Norm one multipliers on $L^p(G)$*

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Introduction

The class of norm one multipliers on $L^p(G)$, G a LCA group, whose norm is an attained value, was introduced by H. S. Shapiro. In the first part of this paper we continue his study and answer some of his questions. In particular we answer a question of L. Carleson and show that the structure of such multipliers is surprisingly similar to that of probability measures.

In the second part of the paper we answer a question of C. Fefferman and H. S. Shapiro by characterizing all faces of codimension one in the unit ball of the space of multipliers on $L^p(G)$, G compact Abelian. This result is used to give a complete description of the isometries between these spaces of multipliers.

Let G be a locally compact Abelian group, and let $\Gamma=\hat{G}$ be its dual. A function v on Γ is called an $L^p(G)$ multiplier if $v\cdot\hat{f}$ is the Fourier transform of an $L^p(G)$ function for each $f\in L^p(G)$. By the closed graph theorem the linear operator $v[f]=(v\hat{f})^{\vee}$ is then bounded on $L^p(G)$, and we identify v with this operator. We denote by $M_p(\Gamma)$ the Banach space of all bounded multipliers on $L^p(G)$ with the operator norm, which we denote by $\|v\|_{M_p(\Gamma)}$.

As it is well known, $M_p(\Gamma)$ for $p=2$ is just $L^\infty(\Gamma)$ while for $p=1$ it is the space of all Fourier transforms of finite Radon measures on G, and if μ is such a measure then $\|\hat{\mu}\|_{M_1(\Gamma)}=\|\mu\|$.

H. S. Shapiro [Sh$_1$] introduced and initiated the study of the classes

$$W_p(\Gamma)=\{v\in M_p(\Gamma)\colon \|v\|_{M_p(\Gamma)}=1=v(e)\}\quad 1<p<\infty,\ p\neq 2.$$

(Here e is the identity element in Γ.) He showed some striking analogies between these multipliers and the Fourier transforms of probability measures on G. (See also [Sh$_2$], especially § 5.)

For example, the main result of [Sh$_1$] is that if $G=\mathbf{T}$, the circle group (hence

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$\Gamma=\mathbf{Z}$ the integers), and if $1<p<\infty$, $p\neq 2$ and $v=''(v(n))\in W_p(\mathbf{Z})$ satisfies $v(k)=1$ for some $k\neq 0$, $k\in\mathbf{Z}$, then $v(kn)=1$ for all $n\in\mathbf{Z}$.

The analogous result for Fourier coefficients of probability measures on $\mathbf{T}$ is, of course, very easy. In fact if $v=\hat{\mu}$ for a probability measure μ, then μ is necessarily supported in the group of roots of unity of order k; hence $\hat{\mu}$ is periodic with period k.

L. Carleson has asked the following question (see the end of § 2 of [Sh$_1$]): Suppose $k>1$ and $v\in W_p(\mathbf{Z})$ satisfies $v(k)=1$. What can be said about the values of $v(n)$ when n is not in the arithmetic progression $k\mathbf{Z}$?

The main result of § 2, Theorem 2.1, is a complete answer to this question, in the context of general locally compact Abelian groups. Surprisingly, the answer is again completely analogous to the case of Fourier transforms of probability measures (although the proof is, of course, completely different). For the particular case of $L^p(\mathbf{T})$, the answer to Carleson's question is that the sequence $v(n)$ must be k-periodic, i.e., be constant on each of the cosets $j+k\mathbf{Z}$ $(j=0, 1, \ldots, k-1)$, and that the sequence of constant values $(v(0), \ldots, v(k-1))$ is a norm one multiplier on $L^p(\mathbf{Z}_k)$, where $\mathbf{Z}_k$ is the group of kth roots of unity, i.e., the cyclic group of order k.

Shapiro and others (see remark (a) at the end of [Sh$_1$]), generalized his theorem from the circle group to arbitrary compact Abelian groups G. They showed that if $v\in W_p(\Gamma)$, then $\Gamma_0=\{\gamma\colon v(\gamma)=1\}$ is a subgroup of Γ. In § 1, we first introduce the notation and the necessary preliminaries for this paper. We then give an alternative, simpler proof of a generalization of Shapiro's theorem to locally compact Abelian groups.

After proving Theorem 2.1 in § 2, we use it to give a further generalization of Shapiro's theorem. We show that if G is a locally compact Abelian group, $1<p<\infty$, $p\neq 2$, and $v\in W_p(\Gamma)$ is continuous, then $\Gamma_1=\{\gamma\in\Gamma\colon |v(\gamma)|=1\}$ is a subgroup of Γ, and that $v|_{\Gamma_1}$ is a character on Γ_1.

In § 3 we show that for every locally compact Abelian group G and $2\leqq p<q<\infty$, *$W_q(\Gamma)$ is strictly contained in $W_p(\Gamma)$*. In fact we construct $v\in W_p(\Gamma)$ so that $v\notin W_q(\Gamma)$ for any $q>p$. This answers a question of H. S. Shapiro (private communication).

In § 4 we consider a problem of a different type. In [FS] C. Fefferman and H. S. Shapiro proved that for $1<p<\infty$, the unit ball of $M_p(\mathbf{Z})$ has many faces of codimension one, namely $F(z, n)=\{v\in M_p(\mathbf{Z})\colon \|v\|=1 \text{ and } v(n)=z\}$ is such a face for each $n\in\mathbf{Z}$ and z with $|z|=1$. They ask whether there are any other codimension one faces. We show, and again in the context of general compact Abelian groups, that the answer is negative.

Thus the codimension one faces of the unit ball of $M_p(\mathbf{Z})$, $1<p<\infty$, $p\neq 2$, are analogous to those in $M_2(\mathbf{Z})=l_\infty$. But the facial structure for $p\neq 2$ is, in fact, very different from the $p=2$ case. Using the results of § 2 we show that if $m\neq n$, and $1<p<\infty$, $p\neq 2$, then $F(z, n)\cap F(w, m)$ is finite dimensional for each $|z|=|w|=1$. This is, of course, quite different from the situation for $p=2$ where $F(z, n)\cap F(w, m)$ is of codimension two.

The results of § 4 are used in § 5 to give the general form of an isometry of $M_p(\Gamma)$ onto $M_p(\Lambda)$ when Γ and Λ are discrete and $p\neq 1, 2, \infty$. In particular if such an isometry exists Γ and Λ must be algebraically isomorphic.

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§ 1. Preliminaries

We shall use standard notation and terminology for locally compact Abelian (*LCA*) groups, multipliers and functional analysis, see, e.g., [HR, R]. We also refer the reader to these books for unexplained terms or standard results which we use without proof.

All integrals will be taken with respect to Haar measure on the appropriate group, which we denote by $d\mu_G$ when it is important to specify the group. Otherwise we shall write $d\mu$ or simply dg or dx. If G is compact, μ is normalized so that $\mu(G)=1$. In general, if H is a closed subgroup of G, the normalization of the Haar measures on G, H and G/H will be such that for all $f\in L^1(G)$

$$\int_G f d\mu_G = \int_{G/H}\int_H f(g+h)\, d\mu_H(h)\, d\mu_{G/H}(\tilde{g}) \tag{1.1}$$

where $\tilde{g}$ is the coset containing g, i.e., $\tilde{g}=g+H$.

The Fourier transform of $f\in L^1(G)$ is $\hat{f}(\gamma)=\int f(g)\overline{\gamma(g)}\, dg$, and the Haar measure on Γ is normalized so that we have the Fourier inversion formula $f(g)=\int \hat{f}(\gamma)\gamma(g)\, d\gamma$ for all $f\in L^1(G)$ satisfying also $\hat{f}\in L^1(\Gamma)$. This is consistent with (1.1) in the sense that if (1.1) holds for a group G, its subgroups and quotient groups, it will also hold for their duals.

We shall use only very simple facts about multipliers: They commute with translations, $\|v\|_{M_p(\Gamma)}=\|v\|_{M_{p'}(\Gamma)}$ for all $v\in M_p(\Gamma)$ (where $1/p+1/p'=1$) and if $2\leq p\leq q$ then $\|v\|_{M_p(\Gamma)}\leq\|v\|_{M_q(\Gamma)}$ for all $v\in M_q(\Gamma)$.

We shall make frequent use of the following two results of S. Saeki [Sa]. The first allows us to reduce the study of a *LCA* group to that of a compact one, namely, its Bohr compactification $\beta(G)$. (Recall that if G is a *LCA* group with dual group Γ, the Bohr compactification $\beta(G)$ of G is the compact group which is the dual of Γ_d — the group Γ with the discrete topology.)

Theorem 1.1 ([Sa], 4.3). *Let G be a LCA group, and let v be a continuous function on Γ. Then $v\in M_p(\Gamma)$ iff $v\in M_p(\Gamma_d)$, and $\|v\|_{M_p(\Gamma)}=\|v\|_{M_p(\Gamma_d)}$.*

Remark. For the theorem to hold, one clearly needs to impose some regularity conditions on v. Instead of continuity Saeki shows that it is enough to assume v is "regulated" (see [Sa], 4.5). The same remark holds for some of our results later on, but we shall not pursue this point.

The second result of Saeki gives the relation between multipliers on $L^p(H)$ and $L^p(G)$ when H is a closed subgroup of G. Let $\Lambda\subset\Gamma$ be the annihilator of H, $\Lambda=H^\perp=\{\gamma\in\Gamma:\ \gamma(h)=1 \text{ for all } h\in H\}$; then $\hat{H}$ is canonically identified with Γ/Λ.

Theorem 1.2 ([Sa], § 3). *Let G be a LCA group, $H\subset G$ a closed subgroup and $\Lambda=H^\perp$. Denote by $\pi\colon \Gamma\to\Gamma/\Lambda$ the quotient map. Let v be a function on Γ/Λ; then v is a bounded multiplier on $L^p(H)$ iff $v\circ\pi$ is a bounded multiplier on $L^p(G)$, and in this case $\|v\|_{M_p(\Gamma/\Lambda)}=\|v\circ\pi\|_{M_p(\Gamma)}$.*

We shall also need the dual result about quotients of G, or equivalently subgroups of Γ. Here the situation is a little more delicate, but for our purposes the case when $\Lambda\subset\Gamma$ is discrete suffices. The next theorem is a combination of results of S. Saeki ([Sa], corollary 4.6) and of A. Figá-Talamanca and G. I. Gaudry ([FG], Theorem 1).

Theorem 1.3. *Let G be a LCA group with dual Γ and let $\Lambda\subset\Gamma$ be a discrete subgroup. Set $H=G/\Lambda^\perp$. If $v\in M_p(\Gamma)$ is continuous and w is its restriction to Λ, then $w\in M_p(\Lambda)$ and $\|w\|_{M_p(\Lambda)}\leqq\|v\|_{M_p(\Gamma)}$. Conversely, if any $w\in M_p(\Lambda)$ is given, there is a continuous v on Γ, $v|_\Lambda=w$ so that $\|v\|_{M_p(\Gamma)}=\|w\|_{M_p(\Lambda)}$.*

We end this section with a generalization of Shapiro's theorem [Sh_1]. The proof is completely different from his.

Theorem 1.4. *Let v be a continuous function in $W_p(\Gamma)$, $1<p<\infty$, $p\neq 2$. Then $\Gamma_0=\{\gamma\in\Gamma:\ v(\gamma)=1\}$ is a closed subgroup of Γ.*

Proof. Since v is continuous, Γ_0 is clearly closed, and to show it is a group we can assume, by theorem 1.1, that G is compact. The operators $v_m=m^{-1}\sum_0^{m-1}v^j$, being a sequence of norm one multipliers on $L^p(G)$, have a limit point w in the weak operator topology. As $m^{-1}\sum_0^{m-1}\alpha^j\to 0$ whenever $|\alpha|\leqq 1$, $\alpha\neq 1$, $w(\gamma)=1$ for $\gamma\in\Gamma_0$ and $w(\gamma)=0$ for $\gamma\notin\Gamma_0$, i.e., w is a norm-one projection onto the subspace $L^p_{\Gamma_0}(G)$ of $L^p(G)$ of all functions whose spectrum is contained in Γ_0. Since $e\in\Gamma_0$, i.e., $1\in L^p_{\Gamma_0}(G)$, the general structure theorem for norm-one projections on L^p (see [L], § 17) says that there is a sub-σ-algebra $\sum$ of the Lebesgue field of G so that $L^p_{\Gamma_0}(G)$ is the space of *all* $\sum$-measurable functions in $L^p(G)$ (and the projection w is just the conditional expectation with respect to $\sum$).

Thus if $\gamma,\beta\in\Gamma_0$, they are $\sum$-measurable, and consequently so are their sum and inverses, i.e., $\gamma+\beta$, $-\gamma\in\Gamma_0$, and Γ_0 is a group.

§ 2

Let G be a LCA group and fix $2<p<\infty$. (The result for $1<p<2$ will follow by duality.) Let v be a continuous function in $W_p(\Gamma)$ and put $\Gamma_0=\{\gamma\in\Gamma\colon v(\gamma)=1\}$. By theorem 1.4 Γ_0 is a closed subgroup of Γ. Let $G_0=\Gamma_0^\perp\subset G$ be the annihilator of Γ_0 and $\pi\colon \Gamma\to\Gamma/\Gamma_0$ the quotient map.

Theorem 2.1. *With the above notation, v is constant on each coset of Γ_0, i.e., there is a function w on Γ/Γ_0 so that $v=w\circ\pi$. This w satisfies $w\in W_p(\Gamma/\Gamma_0)$ and $w(\beta)\neq 1$ for all $\beta\in\Gamma/\Gamma_0$, $\beta\neq e$. Conversely, given such a w, $v=w\circ\pi\in W_p(\Gamma)$ and $\Gamma_0=\{\gamma\colon v(\gamma)=1\}$.*

Proof. The heart of the proof is to show that v is constant on the cosets of Γ_0. Indeed, once this is proved, the rest follows directly from Theorem 1.2. Also, by Theorem 1.1 we can assume that G is compact, i.e., Γ is discrete.

Let $G_1=G/G_0$ and let μ, μ_0 and μ_1 be the Haar measures on G, G_0 and G_1 respectively. Let $q\colon G\to G_1$ be the quotient map.

Fix any $\beta\in\Gamma$. We need to show that v is constant on $\beta+\Gamma_0$, and we assume, for contradiction, that this is not so.

Claim. There is a continuous function φ on G, a neighborhood W_1 of e in G_1 and a constant $c>0$ so that if we put $W=q^{-1}(W_1)\subset G$ then

(i) $\hat{\varphi}$ is supported in $\beta+\Gamma_0$,

(ii) $\varphi|_W\equiv 0$,

(iii) $|v[\varphi](g)|\geqq c$ for all $g\in W$.

We first deduce the theorem assuming the claim holds.

Let $\psi=\varkappa_W$ be the indicator function of W. By (ii) φ and ψ are disjointly supported; thus for all $\lambda>0$

$$\|\psi+\lambda\varphi\|^p = \|\psi\|^p+\lambda^p\|\varphi\|^p. \tag{2.1}$$

Since by its construction, W is G_0-invariant, so is ψ, hence $\hat{\psi}$ is supported on Γ_0. As $v|_{\Gamma_0}\equiv 1$, $v[\psi]=\psi$. Using the fact that $W=q^{-1}(W_1)$ and the identity (1.1) we thus obtain

$$\begin{aligned}\|v[\psi+\lambda\varphi]\|^p &= \|\psi+\lambda v[\varphi]\|^p \geqq \int_W |1+\lambda v[\varphi](g)|^p\,dg \\ &= \int_{W_1}\int_{G_0} |1+\lambda v[\varphi](g+g_0)|^p\,d\mu_0(g_0)\,d\mu_1(\tilde{g}).\end{aligned} \tag{2.2}$$

We now estimate the inner integral. As $\hat{\varphi}$ is supported in $\beta+\Gamma_0$ the same holds for $\widehat{v[\varphi]}$; thus the function $g\to\beta^{-1}(g)v[\varphi](g)$ is G_0-invariant, i.e.,

$$v[\varphi](g+g_0) = \beta(g_0)v[\varphi](g) \quad \text{for all} \quad g\in G,\ g_0\in G_0.$$

Thus the inner integral is equal to

$$\int_{G_0} |1+\lambda\beta(g_0)v[\varphi](g)|^p d\mu_0(g_0). \tag{2.3}$$

We now use the following simple inequality: for each complex number z,

$$\int_{G_0} |1+z\beta(g_0)|^p d\mu_0(g_0) \geqq 1+|z|^2. \tag{2.4}$$

Indeed, the functions 1 and $\beta|_{G_0}$ are orthonormal in $L^2(G_0)$, and since $p>2$ we have

$$\int_{G_0} |1+z\beta(g_0)|^p d\mu_0 \geqq \Big(\int_{G_0} |1+z\beta(g_0)|^2 d\mu_0\Big)^{p/2} = (1+|z|^2)^{p/2} \geqq 1+|z|^2.$$

Substituting $z=\lambda v[\varphi](g)$ in (2.4) and recalling that by (iii) $|v[\varphi](g)|\geqq c$ for all $g\in W$, we see from (2.3) that the inner integral is at least $1+\lambda^2c^2$.

Substituting this value in (2.2), we have

$$\|v[\psi+\lambda\varphi]\|^p \geqq (1+\lambda^2c^2)\mu_1(W_1) = \mu_1(W_1)+K\lambda^2 \tag{2.5}$$

where $K=c^2\mu_1(W_1)$ is a constant. We now compare (2.1) and (2.5): $\|\psi\|^p=\mu(W)=\mu_1(W_1)$, and $\|\varphi\|^p$ is a constant. Since $p>2$, we thus obtain for small enough λ that

$$\|v[\psi+\lambda\varphi]\|^p \geqq \mu_1(W_1)+K\lambda^2 > \mu_1(W_1)+\lambda^p\|\varphi\|^p = \|\psi+\lambda\varphi\|^p,$$

i.e., $\|v[\psi+\lambda\varphi]\|>\|\psi+\lambda\varphi\|$, contradicting $\|v\|_{M_p(\Gamma)}=1$.

It remains to prove the claim:

Assume $\gamma_1, \gamma_2\in\beta+\Gamma_0$ and $v(\gamma_1)\neq v(\gamma_2)$. The trigonometric polynomial $\varphi_0=\gamma_1-\gamma_2$ satisfies

(i′) $\hat{\varphi}_0$ is supported in $\beta+\Gamma_0$,

(ii′) $\varphi_0(e)=0$,

(iii′) $v[\varphi_0](e)=v(\gamma_1)-v(\gamma_2)\neq 0$.

The desired φ will be obtained by approximating φ_0.

The function $\varphi_1(g)=\beta^{-1}(g)\varphi_0(g)$ is G_0-invariant, and can, therefore, be considered as a function — a trigonometric polynomial — on $G_1=G/G_0$, which by (ii′) vanishes at the identity $e\in G_1$. Since points are sets of synthesis, there is a function $\varphi_2\in A(G_1)$ which approximates φ_1 arbitrarily well in $A(G_1)$ and which vanishes in a neighborhood of e. Define $\varphi(g)=\beta(g)\varphi_2(\tilde{g})$. As $\varphi_2\in A(G_1)$ it is continuous, and so is φ. If $\|\varphi_2-\varphi_1\|_{A(G_1)}$ is small enough $v[\varphi](e)\neq 0$, hence by continuity there is a constant c and a neighborhood W_1 of e in G_1, so that we simultaneously have $\varphi(g)=\beta(g)\varphi_2(\tilde{g})=0$ and $|v[\varphi](g)|\geqq c$ whenever $\tilde{g}\in W_1$.

This proves the claim and finishes the proof of theorem.

Before stating our next theorem, we need the following lemma:

Lemma 2.2. *Let G be a compact Abelian group, and assume $1<p<\infty$ $p\neq 2$. Let $v\in W_p(\Gamma)$, and assume $|v(\gamma)|=1$ for some $\gamma\in\Gamma$. Then there is a $g\in G$ so that $\gamma(g)=v(\gamma)$.*

Proof. If γ has infinite order, this is obvious, as $\{\gamma(g): g\in G\}$ is all of the unit circle.

So assume γ has a finite order k, and then $\{\gamma(g): g\in G\}$ is the group of kth roots of unity. Hence we need to show that $(v(\gamma))^k=1$.

Embed G in a group H which is an infinite product of circles. (Equivalently, represent Γ as a quotient of a free group, Λ, with sufficiently many generators.) Let $\Lambda_1\subset\Lambda=\hat{H}$ be the annihilator of G, and identify Γ with Λ/Λ_1. Let $\pi\colon \Lambda\to\Gamma$ be the quotient map. By Theorem 1.2 $w=v\circ\pi$ belongs to $W_p(\Lambda)$.

Fix now any $\beta\in\Lambda$ so that $\pi(\beta)=\gamma$. Since H is connected $\{\beta(h): h\in H\}$ is all of the unit circle; hence we can find an $h\in H$ so that $\beta(h)=w(\beta)=v(\gamma)$. Consider $\hat{\delta}_h$, the Fourier transform of the Dirac measure δ_h at h, i.e., $\hat{\delta}_h(\lambda)=\overline{\lambda(h)}$ for all $\lambda\in\Lambda$. Of course $\hat{\delta}_h\in W_p(\Lambda)$, and the same is true for the product $\hat{\delta}_h\cdot w$. By theorem 1.4 $\Lambda_0=\{\lambda\in\Lambda: \hat{\delta}_h(\lambda)w(\lambda)=1\}$ is a subgroup of Λ. Now $\hat{\delta}_h(\beta)w(\beta)=\overline{\beta(h)}\beta(h)=1$, so $\beta\in\Lambda_0$. Hence also $k\beta\in\Lambda_0$, i.e., $\hat{\delta}_h(k\beta)w(k\beta)=1$. But $\hat{\delta}_h(k\beta)=(\hat{\delta}_h(\beta))^k=(\overline{\beta(h)})^k=(\overline{v(\gamma)})^k$ and $w(k\beta)=v(k\gamma)=1$ because γ has order k. Thus $(v(\gamma))^k=1$.

Theorem 2.3. *Let G be a LCA group, and let $1<p<\infty$, $p\neq 2$. Let v be a continuous function in $W_p(\Gamma)$ and put $\Gamma_1=\{\gamma\in\Gamma: |v(\gamma)|=1\}$. Then Γ_1 is a closed subgroup of Γ and the map $\gamma\to v(\gamma)$ is a character on Γ_1. In particular, there is a $g\in G$ so that $v(\gamma)=\hat{\delta}_g(\gamma)$ for all $\gamma\in\Gamma_1$.*

Proof. By Theorem 1.2 we can assume G is compact.

Assume $|v(\alpha)|=|v(\beta)|=1$, i.e., $\alpha,\beta\in\Gamma_1$. By the lemma there is a $g\in G$ with $\alpha(g)=v(\alpha)$. Consider $\hat{\delta}_g$, i.e., $\hat{\delta}_g(\gamma)=\overline{\gamma(g)}$ for all $\gamma\in\Gamma$. The product $\hat{\delta}_g v$ belongs to $W_p(\Gamma)$. Hence $\Gamma_0=\{\gamma: \hat{\delta}_g(\gamma)v(\gamma)=1\}$ is a subgroup of Γ and $\alpha\in\Gamma_0$ because $\hat{\delta}_g(\alpha)v(\alpha)=\overline{\alpha(g)}\alpha(g)=1$.

By theorem 2.1 $\hat{\delta}_g v$ can be identified as a multiplier on $L^p(H)$ where $H=\Gamma_0^{\perp}\subset G$. Using the lemma again we can find an $h\in H$ so that $\beta(h)=\hat{\delta}_g(\beta)v(\beta)$.

Consider the product $\hat{\delta}_h\hat{\delta}_g v\in W_p(\Gamma)$:

$$\hat{\delta}_h(\beta)\hat{\delta}_g(\beta)v(\beta)=1 \quad \text{by the choice of } h,$$

$$\hat{\delta}_h(\alpha)\hat{\delta}_g(\alpha)v(\alpha)=\overline{\alpha(h)}=1 \quad \text{because} \quad \alpha\in\Gamma_0 \text{ and } h\in\Gamma_0^{\perp}.$$

By theorem 1.4 we must, therefore, also have

$$\begin{aligned}1 &= \hat{\delta}_h(\alpha+\beta)\,\hat{\delta}_g(\alpha+\beta)v(\alpha+\beta)\\ &= \overline{\alpha(h)}\,\overline{\beta(h)}\,\overline{\beta(g)}\,\overline{\alpha(g)}\,v(\alpha+\beta)\\ &= \overline{v(\beta)}\,\overline{v(\alpha)}\,v(\alpha+\beta).\end{aligned}$$

Thus $\alpha+\beta\in\Gamma_1$ and $v(\alpha+\beta)=v(\alpha)v(\beta)$.

§3

It is well-known that for $q>p\geqq 2$ there are norm-one multipliers on $L^p(G)$ whose norm as multipliers on $L^q(G)$ is strictly bigger than one. If G is infinite, bounded $L^p(G)$ multipliers need not even be bounded on $L^q(G)$. Since elements of $W_p(\Gamma)$ behave in a very special way, H. S. Shapiro has asked (private communication) whether these classes must also decrease strictly. In this section we show that this is indeed the case.

Theorem 3.1. *For each* $2\leqq p<\infty$ *and each LCA group* G *there is a multiplier* $v\in W_p(\Gamma)$ *so that* $\|v\|_{M_q(\Gamma)}>1$ *for all* $q>p$.

The case $p=2$ is, of course, immediate. Indeed if $v\in M_2(\Gamma)$ is such that $\Gamma_0=\{\gamma\colon |v(\gamma)|=1\}$ is not a subgroup, or even if it is a subgroup but $v|_{\Gamma_0}$ is not a character on Γ_0, $v\notin M_q(\Gamma)$ for any $q>2$ by Theorem 2.3. Thus we shall assume $p>2$.

It turns out that the most difficult case is when G is finite. To prove the theorem for infinite groups, assume first G is compact. It is well-known that there is a multiplier $w\in M_p(\Gamma)$ so that $w\notin M_q(\Gamma)$ for any $q>p$. By a theorem of C. Fefferman and H. S. Shapiro [FS], if $\varepsilon>0$ is small enough the multiplier v given by

$$v(\gamma)=\begin{cases}1 & \gamma=e\\ \varepsilon w(\gamma) & \gamma\neq e\end{cases}$$

is in $W_p(\Gamma)$. (Their theorem is stated for the circle group, but the proof works for any compact Abelian group.) Obviously v is not even a bounded multiplier for any $q>p$.

For infinite *LCA* groups the result follows by "transplanting" the example constructed for compact groups. By the structure theorem for *LCA* groups, G has either a compact subgroup or a compact quotient. If G has a compact subgroup we use Theorem 1.2, and if it has a compact quotient H, say, then $\hat{H}$ is a discrete subgroup of Γ and we use Theorem 1.3.

As the estimates involved in the known constructions of multipliers in $M_p(\Gamma)\setminus\bigcup_{q>p} M_q(\Gamma)$ are not precise enough for our purposes, we need a completely different approach to prove the theorem for finite G. Thus assume G is finite. Hence all the L^r norms of functions on G are equivalent and bounded sets in $L^p(G)$ are relatively compact. By the theorem of C. Fefferman and H. S. Shapiro quoted above, if $\varepsilon>0$ is small enough then $v_\varepsilon(\gamma)\in W_p(\Gamma)$ where

$$v_\varepsilon(\gamma) = \begin{cases} 1 & \gamma = e \\ i\varepsilon & \gamma \neq e. \end{cases}$$

Let $\varepsilon(p)$ be the maximal value of these ε's, and put $v=v_{\varepsilon(p)}$. We shall show that if $q>p$, $\varepsilon(q)<\varepsilon(p)$, thus $\|v\|_{M_q(\Gamma)}>1$ for all $q>p$.

We first show that $\varepsilon^2(p)\leqq(p-1)^{-1}$, hence, in particular, $\varepsilon(p)<1$.

For $z=x+iy$ a (small) complex number, we have

$$(3.1)\qquad |1+z|^p = (1+2x+x^2+y^2)^{p/2} = 1+px+\frac{p}{2}[(p-1)x^2+y^2]+O(|z|^3).$$

Let $h(g)=x(g)+iy(g)$ be a function on G with $\int h=0$. Then $v[h]=i\varepsilon(p)h$, and since $v(e)=1$, $v[1+h]=1+i\varepsilon(p)h$. Substituting $z=h(g)$ or $z=i\varepsilon(p)h(g)$ in (3.1) and integrating we have

$$(3.2)\qquad \|1+h\|_p^p = 1+\frac{p}{2}[(p-1)\|x\|_2^2+\|y\|_2^2]+O(\|h\|_p^3)$$

and

$$(3.3)\qquad \|v[1+h]\|_p^p = 1+\frac{p}{2}\varepsilon^2(p)[(p-1)\|y\|_2^2+\|x\|_2^2]+O(\|h\|_p^3).$$

As $\|v[1+h]\|_p^p\leqq\|1+h\|_p^p$, we obtain

$$\varepsilon^2(p) \leqq \frac{(p-1)\|x\|_2^2+\|y\|_2^2}{(p-1)\|y\|_2^2+\|x\|_2^2}+O(\|h\|_p).$$

Taking $h\equiv iy$ (i.e., $x\equiv 0$, and h is purely imaginary) and letting $\|h\|\to 0$ we see that $\varepsilon^2(p)\leqq(p-1)^{-1}$.

Fix now a sequence ε_n so that $\varepsilon(p)<\varepsilon_n<1$ and $\varepsilon_n\to\varepsilon(p)$. By the definition of $\varepsilon(p)$, there is a function $f_n\in L^p(G)$ so that

$$(3.4)\qquad \|v_{\varepsilon_n}[f_n]\| > \|f_n\|.$$

As $\|v_{\varepsilon_n}[h]\|=\varepsilon_n\|h\|<\|h\|$ whenever $\int h=0$, (3.4) implies that $\int f_n\neq 0$, so we normalize them so that $f_n=1+h_n$ with $\int h_n=0$. Moreover, by (3.4) again we have

$$1+\varepsilon_n\|h_n\| \geqq \|v[1+h_n]\| > \|1+h_n\| > \|h_n\|-1$$

hence $\|h_n\| \leqq 2(1-\varepsilon_n)^{-1}$. Since $\varepsilon_n \to \varepsilon(p) < 1$ the sequence h_n is bounded and by passing to a subsequence we can thus assume that $h_n \to h$.

We now distinguish two cases:

Case I. $h=0$, i.e., $\|h_n\| \to 0$.

Writing $h_n = x_n + iy_n$ and substituting in (3.2) and (3.3) (with ε_n replacing $\varepsilon(p)$), and using the fact that $\|v_{\varepsilon_n}[1+h_n]\| > \|1+h_n\|$ we now obtain

$$\varepsilon_n^2 \geqq \frac{(p-1)\|x_n\|_2^2 + \|y_n\|_2^2}{(p-1)\|y_n\|_2^2 + \|x_n\|_2^2} + O(\|h_n\|_p) \geqq (p-1)^{-1} + O(\|h_n\|_p)$$

(as $(p-1)|a|+|b|/(p-1)|b|+|a| \geqq (p-1)^{-1}$ whenever $p>2$ and a and b are not both zero).

Letting $n \to \infty$, we obtain $\varepsilon^2(p) \geqq (p-1)^{-1}$, i.e., $\varepsilon^2(p) = (p-1)^{-1}$. Hence if $q>p$, $\varepsilon^2(q) \leqq (q-1)^{-1} < (p-1)^{-1} = \varepsilon^2(p)$ and the result follows.

Case II. If $h \neq 0$, put $f = (1+h)/\|1+h\|$. Then $\|v[f]\| = \|f\| = 1$, and f is non-constant.

If $|f|$ is constant the result follows immediately. Indeed, in this case $\|f\|_q = \|f\|_p = 1$, and if $1/p = \theta/2 + (1-\theta)/q$, Hölder's inequality yields $1 = \|v[f]\|_p \leqq \|v[f]\|_2^\theta \|v[f]\|_q^{1-\theta}$, and since $\|v[f]\|_2 < 1$ (as f is non-constant), $\|v[f]\|_q > 1$.

We shall thus assume $|f|$ is non-constant, and we now estimate $\|v\|_{M_q(\Gamma)}$ by Hadamard's three lines theorem. (This is just a repetition of the basic step in Thorin's proof of the Riesz interpolation theorem; see [BL], p. 3.)

Fix $g \in L^{p'}(G)$ so that $\|g\|_{p'} = 1 = \langle v[f], g \rangle$. If $0 \leqq \mathrm{Re}\, z \leqq 1$, put $1/p(z) = z/q + (1-z)/2$, and $1/p'(z) = z/q' + (1-z)/2$.

Put $\varphi(z) = |f|^{p/p(z)} \operatorname{sign} f$ and $\psi(z) = |g|^{p'/p'(z)} \operatorname{sign} g$, and then for all $t \in \mathbf{R}$, $\|\varphi(it)\|_2 = \|\psi(it)\|_2 = \|\varphi(1+it)\|_q = \|\psi(1+it)\|_{q'} = 1$.

We shall show that

$$A = \sup_t \|v[\varphi(it)]\|_2 < 1.$$

Once this is proved, the result follows immediately. Indeed, put $F(z) = \langle v[\varphi(z)], \psi(z) \rangle$, then $|F(1+it)| \leqq \|v\|_{M_q(\Gamma)}$ while $|F(it)| \leqq A$, and by the three lines theorem if $0<\theta<1$ is such that $p(\theta)=p$, we have

$$1 = \langle v[f], g \rangle = F(\theta) \leqq A^{1-\theta} \|v\|_{M_q(\Gamma)}^\theta.$$

As $A<1$, we obtain that $\|v\|_{M_q(\Gamma)} > 1$.

We now prove that $A<1$.

Write $\varphi(it) = a_t + h_t$ where a_t is constant, and $\int h_t = 0$. Then $v[\varphi(it)] =$

$a_t+i\varepsilon(p)h_t$, and since $\varepsilon(p)<1$ it is enough to show that

$$\inf_t \|h_t\|_2 = \inf_t \operatorname{dist}\{\varphi(it), \mathbf{C}\} > 0$$

where $\mathbf{C}$ is the one-dimensional space of constants in $L^2(G)$.

But for each $t\in\mathbf{R}$,

$$\operatorname{dist}\{\varphi(it), \mathbf{C}\} \geqq \operatorname{dist}\{|\varphi(it)|, \mathbf{C}\} = \operatorname{dist}\{|f|^{p/2}, \mathbf{C}\} > 0$$

as $|f|$ is non-constant.

§ 4

Let G be a compact Abelian group, and fix $1<p<\infty$. C. Fefferman and H. S. Shapiro [FS], proved that if μ is the Haar measure on G and if $\varepsilon>0$ is small enough, then $\|\hat{\mu}+v\|_{M_p(\Gamma)}=1$ whenever $\|v\|_{M_p(\Gamma)}\leqq\varepsilon$ and $v(e)=0$. (They formulated their result for the circle group $\mathbf{T}$, but their proof holds for any compact G.)

Geometrically this means that $\hat{\mu}$ is a relatively interior point of a codimension one face of the unit ball $B_p(\Gamma)$ of $M_p(\Gamma)$: Namely of the face $\{v\in B_p(\Gamma)\colon v(e)=1\}$.

Similarly, for any $\gamma\in\Gamma$ and $|z|=1$, the face

$$F(z,\gamma) = \{v\in B_p(\Gamma)\colon v(\gamma) = z\}$$

has codimension one.

C. Fefferman and H. S. Shapiro ask whether $B_p(\Gamma)$ has any other codimension one faces. As the next theorem shows, the answer is negative. In fact we prove a little more, as we use a somewhat weaker definition of codimension one faces.

If F is a face of $B_p(\Gamma)$, let ϕ be a supporting functional for F, i.e., ϕ is a norm one functional on $M_p(\Gamma)$ and $F=\{v\in B_p(\Gamma)\colon \phi(v)=1\}$. We say that a point $w\in F$ is relatively internal if for any $v\in\operatorname{Ker}\phi$, $w+tv\in F$ provided t is small enough. Clearly if w is a relatively interior point it is relatively internal. If F contains a relatively internal point, the supporting functional is, of course, uniquely determined.

Theorem 4.1. *Let G be a compact Abelian group and $1<p<\infty$. Let w be a relatively internal point of a face F of $B_p(\Gamma)$. Then there are $\gamma\in\Gamma$ and $|z|=1$ so that $w(\gamma)=z$. In particular, the unique supporting functional of F is given by $\phi(v)=\bar{z}v(\gamma)$, i.e., $F=F(z,\gamma)$.*

Proof. Multiplying each $v\in F$ by a fixed appropriate z, $|z|=1$, and replacing ϕ by $\bar{z}\phi$ we can assume $\phi(\hat{\delta}_e)\geqq 0$. (Recall that if $g\in G$, and $\gamma\in\Gamma$ then $\hat{\delta}_g(\gamma)=\overline{\gamma(g)}$. Thus $\hat{\delta}_e$ is identically one.)

Let $f_n\in L^p(G)$, $h_n\in L^{p'}(G)$ satisfy $\|f_n\|_p=\|h_n\|_{p'}=1$ and $\langle w[f_n], h_n\rangle\to 1$. We first show that for each $v\in M_p(\Gamma)$

$$\phi(v) = \lim_n \langle v[f_n], h_n\rangle. \tag{4.1}$$

Indeed, $v-\phi(v)w\in\operatorname{Ker}\phi$, hence there is a $t>0$ so that $\|w\pm t(v-\phi(v)w)\|=1$. Hence also $\|(w\pm t(v-\phi(v)w))[f_n]\|\leqq 1$. As $\|w[f_n]\|\to 1$ and $L^p(G)$ is uniformly convex, we obtain

$$\|(v-\phi(v)w)[f_n]\|\to 0. \tag{4.2}$$

Thus $\langle (v-\phi(v)w)[f_n], h_n\rangle\to 0$, and since $\langle w[f_n], h_n\rangle\to 1$, (4.1) follows.

In particular, upon taking $v=\hat{\delta}_e$ in (4.2) we see that $|\phi(\hat{\delta}_e)|=1$, and by our normalization $\phi(\hat{\delta}_e)=1$.

Next we show that ϕ is a multiplicative functional:

Fix $v_1, v_2\in M_p(\Gamma)$. Then

$$v_1v_2-\phi(v_1)\phi(v_2)w = v_1(v_2-\phi(v_2)w)+\phi(v_2)w(v_1-\phi(v_1)w)+\phi(v_1)\phi(v_2)w(w-\hat{\delta}_e).$$

It follows from (4.2) that the three terms in parentheses on the right hand side, when applied to f_n, converge to zero; thus

$$\langle (v_1v_2-\phi(v_1)\phi(v_2)w)[f_n],\ h_n\rangle\to 0.$$

As $\langle w[f_n], h_n\rangle\to 1$, this together with (4.1) gives that $\phi(v_1v_2)=\phi(v_1)\phi(v_2)$.

In particular the map $g\to\phi(\hat{\delta}_g)$ *is a complex homomorphism on* G. We show it is continuous, i.e., a character on G:

For each fixed n, $v\to\langle v[f_n], h_n\rangle$ is a continuous function with respect to the weak operator topology on $M_p(\Gamma)$. Since by (4.1) these functions converge pointwise to ϕ, a standard corollary to the Baire category theorem gives that ϕ has a point of continuity in the compact (with respect to the weak operator topology) set $\{\hat{\delta}_g\colon g\in G\}$. (See [H], § 42.) Equivalently, the map $g\to\phi(\hat{\delta}_g)$ has a point of continuity in G. Being a homomorphism, this implies it is everywhere continuous, i.e., there is a $\gamma\in\Gamma$ so that $\phi(\hat{\delta}_g)=\gamma^{-1}(g)$.

We conclude the proof by showing that $w(\gamma)=1$:

By (4.2) $\|(\hat{\delta}_g-\gamma^{-1}(g)w)[f_n]\|\to 0$ for all $g\in G$. Multiplying by $\gamma(g)$ and integrating the vector valued function $g\to(\gamma(g)\hat{\delta}_g-w)[f_n]$ with respect to g, we have

$$\left\|\int (\gamma(g)\hat{\delta}_g-w)[f_n]\,dg\right\|\to 0.$$

But

$$\int \gamma(g)\,\hat{\delta}_g[f_n]\,dg=\hat{f}_n(\gamma)\gamma$$

so

$$\|\hat{f}_n(\gamma)\gamma-w[f_n]\|\to 0.$$

Since $\|w[f_n]\|\to 1$, it follows that $|\hat{f}_n(\gamma)|\to 1$. Using the fact that $|\hat{h}(\gamma)|\leqq\|h\|$ for all $h\in L^p(G)$, we thus obtain

$$|1-w(\gamma)||\hat{f}_n(\gamma)| = |\hat{f}_n(\gamma)-w(\gamma)\hat{f}_n(\gamma)|\leqq\|\hat{f}_n(\gamma)\gamma-w[f_n]\|.$$

As the latter converges to zero, while $|\hat{f}_n(\gamma)|\to 1$, it follows that $w(\gamma)=1$.

Remark. The facial structure of $B_p(\Gamma)$ for $p\neq 2$ is in fact very different from that of $B_2(\Gamma)$, despite the similarity of their codimension one faces. For example, the intersection of two codimension one faces

$$F(z,\gamma)\cap F(w,\beta) = \{v\in B_p(\Gamma)\colon v(\gamma) = z \quad\text{and}\quad v(\beta) = w\}$$

when $\beta\neq\gamma$ is a face of codimension two in $B_2(\Gamma)$. But by theorem 2.3 for v to belong to this face requires very strong conditions: there is a subgroup $\Gamma_0\subset\Gamma$ so that both β and γ belong to the same Γ_0-coset. Moreover, if we normalize so that, say, $\beta=e$ and $w=1$, then $v|_{\Gamma_0}$ must be a character on Γ_0. For example, since subgroups of $\mathbf{Z}$ have finite index, it follows from theorem 2.1 that when $G=\mathbf{T}$, the intersection is finite dimensional.

If we intersect three codimension one faces corresponding to three different characters γ_1, γ_2, γ_3, we obtain in $B_2(\Gamma)$ a face of codimension three, but for $p\neq 2$ it follows from the above that the intersection is usually empty.

§ 5

When $p=1$ or $p=2$ the algebraic structure of G (or Γ) does not reflect at all in the structure of $M_p(\Gamma)$ as a Banach space. Indeed, in these cases $M_p(\Gamma)$ is just the space of measures on G, or $L^\infty(\Gamma)$ respectively, and the structure of these Banach spaces depends only on G and Γ as abstract measure spaces. It follows from the next theorem that for $1<p<\infty$, $p\neq 2$, and G compact, $M_p(\Gamma)$ uniquely determines the group Γ.

There are four natural isometries on $M_p(\Gamma)$:

(i) $v\to zv$ for some fixed $|z|=1$.
(ii) $v\to\hat{\delta}_g\cdot v$ for some fixed $g\in G$.
(iii) $v\to v_{\gamma_0}$ for some fixed $\gamma_0\in\Gamma$, where $v_{\gamma_0}(\gamma)=v(\gamma-\gamma_0)$.
(iv) $v\to v\circ\pi$ where π is an isomorphism of Γ onto itself.

It turns out that when G is compact and $1<p<\infty$, $p\neq 2$, every isometry of $M_p(\Gamma)$ is obtained by composing such isometries. We do not know if an analogous result holds for *LCA* groups.

Theorem 5.1. *Let G and H be compact Abelian groups with duals Γ and Λ respectively. For $1<p<\infty$, $p\neq 2$ let $T\colon M_p(\Gamma)\to M_p(\Lambda)$ be an onto isometry. Then there are $|z_0|=1$, $h_0\in H$, $\gamma_0\in\Gamma$ and an isomorphism π of Λ onto Γ such that for all $v\in M_p(\Gamma)$ and $\lambda\in\Lambda$*

$$(Tv)(\lambda) = z_0\hat{\delta}_{h_0}(\lambda)v(\pi(\lambda)-\gamma_0).$$

Proof. The codimension one faces of the unit ball $B_p(\Gamma)$ of $M_p(\Gamma)$ are mapped by T in a one-one way onto the codimension one faces of $B_p(\Lambda)$. Hence there is a map $\gamma \to \varphi(\gamma)$ from Γ to Λ, and a function $z(\gamma)$ so that T maps $F(\gamma, 1)$ to $F(\varphi(\gamma), z(\gamma))$.

Replacing T by the isometry obtained by composing it with a translation of Γ by $\gamma_0 \varepsilon \varphi^{-1}(e)$, and multiplying it by $z_0=\overline{z(\gamma_0)}$, we can assume that $\varphi(e)=e$ and that $z(e)=1$.

Let $v_0=T\hat{\delta}_e$. As $\hat{\delta}_e \in \bigcap_\gamma F(\gamma, 1)$, $v_0 \in \bigcap_\gamma F(\varphi(\gamma), z(\gamma))$, i.e. $|v_0(\lambda)|=1$ for all λ in the range of φ. As $v_0(e)=1$, theorem 2.3 implies that there is an $h_0 \in H$ so that $\hat{\delta}_h(\lambda)=\overline{v_0(\lambda)}$ for all λ in the range of φ. Composing T with the isometry $w \to \hat{\delta}_h \cdot w$ of $M_p(\Lambda)$, we can thus assume that $z(\gamma)\equiv 1$.

It now follows that φ is one-one. Indeed if $\gamma_1 \neq \gamma_2$, the two different faces $F(\gamma_1, 1)$ and $F(\gamma_2, 1)$ cannot be mapped by T to a single face $F(\lambda, 1)$. It is also onto, because if $\lambda \in \Lambda \setminus \varphi(\Gamma)$, then $F(\lambda, 1)$ is different from, and intersects all the faces $F(\varphi(\gamma), 1)$ in $B_p(\Lambda)$. But there is no codimension one face in $B_p(\Gamma)$ which is different from and intersects all the $F(\gamma, 1)$'s.

Taking $\pi=\varphi^{-1}$ we have by our normalization that

$$T(F(\pi(\lambda), 1)) = F(\lambda, 1) \quad \text{for all} \quad \lambda \in \Lambda. \tag{5.1}$$

If we denote by ϕ_λ and ψ_γ the supporting functionals of $F(\lambda, 1)$ and $F(\gamma, 1)$ respectively it follows from (5.1) that $T^*\phi_\lambda=\psi_{\pi(\lambda)}$. Hence

$$Tv(\lambda) = \langle Tv, \phi_\lambda\rangle = \langle v, T^*\phi_\lambda\rangle = \langle v, \psi_{\pi(\lambda)}\rangle = v(\pi(\lambda)). \tag{5.2}$$

It remains to show that π is an algebraic isomorphism, i.e., that for each $\lambda, \mu \in \Lambda$ and $g \in G$

$$(\pi(\mu)+\pi(\lambda))(g) = (\pi(\mu+\lambda))(g). \tag{5.3}$$

Indeed, given g, (5.2) implies that $T\hat{\delta}_g$ is a norm one multiplier of absolute value one on $M_p(\Lambda)$, with $T\hat{\delta}_g(e)=1$. By Theorem 2.3 there is an $h \in H$ so that $T\hat{\delta}_g=\hat{\delta}_h$. Taking $v=\hat{\delta}_g$ in (5.2) now yields

$$\pi(\lambda)(g) = \overline{\hat{\delta}_g(\pi(\lambda))} = \overline{T\hat{\delta}_g(\lambda)} = \overline{\hat{\delta}_h(\lambda)} = \lambda(h)$$

and similarly $\pi(\mu)(g)=\mu(h)$ and $(\pi(\lambda+\mu))(g)=(\lambda+\mu)(h)$. Thus

$$(\pi(\mu)+\pi(\lambda))(g) = \pi(\mu)(g)\cdot\pi(\lambda)(g) = \mu(h)\lambda(h) = (\mu+\lambda)(h) = (\pi(\lambda+\mu))(g).$$

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