}_n)\| \leqq c\|\psi_j f|F^s_{p,q}(M)\|,$$

provided that $\varepsilon>0$ is chosen sufficiently small. (Cf. Remark 20 below where we give an estimate of the admissible ε's.) The proof of (65) is complete.

Step 3. We prove

$$(84) \qquad \sum_{j=1}^{\infty} \|\psi_j f|F^s_{p,q}(M)\|^p \leqq c\|f|F^s_{p,q}(M)\|^p$$

under the above assumptions for s and L, cf. in particular (73). Recall that $\|f|F^s_{p,q}(M)\|$ has the same meaning as in Step 1. As in Step 2 all calculations can be done within the local charts $(\Omega_{P_j}(r'), \exp^{-1}_{P_j})$. For fixed j we use (66), (67) with

$$(85) \qquad h(C(x, X, t)) = \psi_j \circ \exp_{P_j}(C(x, X, t)) f \circ \exp_{P_j}(C(x, X, t)).$$

We use (69), but in contrast to Step 2 only in the factor $\psi(C(x, X, t))$ with $\psi = \psi_j \circ \exp_{P_j}$. We have (70) and (74) with ψ instead of h, where we now incorporate tX in $\sum_{2\leqq|\alpha|\leqq 2L-1}$, i.e. we have the sum $\sum_{1\leqq|\alpha|\leqq 2L-1} t^{|\alpha|} b_\alpha(x) X^\alpha$, and $x+tX$ on the right-hand side of (74) is replaced by x. The term of interest is $\beta=0$ in this modified formula (74): it is simply $\psi(x)$. We put h from (85) with $(\psi_j \circ \exp_{P_j})(x)$ instead of $\psi_j \circ \exp_{P_j}(C(x, X, t))$ in the second term on the right-hand side of (66).

The resulting term can be estimated from above by

(86)

$$c\left\|\left(\int_0^r t^{-sq}|K(k_N, t)(f\circ \exp_{P_j})(C(x, X, t))|^q \frac{dt}{t}\right)^{1/q} |L_p(\operatorname{supp}(\psi_j\circ \exp_{P_j}))\right\|.$$

Summation of the p-th power of (86) with respect to j and the properties of the resolution of unity $\{\psi_j\}$ yield the right-hand side of (84). The remaining terms come from the above-mentioned counterparts of (70) and (74) with $|\beta|>0$. In the respective counterparts of (85) we can replace $f\circ\exp_{P_j}$ by $(\tilde{\psi}_j f)\circ\exp_{P_j}$ with $\tilde{\psi}_j=\sum \psi_l$ where the sum is taken over all l with $\Omega_{P_j}(r')\cap \operatorname{supp}\psi_l\neq\emptyset$. In particular we have

$$(\psi_j\circ\exp_{P_j})(x)=(\psi_j\circ\exp_{P_j})(x)\tilde{\psi}_j(C(x, X, t))$$

for all x, X, t of interest and $\psi_j=\psi_j\tilde{\psi}_j$. This justifies this replacement. Furthermore, the number of elements in $\sum\psi_l$ can be estimated from above by a natural number which is independent of j. Now we expand $(\tilde{\psi}_j f)\circ\exp_{P_j}$ in the same way as in the second step. Together with the remaining expressions and factors from the modified expansion (74) it follows that all the other terms can be estimated from above by

$$\eta\|\tilde{\psi}_j f|F^s_{p,q}(M)\|+c_\eta\|K(k_0, \varepsilon)\tilde{\psi}_j f\circ\exp_{P_j}|L_p(U)\| \tag{87}$$

where $\eta>0$ is at our disposal, $\varepsilon>0$ is small, and c_η is independent of ε. We use (69) with $t=\varepsilon$ in order to expand $\tilde{\psi}_j(C(x, X, \varepsilon))$ in the second summand in (87). Then we obtain a term

$$\|K(k_0, \varepsilon)f\circ\exp_{P_j}|L_p(\operatorname{supp}(\tilde{\psi}_j\circ\exp_{P_j}))\| \tag{88}$$

and remainder terms with a factor ε in front which can be treated as above. We choose $\varepsilon>0$ and $\eta>0$ in (87) small. Summation over j yields

$$\sum_{j=1}^\infty\|\psi_j f|F^s_{p,q}(M)\|^p\leqq c\|f|F^s_{p,q}(M)\|^p+\tfrac{1}{2}\sum_{j=1}^\infty\|\psi_j f|F^s_{p,q}(M)\|^p.$$

The proof of (84) is complete.

Step 4. (64), (84) and (65) prove Theorem 2(i) if $p<\infty$ and s is large, cf. Definition 3(i). As had been said the proof of Theorem 2(i) with $p=q=\infty$ and s large is essentially the same.

Remark 18. Let the above conditions for p, q, s be satisfied, cf. in particular (73). By Proposition B(ii) one can replace $\|K^e(k_0, \varepsilon)f|L_p(\mathbf{R}_n)\|$ in (8) by $\|f|L_p(\mathbf{R}_n)\|$ (equivalent quasi-norms). Similarly one can try to replace $\|K(k_0, \varepsilon)f|L_p(M)\|$ in (25) by $\|f|L_p(M)\|$. Let us denote the corresponding quasi-norm $\|f|F^s_{p,q}(M)\|^{k_N}_r$ from (31) temporarily by $\|f|F^s_{p,q}(M)\|_*$ (k_N remains unchanged). Obviously we have (64) with $\|\cdot\|_*$ instead of $\|\cdot\|$. Furthermore, the proof of (65) with $\|\cdot\|_*$

instead of $\|\cdot\|$ on the left-hand side of (65) is essentially the same as in Step 2 (some details are simpler now). The counterpart of (82) with $\|h|L_p(\mathbf{R}_n)\|$ instead of $\|K^\varepsilon(k_0, \varepsilon)h|L_p(\mathbf{R}_n)\|$ may be found in [29, (37) in 2.5.9]. Finally, one proves (84) with $\|\cdot\|_*$ instead of $\|\cdot\|$ in the same way as in Step 3. In particular, $\|f|F^s_{p,q}(M)\|_*$ is an equivalent quasi-norm in $F^s_{p,q}(M)$. In other words, we just proved a special case of Theorem 3 (i).

Remark 19. We outline a proof of (82). Let $\hat{k}_0 = Fk_0$ and $h^-(z) = h(-z)$. We have

(89) $$[K^\varepsilon(k_0, \varepsilon)h](x) = F^{-1}[\hat{k}_0(\varepsilon\cdot)Fh^-](-x),$$

cf. (6), and

(90) $$(F^{-1}\varphi_b Fh)(x) = \left(F^{-1}\frac{\varphi_b}{\hat{k}_0(\varepsilon\cdot)}FF^{-1}\hat{k}_0(\varepsilon\cdot)Fh\right)(x),$$

where we omitted some brackets. Recall $\hat{k}_0(\varepsilon y) \neq 0$, cf. (5). Let $1 \leqq p \leqq \infty$. Then we obtain

(91) $$\|F^{-1}\varphi_b Fh|L_p(\mathbf{R}_n)\| \leqq \left\|F^{-1}\frac{\varphi_b}{\hat{k}_0(\varepsilon\cdot)}|L_1(\mathbf{R}_n)\right\| \|K^\varepsilon(k_0, \varepsilon)h|L_p(\mathbf{R}_n)\|$$
$$\leqq \left\|\frac{\varphi_b}{\hat{k}_0(\varepsilon\cdot)}|W_2^{1+[n/2]}(\mathbf{R}_n)\right\| \|K^\varepsilon(k_0, \varepsilon)h|L_p(\mathbf{R}_n)\|.$$

Now (5) yields (82), even with $\sigma=0$, where c_σ is independent of ε. If $0<p<1$ then the necessary modifications have been described in [31, (42), (43)]. We omit the details. But we wish to emphasize that in this case (at least in our proof) we need the first term on the right-hand side of (82).

Remark 20. If p, q and the kernels k_0, k_N are fixed, then an estimate of the admissible ε's in dependence on s is useful. If one puts (82) in (81) then one has to choose $\sigma \sim 2^{-s^2B'}$ with some $B'>0$. Consequently $c_\sigma \sim 2^{s^2A}$ for some $A>0$ (this follows from the proof of (82) in [31]). Hence by (79) we have $0<\varepsilon<2^{-s^2C}$ for some $C>0$, where C is independent of s, and s has the above meaning.

4.3. Lifting properties. The proof of Theorem 2 (i) for arbitrary values of s (where p and q are fixed) is based on Theorem 2 (i) for large values of s and a lifting procedure. Recall that Δ stands for the Laplace—Beltrami operator, cf. (36). Let k_0 and k_N be the same kernels as in 3.2, cf. in particular (3)—(5) and (19), (20). Furthermore, $K(k_0, \varepsilon)f$ and $K(k_N, t)f$ are the corresponding means, cf. (21). Let E be the identity. We prove a lifting property with respect to the operator $(E-\Delta)^G$, where G is a natural number.

Proposition 1. *Let* $-\infty<s<\infty$. *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Let G be a natural number such that the proof given in* 4.2 *can be applied to* $F^{s+2G}_{p,q}(M)$.

Let $\varepsilon>0$ *be an admissible number and let* k_0, k_N *be admissible kernels for the space* $F^{s+2G}_{p,q}(M)$. *Cf. Remark 20 and Definition 2. Then*

$$\text{(92)}\quad \|K(k_0,\varepsilon)f|L_p(M)\|+\left\|\left(\int_0^r t^{-(s+2G)q}|(K(k_{N+G},t)f)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(M)\right\|$$

$$\sim \|K(k_0,\varepsilon)(E-\Delta)^G f|L_p(M)\|+\left\|\left(\int_0^r t^{-sq}|(K(k_N,t)(E-\Delta)^G f)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(M)\right\|$$

(equivalent quasi-norms in $F^{s+2G}_{p,q}(M)$*). (Modification if* $q=\infty$.*)*

Proof. Step 1. We prove (92) with $G=1$. In normal geodesic coordinates with respect to a given point $P\in M$ we have

$$\text{(93)}\quad (K(k_N,t)\Delta f)(P)=\int_{\mathbf{R}_n}k_N(X)(\Delta f)(tX)\,dX$$

$$=t^{-2}\int_{\mathbf{R}_n}k_N(X)\frac{1}{\sqrt{|\det g(tX)|}}\frac{\partial}{\partial X^j}\left(\sqrt{|\det g(tX)|}\,g^{ij}(tX)\frac{\partial f(tX)}{\partial X^i}\right)dX$$

$$=t^{-2}\int_{\mathbf{R}_n}\frac{\partial}{\partial X^i}\left(g^{ij}(tX)\sqrt{|\det g(tX)|}\frac{\partial}{\partial X^j}\frac{k_N(X)}{\sqrt{|\det g(tX)|}}\right)f(tX)\,dX$$

$$=t^{-2}\int_{\mathbf{R}_n}k_{N+1}(X)f(tX)\,dX+\dots$$

where we used the Taylor expansions with respect to tX for the functions $g^{ij}(tX)$ and $\det g(tX)$, cf. (14). As in the preceding subsections one can give the explicit terms in (93) an invariant meaning, hence also the term $+\dots$ This term $+\dots$ can be represented as a finite sum of terms with factors $t^{|\alpha|-2}X^\alpha$ where $2\leqq|\alpha|\leqq L$, or $t^{|\beta|-1}X^\beta$ where $1\leqq|\beta|\leqq L$, or $t^{|\gamma|}X^\gamma$ where $0\leqq|\gamma|\leqq L$, multiplied with appropriate kernels and (in local coordinates x) with smooth functions of x, and a remainder term, say, with t^{L-2} in front. If L is large (as in 4.2) then all these terms can be treated in the same way as in 4.2 with the help of the previous resolution of unity $\{\psi_j\}$ and the reduction to the Euclidean case. (Recall that we assumed that the hypotheses of 4.2 with $s+2$ instead of s are satisfied.) In particular we have

$$\text{(94)}\quad \left\|\left(\int_0^r t^{-sq}|(+\dots)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(M)\right\|\leqq c\|f|F^{s+2-\delta}_{p,q}(M)\|$$

for some $\delta>0$. The term $(K(k_0,\varepsilon)\Delta f)(P)$ can be treated in the same way. Now it follows that the right-hand side of (92) can be estimated from above by the left-hand side and an additional term $c\|f|F^{s+2-\delta}_{p,q}(M)\|$. We may assume that 4.2 is not only applicable to $F^{s+2}_{p,q}(M)$ but also to $F^{s+2-\delta}_{p,q}(M)$. In particular the left-hand side of (92) is an equivalent quasi-norm in $F^{s+2}_{p,q}(M)$, and $\|f|F^{s+2-\delta}_{p,q}(M)\|$ can be estimated from above by the left-hand side. Hence, the right-hand side of (92) can be estimated from above by the left-hand side. We prove the converse assertion. For this purpose we replace $k_0(X)=k_0'(|X|)$ in (92) by $\tilde{k}_0(X)=k_0'(|X|)-k_1'(|X|)$, cf. (19), (20) with $k=k_0$. Furthermore (3), (5) are satisfied for $\tilde{k}_0$, too. This replacement can be done because the left-hand side of (92) with $\tilde{k}_0$ instead of k_0 is again an equivalent quasi-norm in $F^{s+2}_{p,q}(M)$. By partial integration as in (93) we obtain

$$(95)\qquad \|K(\tilde{k}_0,\varepsilon)f|L_p(M)\| \leqq c\|K(k_0,\varepsilon)(E-\Delta)f|L_p(M)\|+c\varepsilon\|f|F^{s+2}_{p,q}(M)\|,$$

where c is independent of ε. Consequently we have for small ε

$$(96)\qquad \|f|F^{s+2}_{p,q}(M)\| \leqq c\|K(k_0,\varepsilon)(E-\Delta)f|L_p(M)\|$$
$$+\left\|\left(\int_0^r t^{-sq}|(K(k_N,t)(E-\Delta)f)(\cdot)|^q\frac{dt}{t}\right)^{1/q}|L_p(M)\right\|+c\|f|F^{s+2-\delta}_{p,q}(M)\|,$$

where we used (94), (95). Next we use

$$(97)\qquad \|f|F^{s+2-\delta}_{p,q}(M)\| \leqq \eta\|f|F^{s+2}_{p,q}(M)\|+c_\eta\|K(\tilde{k}_0,\varepsilon)f|L_p(M)\|,$$

where $\eta>0$ is at our disposal. (97) follows essentially from (81), (82) and (79) (under the assumption that ε is small), and the technique of estimates used in 4.2. We put (97) in (96) and use afterwards (95). We choose $\varepsilon>0$ sufficiently small. Then it follows that the left-hand side of (92) can be estimated from above by the right-hand side.

Step 2. The proof of (92) for arbitrary natural numbers G is essentially the same as for $G=1$. First one must iterate (93). For the remainder term one has an obvious counterpart of (94). Then it follows that the right-hand side of (92) can be estimated from above by the left-hand side. Furthermore one has corresponding counterparts of (95)—(97), where now $\tilde{k}_0$ must be chosen in an appropriate way. This yields the desired estimate.

Remark 21. Let the hypotheses of Proposition 1 be satisfied and let in addition $s>\tilde{\sigma}_p$. $\left(\text{Recall } \tilde{\sigma}_p=0 \text{ if } 1\leqq p\leqq\infty \text{ and } \tilde{\sigma}_p=n\left(\frac{1}{p}-1\right) \text{ if } 0<p<1\right)$. Then $\|K(k_0,\varepsilon)f|L_p(M)\|$ and $\|K(k_0,\varepsilon)(E-\Delta)^G f|L_p(M)\|$ in (92) can be replaced by $\|f|L_p(M)\|$ and $\|(E-\Delta)^G f|L_p(M)\|$, respectively. We prove this claim. By Remark 18 one can replace $\|K(k_0,\varepsilon)f|L_p(M)\|$ in (25), with $s+2G$ instead of s, by $\|f|L_p(M)\|$. Because $s+2G>\tilde{\sigma}_p+2G$ one can even replace $\|K(k_0,\varepsilon)f|L_p(M)\|$ by $\|(E-\Delta)^G f|L_p(M)\|$. This follows by the same arguments as in Remark 18 and the known $\mathbf{R}_n$-counterpart. Now one can follow the above proof with the just indicated modified quasi-norms, cf. also Proposition B(ii). Furthermore there exists a counterpart of (97) with $\|f|L_p(M)\|$ instead of $\|K(k_0,\varepsilon)f|L_p(M)\|$. The corresponding assertion with $\mathbf{R}_n$ instead of M may be found in [29, (37) in 2.5.9]. The extension from $\mathbf{R}_n$ to M follows from (27). We assume $s-\delta>\tilde{\sigma}_p$ and obtain

$$\|f|F^{s-\delta}_{p,q}(M)\|^{\psi} \leqq \eta\|f|F^{s}_{p,q}(M)\|+c_\eta\|f|L_p(M)\|. \tag{98}$$

We replace f by $(E-\Delta)^G f$. By Theorem 6 (which will be proved later on independently of this remark) and 4.2 we obtain

$$\|f|F^{s+2G-\delta}_{p,q}(M)\| \leqq \eta\|f|F^{s+2G}_{p,q}(M)\|+c_\eta\|(E-\Delta)^G f|L_p(M)\|. \tag{99}$$

This is the desired substitute of (97). This completes the proof of the above claim (under the assumption that Theorem 6 is proved).

Remark 22. We discuss the assumptions for the numbers N and ε_0 in Definition 2. In 4.2 we used (73) and $N>s+\tilde{\sigma}_p$, cf. also the remarks after (73). In particular all s with $s>3+2n/p$ are covered. Hence in Proposition 1 we may assume

$$5+2n/p \geqq s+2G > 3+2n/p, \tag{100}$$

at least as far as small values of s are concerned (for large values of s a discussion of the possible values of N and ε_0 is not necessary, cf. Remark 20). Now (100) explains $N>\max(s, 5+2n/p)+\tilde{\sigma}_p$ in Definition 2. The considerations in 4.2 and in Proposition 1 are the basis in order to prove Theorem 2(i) for all values of s. Afterwards the spaces $B^s_{p,q}(M)$ are incorporated via real interpolation. There are no new conditions for N and ε. As far as ε_0 is concerned we use Remark 20. It follows that $\varepsilon_0=c2^{-C\max(0,s^2)}$ for some positive numbers c and C (which may depend on p and q) is sufficient.

4.4. Mapping properties of the Laplace—Beltrami operator. In order to prove Theorem 2(i) for arbitrary values of s we need a second preparation, beside Proposition 1. Recall that Δ stands always for the Laplace—Beltrami operator from (36). Furthermore, M is always the complete Riemannian manifold from 3.1. It is known that $-\Delta$ with $D(M)=C_0^\infty(M)$ as its domain of definition is a positive essentially self-adjoint operator with respect to the Hilbert space $L_2(M)$. We refer to [24, Section 2] and the papers mentioned there. Let G be a natural number. If $f\in D(M)$ is given then

$$(E-\Delta)^G g = f \quad \text{with} \quad g\in L_2(M)\cap C^\infty(M), \tag{101}$$

where g is the unique L_2-solution. The assertion $g\in C^\infty(M)$ follows from a local smoothness theory, which is Euclidean. We claim $g\in F^s_{2,2}(M)$ for any s, where $F^s_{2,2}(M)$ is given by (27). We have

$$\begin{aligned} &\|\psi_j g\circ\exp_{P_j}|F^s_{2,2}(\mathbf{R}_n)\| \\ &\leqq c\|(E-\Delta)^G\psi_j g\circ\exp_{P_j}|F^{s-2G}_{2,2}(\mathbf{R}_n)\|+c\|\psi_j g\circ\exp_{P_j}|F^{s-2G}_{2,2}(\mathbf{R}_n)\| \\ &\leqq c'\|\psi_j f\circ\exp_{P_j}|F^{s-2G}_{2,2}(\mathbf{R}_n)\|+c\sum\|\psi_l g\circ\exp_{P_j}|F^{s-1}_{2,2}(\mathbf{R}_n)\| \end{aligned} \tag{102}$$

where we may assume that the sum $\sum$ is restricted to those l with $\operatorname{supp} \psi_l \cap \operatorname{supp} \psi_j \neq \emptyset$. In (102) we used well-known properties for the corresponding spaces on $\mathbf{R}_n$. Hence, $g \in F^s_{2,2}(M)$ if $g \in F^{s-1}_{2,2}(M)$. Consequently the above claim follows by mathematical induction beginning with $s=1$.

Proposition 1. *Let either* $0<p<\infty$, $0<q\leqq\infty$ *or* $p=q=\infty$. *Let* $-\infty<s<\infty$. *If* f *and* g *have the above meaning then* $g \in F^s_{p,q}(M)$.

Proof. Step 1. If $p=q=2$ then the desired assertion is proved. Let $2<p<\infty$. Recall the embedding

$$F^{s_0}_{2,2}(\mathbf{R}_n) \subset F^{s_1}_{p,q}(\mathbf{R}_n), \quad 0<q\leqq\infty, \quad s_0-n/2>s_1-n/p, \tag{103}$$

and a corresponding assertion if $p=q=\infty$, cf. [29, 2.7.1]. Furthermore $l_2 \subset l_p$. Then it follows $g \in F^s_{p,q}(M)$, cf. (27) and the above considerations.

Step 2. We extend this assertion to $p\leqq 2$. Let $P_0 \in M$ be a fixed point and let $\Omega_{P_0}(\varrho)$ be the collection of all $P \in M$ with $\operatorname{dist}(P_0, P)<\varrho$. There exist functions $\lambda_l \in D(M)$ with $\lambda_l(P)=1$ if $P \in \Omega_{P_0}(2^l)$, $\operatorname{supp} \lambda_l \subset \Omega_{P_0}(2^{l+1})$ and

$$|D^\alpha(\lambda_l \circ \exp_P)| \leqq c_\alpha 2^{-l} \quad \text{in} \quad B(r), \quad P \in M, \tag{104}$$

cf. (13), where α is an arbitrary multi-index and c_α is independent of $l=1, 2, 3, \dots$. We refer to [1, 24, 33] where Lipschitz-continuous functions λ_l have been constructed which satisfy (104) for first-order derivatives and which have the other listed properties. Under our assumptions for M these Lipschitz-continuous functions can be mollified, e.g. via an invariant version of Sobolev's method, in a uniform way (with respect to the local charts). As far as an invariant version of Sobolev's mollification method is concerned one can use $K(k_0, t)f$ from (21) with the additional properties $k_0'(\|X\|)\geqq 0$ and $\int_{T_P M} k_0'(\|X\|)\,dX=1$. Then one obtains the above functions λ_l. Furthermore we remark that $\operatorname{vol} \Omega_{P_0}(2^l)\leqq c2^{l\nu}$, where c is independent of l, cf. (15′). Let again $\|\cdot|F^s_{p,q}(M)\|=\|\cdot|F^s_{p,q}(M)\|^{k_0, k_N}_{\varepsilon, r}$, cf. (25), where we know at this moment only for large values of s that $\|\cdot|F^s_{p,q}(M)\|$ is an equivalent quasi-norm in $F^s_{p,q}(M)$. Let $s<0$. We apply Proposition 1 to $\lambda_l g$ (instead of f). Then we have for large values of l

$$\|\lambda_l g|F^{s+2G}_{p,q}(M)\| \leqq c\|f|F^s_{p,q}(M)\|+c2^{-l}\sum_{k=0}^{2G-1}\Big\|\sup_{\operatorname{dist}(P,Q)\leqq r}|(\nabla^k g)(Q)|\,\Big|L_p(V_l)\Big\| \tag{105}$$

with $\operatorname{vol} V_l\leqq c2^{l\nu}$, cf. (35), where c is independent of l. We apply Hölder's inequality with respect to

$$\frac{1}{p}=\frac{1}{\nu}+\frac{1}{v} \tag{106}$$

to the last terms on the right-hand side of (105). Then we obtain

$$(107)\quad \|\lambda_l g|F^{s+2G}_{p,q}(M)\| \leqq c\|f|F^s_{p,q}(M)\|+c\sum_{k=0}^{2G-1}\Big\|\sup_{\mathrm{dist}(P,Q)\leqq r}|(\nabla^k g)(Q)|\Big|L_v(M)\Big\|.$$

Let $v\leqq 2$. Then it follows from Step 1 and (11) that the right-hand side of (107) is finite. Now $l\to\infty$ yields $g\in F^{s+2G}_{p,q}(M)$, cf. (25). (By a small modification of the above arguments it follows that $\{\lambda_l g\}_{l=1}^{\infty}$ is a fundamental sequence in $F^{s+2G}_{p,q}(M)$.) Step 1, (106) and mathematical induction prove the proposition.

Remark 23. Here we used for the first and the last time (15′). Step 1 is independent of this assumption. By duality arguments, similar as in Remark 24 below, we can extend Step 1 to all spaces $F^s_{p,q}(M)$ with $1\leqq p\leqq\infty$. In other words, (15′) is needed only for the case $p<1$. This justifies our claim in the footnote to (15′).

4.5. Proof of Theorem 2(i): the general case. The proof is a combination of the considerations in 4.2 and the two preceding propositions. We assume that either $0<p<\infty$, $0<q\leqq\infty$ or $p=q=\infty$. (In explicit formulations we mostly assume $p<\infty$.) Furthermore s is an arbitrary real number. The other notations have the same meaning as in 4.2.

Step 1. The proof of (64), i.e.

$$(108)\qquad \|f|F^s_{p,q}(M)\|^p \leqq c\sum_{j=1}^{\infty}\|\psi_j f|F^s_{p,q}(M)\|^p,$$

remains unchanged (modification if $p=\infty$). Recall that we put $\|\cdot|F^s_{p,q}(M)\|=\|\cdot|F^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r}$ for sake of brevity. Next we assume that $f\in F^s_{p,q}(M)$ can be represented as $f=(E-\Delta)^G g$ with $g\in F^{s+2G}_{p,q}(M)$. Then one obtains in the same way as in Proposition 1

$$(109)\qquad \sum_{j=1}^{\infty}\|\psi_j f|F^s_{p,q}(M)\|^p \leqq c\|g|F^{s+2G}_{p,q}(M)\|^p \leqq c'\|f|F^s_{p,q}(M)\|^p$$

where the last inequality comes from (92). Hence, (108) and (109) prove (29), under the assumption $f=(E-\Delta)^G g$ with $g\in F^{s+2G}_{p,q}(M)$. In the same way one proves (30).

Step 2. Let $\|f|F^s_{p,q}(M)\|^{\psi}$ be the quasi-norm from (27) where we assume that $\|\cdot|F^s_{p,q}(R_n)\|$ is given by the Euclidean counterpart (8) (modification if $p=\infty$) of the above quasi-norm $\|\cdot|F^s_{p,q}(M)\|$. Let again $f=(E-\Delta)^G g$, $g\in F^{s+2G}_{p,q}(M)$. We can replace $\|\psi_j f|F^s_{p,q}(M)\|$ in (109) by $\|\psi_j f\circ\exp_{P_j}|F^s_{p,q}(R_n)\|$ (because all the necessary information for the spaces $F^{s+2G}_{p,q}(M)$ is known). Hence we have

$$(110)\qquad \|f|F^s_{p,q}(M)\|^{\psi} \leqq c\|f|F^s_{p,q}(M)\|.$$

We prove the converse assertion of (110), again under the assumption $f=(E-\Delta)^G g$,

$g\in F^{s+2G}_{p,q}(M)$. We have

(111) $$\|g|F^{s+2G}_{p,q}(M)\|^p \leqq c\sum_{j=1}^{\infty}\|\psi_j g\circ \exp_{P_j}|F^{s+2G}_{p,q}(\mathbf{R}_n)\|^p$$
$$\leqq c'\sum_{j=1}^{\infty}\|\psi_j f\circ \exp_{P_j}|F^{s}_{p,q}(\mathbf{R}_n)\|^p + c''\|g|F^{s+2G-\delta}_{p,q}(M)\|^p$$

where we used $\psi_j f=(E-\Delta)^G\ \psi_j g+\dots$, the Euclidean version of Proposition 1 (but with the above operator Δ) and the above technique of estimates as far as the remainder term is concerned. As in (97) we have

(112) $$\|g|F^{s+2G-\delta}_{p,q}(M)\| \leqq \eta\|g|F^{s+2G}_{p,q}(M)\| + c_\eta\|K(\bar{k}_0, \varepsilon)g|L_p(M)\|.$$

If we choose $\bar{k}_0$ in an appropriate way (cf. Step 1 in 4.3 as far as the case $G=1$ is concerned) then the last term in (112) can be estimated from above by

(113) $$c'_\eta\|K(k_0, \varepsilon)f|L_p(M)\| + \sigma\|g|F^{s+2G}_{p,q}(M)\|,$$

where σ is small if ε is small. Now (111)—(113) yield

(114) $$\|f|F^{s}_{p,q}(M)\| \sim \|g|F^{s+2G}_{p,q}(M)\| \leqq c\|f|F^{s}_{p,q}(M)\|^{\psi}$$

where the equivalence relation comes again from Proposition 1. But (110) and (114) complete the proof of Theorem 2(i) (under the assumption $f=(E-\Delta)^G g$, $g\in F^{s+2G}_{p,q}(M)$) if $p<\infty$. One argues similar if $p=q=\infty$.

Step 3. Let $p<\infty$ and $q\leqq\infty$. By Definition 3(i) and the proof of Theorem 1(i) in 4.1 the distributions $f\in F^s_{p,q}(M)$ with compact support are dense in $F^s_{p,q}(M)$. Recall that $D(\mathbf{R}_n)=C_0^\infty(\mathbf{R}_n)$ is dense in $F^s_{p,q}(\mathbf{R}_n)$ if $p<\infty$ and $q<\infty$, cf. [29, 2.3.3]. Then it follows that $D(M)$ is dense in $F^s_{p,q}(M)$ if $p<\infty$ and $q<\infty$. On the other hand by Proposition 2, Step 1 and Step 2 we have

(115) $$\|f|F^{s}_{p,q}(M)\|^{\psi} \sim \|g|F^{s+2G}_{p,q}(M)\|^{\psi} \sim \|f|F^{s}_{p,q}(M)\|,$$
$$(E-\Delta)^G g = f\in D(M),\quad f\in F^{s+2G}_{p,q}(M).$$

By completion (115) can be extended to all $f\in F^s_{p,q}(M)$. Hence the above considerations are applicable to every $f\in F^s_{p,q}(M)$. This completes the proof of Theorem 2(i) if $p<\infty$, $q<\infty$. Let $p<\infty$ and $q=\infty$. Again the distributions $f\in F^s_{p,\infty}(M)$ with compact support are dense in $F^s_{p,\infty}(M)$. We approximate $f\in F^s_{p,\infty}(M)$ with $\operatorname{supp} f$ compact in $D'(M)$ by functions $f^l\in D(M)$, where we may assume $\operatorname{supp} f^l\subset V$, where V is a compact subset of M. This is essentially an Euclidean procedure. Note $K(k_N, t)f^l\to K(k_N, t)f$ pointwise. Let $(E-\Delta)^G g^l=f^l$ and $(E-\Delta)^G g=f$. This makes sense by the above results and elementary embeddings $F^s_{p,\infty}(M)\subset F^{s-\varepsilon}_{p,q}(M)$, $\varepsilon>0$. Then $g^l\to g$ in $D'(M)$ and $K(k_N, t)g^l\to K(k_N, t)g$ pointwise. We have (92) with g^l and f^l instead of f and $(E-\Delta)^G f$, respectively. Under our assumptions the right-hand side of this modified equivalence (92) may be reduced to the Euclidean case (but with the above operator $(E-\Delta)^G$). Then it is clear that the right-hand side of this equivalence is uniformly bounded, i.e.

$$\|g^l|F^{s+2G}_{p,\infty}(M)\| \leqq c < \infty,$$

where c is independent of l. By Fatou's lemma we have $g\in F^{s+2G}_{p,\infty}(M)$. The rest is the same as above. This completes the proof of Theorem 2(i) if $p<\infty$ and $q\leqq\infty$.

Step 4. We complete the proof in the case $p=q=\infty$. Because $D(M)$ is dense in $F^s_{1,1}(M)$, the dual space $(F^s_{1,1}(M))'$ of $F^s_{1,1}(M)$ can be interpreted in the usual way in the sense of the dual pairing $(D(M), D'(M))$. We have

$$(F^s_{1,1}(M))' = F^{-s}_{\infty,\infty}(M), \quad -\infty < s < \infty, \tag{116}$$

as in the Euclidean case. We outline a proof of (116) in Remark 24 below. By the above considerations $(E-\Delta)^G$ yields an isomorphic map from $F^{s+2G}_{1,1}(M)$ onto $F^s_{1,1}(M)$ (under the above assumptions for s and G, cf. Proposition 1). Cf. also Step 2 in 4.8. Furthermore by the arguments in Step 3 in 4.9 (proof of Theorem 6) $(E-\Delta)^G$ is an isomorphic map from $F^{s+2G}_{1,1}(M)$ onto $F^s_{1,1}(M)$ for all real s and natural G. Recall that $(E-\Delta)^G$ is formally self-adjoint. Then (116) yields that $(E-\Delta)^G$ is also an isomorphic map from $F^{-s}_{\infty,\infty}(M)$ onto $F^{-s-2G}_{\infty,\infty}(M)$ for all real s and natural G. Hence, by Step 1 and Step 2 the proof is finished in the case $p=q=\infty$, too.

Remark 24. For sake of completeness we outline a proof of (116). We have

$$(F^s_{1,1}(\mathbf{R}_n))' = F^{-s}_{\infty,\infty}(\mathbf{R}_n), \quad -\infty < s < \infty, \tag{117}$$

cf. [29, 2.11.2]. Let $\{\psi_j\}$ be the above resolution of unity and let $\psi_J=\sum \psi_l$ where the sum is taken over all l with $\operatorname{supp}\psi_l\cap\operatorname{supp}\psi_j\neq\emptyset$. Let $g\in(F^s_{1,1}(M))'\subset D'(M)$. For any $f\in F^s_{1,1}(\mathbf{R}_n)$ we have

$$\begin{aligned}|(\psi_j g)(f)| &= |(\psi_j g)(\psi_J f)| = |g(\psi_j f)| \\ &\leqq \left\|g|(F^s_{1,1}(M))'\right\| \|\psi_j f|F^s_{1,1}(M)\| \leqq c\|f|F^s_{1,1}(\mathbf{R}_n)\|\end{aligned} \tag{118}$$

where one has to choose appropriate interpretations (we omitted $\exp_{P_j}$ and $\exp^{-1}_{P_j}$). By (117) we have $\psi_j g\in F^{-s}_{\infty,\infty}(\mathbf{R}_n)$. Furthermore, $\|\psi_j g|F^{-s}_{\infty,\infty}(\mathbf{R}_n)\|$ is uniformly bounded. Hence, $g\in F^{-s}_{\infty,\infty}(M)$. Conversely let $g\in F^{-s}_{\infty,\infty}(M)$. If $f\in F^s_{1,1}(M)$ then we have

$$|g(f)| = \left|\sum_{j=1}^\infty (\psi_j g)(\psi_J f)\right| \leqq c \sup_j \|\psi_j g|F^{-s}_{\infty,\infty}(\mathbf{R}_n)\| \sum_{k=1}^\infty \|\psi_k f|F^s_{1,1}(\mathbf{R}_n)\|$$

$$\leqq c\|g|F^{-s}_{\infty,\infty}(M)\| \, \|f|F^s_{1,1}(M)\|,$$

(distributions on M or on $\mathbf{R}_n$). Hence, $g\in(F^s_{1,1}(M))'$. The proof of (116) is complete.

4.6. Proof of Theorem 1 (ii) and Theorem 2 (iii) .

Step 1. (Theorem 2(iii).) We use the operators Λ and Ψ from (62) and (63), respectively. Recall $\Lambda\Psi=\mathrm{id}$. This is a standard situation in interpolation theory, the method of retraction-coretraction. Let $0<p\leqq\infty$, $-\infty<s_0<s_1<\infty$, $0<\Theta<1$ and $s=(1-\Theta)s_0+\Theta s_1$ be given. Let $l_p(F^s_{p,p}(\mathbf{R}_n))$ be the same space as in Step 2 of 4.1, cf. in particular (61). We have

$$(119)\qquad \left(l_p(F^{s_0}_{p,p}(\mathbf{R}_n)), l_p(F^{s_1}_{p,p}(\mathbf{R}_n)\right)_{\Theta,p} = l_p\left((F^{s_0}_{p,p}(\mathbf{R}_n), F^{s_1}_{p,p}(\mathbf{R}_n))_{\Theta,p}\right) = l_p\left(F^s_{p,p}(\mathbf{R}_n)\right)$$

where the first equality comes from [28, 1.18.1] (extended to quasi-Banach spaces and to $p=\infty$, cf. Remark 4 in [28, 1.18.1]) and the second from [29, 2.4.2]. Recall

$$(120)\qquad B^s_{p,p}(M) = \left(F^{s_0}_{p,p}(M), F^{s_1}_{p,p}(M)\right)_{\Theta,p},$$

cf. (28). The interpolation property yields

$$(121)\qquad \left\|\Psi f|l_p(F^s_{p,p}(\mathbf{R}_n))\right\| \leqq c\|f|B^s_{p,p}(M)\|$$

if $f\in B^s_{p,p}(M)$ and

$$(122)\qquad \|\Lambda\{f_l\}|B^s_{p,p}(M)\| \leqq c\left\|\{f_l\}|l_p(F^s_{p,p}(\mathbf{R}_n))\right\|$$

if $\{f_l\}\in l_p(F^s_{p,p}(\mathbf{R}_n))$. If we choose $\{f_l\}=\Psi f$ with $f\in B^s_{p,p}(M)$ in (122) then $\Lambda\Psi=\mathrm{id}$ yields the reversion of (121). Hence we have $B^s_{p,p}(M)=F^s_{p,p}(M)$. In particular, $B^s_{p,p}(M)$ is independent of the above numbers s_0 and s_1.

Step 2. (Theorem 1(ii).) We use the reiteration theorem of interpolation theory for the spaces $(\cdot,\cdot)_{\Theta,q}$ with $0<\Theta<1$ and $0<q\leqq\infty$, cf. [28, 1.10] which holds also for quasi-Banach spaces. Then it follows from Step 1, i.e. $B^s_{p,p}(M)=F^s_{p,p}(M)$, that (28) makes sense: the definition of $B^s_{p,q}(M)$ is independent of the chosen numbers s_0 and s_1. Furthermore, as an interpolation space of complete spaces, $B^s_{p,q}(M)$ is complete, too.

4.7. Proof of Theorem 2(ii): the case of large values of s. Let $0<p\leqq\infty$ and $0<q\leqq\infty$ be given. In this subsection we assume that s is large enough in order to apply the technique developed in 4.2, cf. (73).

Step 1. We prove

$$(123)\qquad \|f|B^s_{p,q}(M)\| \leqq c\|f|B^s_{p,q}(M)\|^{k_0,k_N}_{\varepsilon,r},\quad f\in B^s_{p,q}(M),$$

where the quasi-norm on the left-hand side comes from (28) and the right-hand side is given by (26). (These abbreviations contrast somewhat the abbreviation $\|\cdot|F^s_{p,q}(M)\|$ used in 4.2, but there is no danger of confusion). Of course, c in (123) should be independent of f. We decompose the resolution of unity $\{\psi_j\}_{j=1}^\infty$ from Proposition C and Remark 6 in a finite number of families $\{\psi_{j,a}\}_{j=1}^\infty$ with $a=1, \ldots, A$ where

(124) $$\operatorname{dist}\left(\Omega_{P_{j,a}}(\delta), \Omega_{P_{k,a}}(\delta)\right) \geqq 4\delta, \quad j \neq k,$$

cf. (17) where one has to replace ψ_j and P_j by $\psi_{j,a}$, and $P_{j,a}$, respectively, $\{\psi_j\}=\bigcup_{a=1}^A \{\psi_{j,a}\}$. Let $\psi^a=\sum_{j=1}^\infty \psi_{j,a}$ and $f^a=\psi^a f$ with $f\in B_{p,q}^s(M)$. Then we have

(125) $$\operatorname{supp} f^a \subset \bigcup_{j=1}^\infty \Omega_{P_{j,a}}(\delta) \subset \bigcup_{j=1}^\infty \Omega_{P_{j,a}}(2\delta) = M^a.$$

Of course, $M=\bigcup_{a=1}^A M^a$. In order to make the next conclusions more transparent we combine the map $\exp_{P_j}$ in (27) with a translation T_j: $x \to x-x^j$ in $\mathbf{R}_n$. In other words we replace $\psi_j f \circ \exp_{P_j}$ in (27) by $\psi_j f \circ \exp_{P_j} \circ T_j$. This is completely immaterial in (27) but of great service for us. $\exp_{P_j} \circ T_j$ maps a ball B_j in $\mathbf{R}_n$ centered at x^j and with radius 2δ onto $\Omega_{P_j}(2\delta)$. Restricted to the subsequence $\{P_{j,a}\}_{j=1}^\infty$ of the points $\{P_j\}_{j=1}^\infty$ we assume that the corresponding balls $B_{j,a}$ are located in $\mathbf{R}_n$ in such a way that the obvious counterpart of (124) is satisfied. Let $B^a=\bigcup_{j=1}^\infty B_{j,a}$. Let $\tilde{F}_{p,q}^s(M^a)$ be given by (27) where only distributions $f\in D'(M)$ with $\operatorname{supp} f \subset M^a$ are allowed. By the above considerations it follows that one has a one-to-one map from $\tilde{F}_{p,q}^s(M^a)$ onto

$$\tilde{F}_{p,q}^s(B^a) = \{g \mid g\in F_{p,q}^s(\mathbf{R}_n),\ \operatorname{supp}\ g\subset B^a\}$$

generated by $\{\exp_{P_{j,a}} \circ T_{j,a}\}_{j=1}^\infty$. Now we apply the real interpolation procedure which we described in Step 1 in 4.6 with $\tilde{F}_{p,q}^s(B^a)$ instead of $F_{p,q}^s(\mathbf{R}_n)$. Recall

$$B_{p,q}^s(\mathbf{R}_n) = \left(F_{p,p}^{s_0}(\mathbf{R}_n), F_{p,p}^{s_1}(\mathbf{R}_n)\right)_{\Theta,q}$$

with the same conditions for the parameters involved as in (28). We assume that $B_{p,q}^s(\mathbf{R}_n)$ is quasi-normed by (7). We use the interpolation property and the same technique as in 4.2 (as far as the incorporation of the geodesics and estimates with respect to ψ^a are concerned). We obtain

(126) $$\|f^a | B_{p,q}^s(M)\| \leqq c\|f | B_{p,q}^s(M)\|_{\varepsilon,r}^{k_0; k_N} + \ldots$$

where the remainder terms $+\ldots$ can be estimated from above by terms of the type

(127) $$\left(\int_0^r t^{-sq+\varrho q}\|\ldots | L_p(M)\|^q \frac{dt}{t}\right)^{1/q} \leqq c\left\|\sup_{0<t<r} t^{-s+\varrho'}|\ldots| \big| L_p(M)\right\|$$

with $0<\varrho'<\varrho$ for some ϱ. Here ... indicates terms of the same type as in the Steps 2 and 3 in 4.2. However the right-hand side of (127) can be estimated from above by $c\|f | F_{p,\infty}^{s-\varrho'}(M)\|$. We use an inequality of type (112), where $\|\cdot | F_{p,q}^{s+2G}(M)\|$ can be replaced by $\|\cdot | B_{p,q}^{s+2G}(M)\|$ (as a consequence of interpolation properties). Hence

$$\|f | F_{p,\infty}^{s-\varrho'}(M)\| \leqq \eta\|f | B_{p,q}^s(M)\| + c_\eta\|f | B_{p,q}^s(M)\|_{\varepsilon,r}^{k_0, k_N}$$

where η is at our disposal. We incorporate the last estimate in (126). Summation over a yields (123).

Step 2. We prove

$$\|f|B^s_{p,q}(M)\|^{k_0;k_N}_{\varepsilon,r} \leqq c\|f|B^s_{p,q}(M)\|, \tag{128}$$

where c is independent of $f\in B^s_{p,q}(M)$. By Step 1 we have

$$\|f^a|B^s_{p,q}(M)\|^{k_0;k_N}_{\varepsilon,r} \leqq c\|f^a|B^s_{p,q}(M)\|+\dots \tag{129}$$

where the remainder terms $+\dots$ can be estimated as in (127). Recall $f^a=\psi^a f$ and

$$\|\psi^a f|F^s_{p,p}(M)\| \leqq c\|f|F^s_{p,p}(M)\|.$$

Then it follows from (28) and the interpolation property that the first term on the right-hand side of (129) can be estimated from above by $c'\|f|B^s_{p,q}(M)\|$. Furthermore the remainder terms in (129) can be estimated from above by $c\|f|F^{s-\varrho}_{p,p}(M)\|$ for some $\varrho>0$ and hence by $c'\|f|B^s_{p,q}(M)\|$, cf. (28). This completes the proof of (128).

4.8. Proof of Theorem 2(ii): the general case. We reduce the proof of Theorem 2(ii) for general values of s via lifting procedures to Theorem 2(ii) for large values of s.

Step 1. (Mapping properties.) Let $0<p\leqq\infty$, $0<q\leqq\infty$ and $-\infty<s<\infty$. Let k_0, k_N be the same kernels and G, ε, r be the same numbers as in Proposition 1, cf. also Remarks 20 and 22. The counterpart of (92) reads as follows,

$$\begin{aligned}\|K(k_0,\varepsilon)f|L_p(M)\|+\left(\int_0^r t^{-(s+2G)q}\|K(k_{N+G},t)f|L_p(M)\|^q\frac{dt}{t}\right)^{1/q}\\ \sim \|K(k_0,\varepsilon)(E-\Delta)^G f|L_p(M)\|\\ +\left(\int_0^r t^{-sq}\|K(k_N,t)(E-\Delta)^G f|L_p(M)\|^q\frac{dt}{t}\right)^{1/q}\end{aligned} \tag{130}$$

(equivalent quasi-norms in $B^{s+2G}_{p,q}(M)$). (Modification if $q=\infty$.) Recall that Δ stands for the Laplace—Beltrami operator, cf. (36). The equivalence (130) can be proved in the same way as in the proof of Proposition 1, cf. in particular (93) where the remainder terms can be treated in the same way as above.

Step 2. (The lift.) Let $0<p\leqq\infty$, $0<q\leqq\infty$ and $-\infty<s<\infty$. Let G be the same number as in Step 1. Then it follows from 4.5 that $(E-\Delta)^G$ yields an isomorphic map from $F^{s+2G}_{p,p}(M)$ onto $F^s_{p,p}(M)$. By interpolation, cf. (28), $(E-\Delta)^G$ yields also an isomorphic map from $B^{s+2G}_{p,q}(M)$ onto $B^s_{p,q}(M)$.

Step 3. Let p, q, s, G be the same numbers as above. 4.7 can be applied to $B^{s+2G}_{p,q}(M)$. Then the proof of Theorem 2(ii) for the spaces $B^s_{p,q}(M)$ is a consequence of 4.7 and the two preceding steps.

4.9. Proof of the remaining assertions.

Step 1. (Discrete version.) If one replaces M in (50) and (51) by $\mathbf{R}_n$ and K by K^e then one obtains equivalent quasi-norms in $F^s_{p,q}(\mathbf{R}_n)$ and $B^s_{p,q}(\mathbf{R}_n)$, respectively. These are the discrete versions of (8) and (7), respectively. If one uses these discrete quasi-norms for the spaces on $\mathbf{R}_n$ in the above arguments then one obtains the equivalent quasi-norms (50) and (51) instead of Theorem 2(i, ii).

Step 2. (Embeddings.) Now the embeddings (40), (45) and (46) are easy consequences of the proved parts of Theorem 7, cf. also the proof of Theorem 2.3.2(d) in [28]. Recall

$$F^s_{p,q}(\mathbf{R}_n)\subset B^s_{p,\infty}(\mathbf{R}_n)\subset \mathscr{C}^\sigma(R_n),\ 0<p\leqq\infty,\ 0<q\leqq\infty,\quad 0<\sigma\leqq s-n/p,$$

cf. [29, 2.7.1] where we put $\mathscr{C}^\sigma=B^\sigma_{\infty,\infty}$. By Definition 3(i) and (33) we have

$$F^s_{p,q}(M)\subset\mathscr{C}^\sigma(M). \tag{131}$$

Now (41) is a consequence of (131), (28), Theorem 2(iii), and well-known properties of the real interpolation method. Furthermore,

$$F^s_{p,q}(M)\subset L_1(M),\quad 0<p\leqq 1,\quad 0<q\leqq\infty,\quad s>n\left(\frac{1}{p}-1\right), \tag{132}$$

follows from a corresponding assertion with $\mathbf{R}_n$ instead of M and Definition 3(i). Now (42) is a consequence of (132) and (46).

Step 3. (Theorem 6.) By Step 2 in 4.8 we know that $(E-\Delta)^G$ yields an isomorphic map from $F^{s+2G}_{p,q}(M)$ onto $F^s_{p,q}(M)$ and from $B^{s+2G}_{p,q}(M)$ onto $B^s_{p,q}(M)$ provided that the hypotheses of Proposition 1 are satisfied. If p, q, s are given then we choose G such that these assumptions are satisfied and apply the above assertion to $(E-\Delta)^G$ and $(E-\Delta)^{G+1}$. Let $f\in F^{s+2}_{p,q}(M)$ and $f=(E-\Delta)^G g$ with $g\in F^{s+2+2G}_{p,q}(M)$. Then we have

$$\|(E-\Delta)f|F^s_{p,q}(M)\|\sim\|g|F^{s+2+2G}_{p,q}(M)\|\sim\|f|F^{s+2}_{p,q}(M)\|.$$

Now the claimed lift property is an easy consequence of the last assertion.

Step 4. (Interpolation.) In Step 1 in 4.6 we described the method of retraction-coretraction in order to prove $B^s_{p,p}(M)=F^s_{p,p}(M)$. In the same way one proves

(47) where the counterpart of (119) is given by

$$[l_{p_0}(F^{s_0}_{p_0,q_0}(\mathbf{R}_n)), l_{p_1}(F^{s_1}_{p_1,q_1}(\mathbf{R}_n))]_\Theta$$
$$= l_p([F^{s_0}_{p_0,q_0}(\mathbf{R}_n), F^{s_1}_{p_1,q_1}(\mathbf{R}_n)]_\Theta) = l_p(F^s_{p,q}(\mathbf{R}_n)).$$

The numbers involved have the same meaning as in (47), (48). Cf. [28, 1.18.1, 2.4.2]. Furthermore, (44) follows from (40), (28), and the reiteration theorem of interpolation theory.

Step 5. (Theorem 5(i): density.) Let $0<p<\infty$, $0<q<\infty$ and $-\infty<s<\infty$. Recall that $D(\mathbf{R}_n)=C_0^\infty(\mathbf{R}_n)$ is dense in $F^s_{p,q}(\mathbf{R}_n)$. Then it follows from (27) that $D(M)$ is dense in $F^s_{p,q}(M)$. Furthermore, $D(M)$ is also dense in $B^s_{p,q}(M)$. This follows from (28) and well-known properties of interpolation theory, cf. [28, 1.6.2] (which holds also for quasi-Banach spaces).

Step 6. (Theorem 4(ii).) Let $\{\psi_j\}_{j=1}^\infty$ be the above resolution of unity, cf. Proposition C and Remark 6, in particular (18). Let $1<p<\infty$ and $k=0, 1, 2, \ldots$. Then it follows easily

(133) $$\|f|W^k_p(M)\| \sim \left(\sum_{j=1}^\infty \|\psi_j f|W^k_p(M)\|^p\right)^{1/p}.$$

However

(134) $$\|\psi_j f|W^k_p(M)\| \sim \|\psi_j f\circ \exp_{P_j}|W^k_p(\mathbf{R}_n)\|,$$

where we may assume that $\psi_j f\circ\exp_{P_j}$ is extended outside of $B(r)$ (cf. (13)) by zero. Recall $W^k_p(\mathbf{R}_n)=F^k_{p,2}(\mathbf{R}_n)$, cf. [28, 2.3.3] or [29, 2.5.6]. Then (133), (134) and (27) yield

$$\|f|W^k_p(M)\| \sim \|f|F^k_{p,2}(M)\|.$$

Furthermore, $D(M)$ is dense in $F^k_{p,2}(M)$, cf. Step 5, and

$$D(M)\subset\{h|h\in C^\infty(M), \|h|W^k_p(M)\| <\infty\}\subset F^k_{p,2}(M).$$

Consequently we have $W^k_p(M)=F^k_{p,2}(M)$, cf. Definition 4(i). In particular Step 3 yields that $(E-\Delta)^k$ is an isomorphic map from $W^k_p(M)$ onto $L_p(M)$. Consequently, $W^k_p(M)=H^k_p(M)$, cf. Definition 4(ii). Hence (39) is proved.

Step 7. (Theorem 4(i).) Let k be a natural number, $1<p<\infty$ and $0<\Theta<1$. By (39) we have

(135) $$[L_p(M), W^k_p(M)]_\Theta = F^{k\Theta}_{p,2}(M),$$

cf. (47), and

(136) $$[L_p(M), W^k_p(M)]_\Theta = H^{k\Theta}_p(M),$$

cf. [24, Corollary 4.6]. Now (38) with $s>0$ is a consequence of (135) and (136). Finally, (38) with $s\leqq 0$ follows from (38) with $s>0$ and the lift property, which holds both for $F^s_{p,2}(M)$, cf. Step 3, and $H^s_p(M)$ by definition.

Step 8. (Theorem 3, Theorem 7.) Let $s > \tilde{\sigma}_p$. Then the assertion of Theorem 3 for the spaces $F^s_{p,q}(M)$ is a consequence of Remark 18, Remark 21 and the lifting property from Theorem 6, cf. Step 3. Afterwards one can follow the arguments given in 4.7 and 4.8 where one replaces $\|K(k_0, \varepsilon)f|L_p(M)\|$ in (25) and (26) by $\|f|L_p(M)\|$. This proves Theorem 3 for the spaces $B^s_{p,q}(M)$. The discrete versions from Theorem 7 have been proved in Step 1. By the same arguments as above one can replace $\|K(k_0, \varepsilon)f|L_p(M)\|$ in (50) and (51) by $\|f|L_p(M)\|$ if $s > \tilde{\sigma}_p$.

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Added in proof (June 1986). Condition (15′) is not necessary. This will be proved in [32]. Cf. also the footnote on p. 306.