A lower bound for projection operators on L^1 $(-1, 1)$

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1. Introduction and results

Given a fundamental set in a Banach space which is not a basis, it is often difficult to decide whether or not there exists a uniformly bounded sequence of linear projections onto the subspaces spanned by finitely many fundamental functions. In several classical cases, like the trigonometric polynomials as a fundamental set in the spaces $C_{2\pi}$, $L^1_{2\pi}$ of periodic functions and the algebraic polynomials in the spaces $C[-1, 1]$ or $L^1_\varrho(-1, 1)$ with Chebyshev weight $\varrho(x)=(1-x^2)^{-1/2}$, such projections do not exist, as a consequence of the Kharshiladze—Lozinski theorem (cf. [6, § 6.5], [28, Appendix 3]).

Theorem A. *For each* $n\in \mathbf{P}=\{0, 1, 2, \dots\}$ *let there be given a bounded linear projection* P_n *of* $C_{2\pi}$ *onto* $\Pi_n=\text{span}\,\{e^{ikx};\ |k|\leqq n,\ |k|\in \mathbf{P}\}$, *or of* $L^1_{2\pi}$ *onto* Π_n, *or of* $C[-1, 1]$ *onto* $\mathcal{P}_n=\text{span}\,\{x^k;\ 0\leqq k\leqq n,\ k\in\mathbf{P}\}$, *or of* $L^1_\varrho(-1, 1)$ *onto* $\mathcal{P}_n$. *Then*

$$\limsup_{n\to\infty} \|P_n\|_{[X]} = +\infty,$$

where X *denotes the respective space and* $\|\cdot\|_{[X]}$ *is the norm of an operator from* X *into* X.

Adopting a terminology of Kadec [21] (cf. [32, pp. 194, 212]), one may say that the sets of trigonometric or algebraic polynomials form Λ systems (Lozinski—Kharshiladze systems) on the respective spaces X. Further Λ systems exist on spaces $C(Y)$ and $L^p(Y)$ for certain p, where Y denotes the compact torus T^d, the sphere S_{d-1} in the real Euclidean space $\mathbf{R}^d$, $d>2$, or a projective space P_d of dimension d, cf. [9], [10], [11] and the literature cited there. But in numerous other cases no good lower bound for the norm of an arbitrary projection is known, including the algebraic

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polynomial projections in L^1 spaces with Jacobi weights $w_{\alpha,\beta}(x)=(1-x)^\alpha(1+x)^\beta$, $-1<x<1$, $\alpha,\beta\geqq -1/2$, $\alpha+\beta>-1$ (cf. (1.1) below). The purpose of the present paper is to solve this problem for the most interesting case of unit weight ($\alpha=\beta=0$) as well as for the weights $w_{0,-1/2}$ and $w_{-1/2,0}$.

In contrast to the situation of Theorem A, now the partial sum operators of the "natural" orthogonal expansions, i.e. the corresponding Jacobi expansions, are no longer candidates for minimal or nearly minimal projections. (A projection Q_n is called minimal if $\|Q_n\|\leqq\|P_n\|$ for any projection P_n, and it is called nearly minimal if there exists a constant C such that $\|Q_n\|\leqq C\|P_n\|$ for any P_n.) Instead, in the three cases mentioned it turns out that the role of a nearly minimal projection can be taken over by some other Jacobi partial sum with shifted parameters, i.e., our lower bounds will be

$$\|S_n^{1/2,1/2}\|_{[L^1]},\ \|S_n^{1/2,-1/2}\|_{[L^1_{w(0,-1/2)}]},\ \|S_n^{-1/2,1/2}\|_{[L^1_{w(-1/2,0)}]},$$

respectively (cf. (2.5)).

Using the notation

$$L^1_{w(\alpha,\beta)}=\Big\{f;\ \|f\|_{L^1_{w(\alpha,\beta)}}=\int_{-1}^1|f(x)|\,w_{\alpha,\beta}(x)\,dx<\infty\Big\}, \tag{1.1}$$

thus $L^1_{w(0,0)}=L^1$, $L^1_{w(-1/2,-1/2)}=L^1_\varrho$, our results are the following.

Theorem 1. *For each $n\in\mathbf{P}$ let P_n be a bounded linear projection of $L^1(-1,1)$ onto $\mathscr{P}_n$. Then*

$$\|P_n\|_{[L^1]}\geqq\frac{2}{\pi^2}\log n+\mathcal{O}(1),\quad n\to\infty. \tag{1.2}$$

Theorem 2. *Let X be one of the spaces $L^1_{w(0,-1/2)}$ or $L^1_{w(-1/2,0)}$, and for each $n\in\mathbf{P}$ let P_n be a bounded linear projection of X onto $\mathscr{P}_n$. Then*

$$\|P_n\|_{[X]}\geqq\frac{2}{\pi^2}\log n+\mathcal{O}(1),\quad n\to\infty.$$

Theorems 1 and 2 cannot be established by a straightforward extension of the proof of Theorem A, since the Berman—Marcinkiewicz identity, which is the main tool there, does no longer work here. In the trigonometric case, this is the identity

$$S_nf=\frac{1}{2\pi}\int_{-\pi}^{\pi}T_{-t}P_nT_tf\,dt\quad(f\in L^1_{2\pi}). \tag{1.3}$$

(See Berman [4, (10)], Marcinkiewicz [26, Theorem 1], Faber [15, (24)].) Here S_nf, P_n, and T_t denote the nth partial sum of the Fourier series of f, a projection of $C_{2\pi}$ or $L^1_{2\pi}$ onto Π_n, and the ordinary translation operator defined by $T_t(f;x)=f(x+t)$, respectively. In the algebraic case of Theorem A, the corresponding identity is

$$S_n^{-1/2,-1/2}(f;\ x)=\frac{2}{\pi}\int_{-1}^1(\tau_tP_n\tau_tf)(x)\varrho(t)\,dt-\frac{1}{\pi}\int_{-1}^1f(u)\,\varrho(u)\,du, \tag{1.4}$$

where $f\in L^1_\varrho$. Here $S_n^{-1/2,-1/2}f$ is the nth partial sum of the Chebyshev expansion of f, P_n a projection of $C[-1,1]$ or L^1_ϱ onto $\mathscr{P}_n$, and τ_t the Chebyshev translation operator, defined by

$$(1.5)\quad \tau_t(f;\ x) = \tfrac{1}{2}\{f(\cos[\arccos x+\arccos t])+f(\cos[\arccos x-\arccos t])\},\qquad (x, t\in[-1, 1]).$$

The right-hand sides of (1.3) and (1.4) may be interpreted as symmetrizations of the given arbitrary projections P_n relative to the corresponding translation operators [9], [31]. The symmetrization principle was also used by Newman and, more generally, by Rudin (cf. [30]) to prove that there is no bounded linear projection of $L^1_{2\pi}$ onto the subspace H^1.

Extending now this approach to a space $L^1_{w(\alpha,\beta)}$ with $(\alpha,\beta)\neq(-1/2, -1/2)$ would lead to a Berman—Marcinkiewicz type identity which has

$$N_nf = \int_{-1}^{1} T_t^{\alpha,\beta}P_nT_t^{\alpha,\beta}f\,w_{\alpha,\beta}(t)\,dt$$

on its right-hand side, where $T_t^{\alpha,\beta}$ is the Jacobi translation operator (see [3], [18]). Unfortunately, the Jacobi partial sum $S_n^{\alpha,\beta}f$ is not identical with N_nf but rather with $M_n(N_nf)$, where M_n is a multiplier operator arising from the normalization constants. The operator norm of M_n increases with n too rapidly in order to derive a lower bound for $\|P_n\|$. So in the Legendre case, for example, one would only obtain

$$cn^{1/2} \leqq \|S_n^{(0,0)}\|_{[L^1]} \leqq Cn\|P_n\|_{[L^1]}$$

for certain constants $c, C>0$. One might think of improving the lower estimate by factoring the operator N_n through several different spaces, but, as already mentioned in [19], this will not suffice to prove the above theorems.

Instead, the main idea in proving Theorems 1 and 2 will be to divide the problem into two parts. On the one hand we study, under a more general aspect, the main ingredients of the proof of a Kharshiladze—Lozinski type theorem (Proposition 1). This leads to a new sort of a Berman—Marcinkiewicz type identity which employs the Chebyshev translation (1.5), though this is not the proper translation for the orthogonal systems involved. It is based on a product formula of an unfamiliar type (see (2.9) below) which involves two different sets of orthogonal functions which may not consist of polynomials. On the other hand we isometrically transform a given algebraic polynomial projection on $L^1_{w(\alpha,\beta)}$ into a projection of L^1_ϱ onto some set of generalized polynomials (Proposition 2). The actual proof then consists in showing that the two propositions fit together nicely in the three cases under consideration.

Before proceeding with the proofs of Theorems 1 and 2 we note some of their consequences. As analogues of the classical theorems of Nikolaev and Berman (cf. [6, p. 215]) one deduces the non-existence of a sequence of orthogonal projections of $L^1(-1, 1)$ onto $\mathcal{P}_n$ which is uniformly bounded in n, as well as the non-existence of linear operators R_n from L^1 into $\mathcal{P}_n$ such that $\|f-R_nf\|/\|f-p_n^*\|_{L^1}=\mathcal{O}(1)$, $n\to\infty$, for each $f\in L^1$, where p_n^* denotes a polynomial of best approximation of f in L^1. There is also a counterpart of the classical Faber theorem [15]. The latter says that, given a triangular matrix Δ of interpolation nodes in $[-1, 1]$ there always exists an $f\in C[-1, 1]$ such that the associated Lagrange interpolation polynomials $L_nf=L_n(\Delta;f)$ do not converge to f in the uniform norm. The situation is different if convergence in the L^p_ω norm for any non-negative weight ω is considered and if one chooses $\Delta=\Delta_\omega$, the matrix of zeros of the orthogonal polynomials corresponding to ω. The case $p=2$ has been treated by Erdős and Turán [13]. Other cases and particular weights have been considered by Erdős and Feldheim [12], Marcinkiewicz [25], Holló [20], Szegő [33, § 14.3], Askey [1], [2], Nevai [29], and Vértesi [34]. We only mention Holló's result: For the Jacobi abscissas $\Delta=\Delta(\alpha,\beta)$ and each $f\in C[-1, 1]$ one has $\|L_n(\Delta,f)-f\|_{L^1}\to 0$, $n\to\infty$, provided that $\max(\alpha,\beta)<3/2$. All these results deal with $C[-1, 1]$ functions or with properly Riemann integrable functions, at least. There are also a few L^2 convergence results for functions which are improperly Riemann square integrable (e.g. Freud [17, Chapter III. 2], Esser [14]). If the class of functions is enlarged to $CL^1(-1, 1)=C(-1, 1)\cap L^1(-1, 1)$, however, Theorem 1 implies that the result turns into the negative again (in Theorem 1 the domain of P_n may be restricted to the dense subspace CL^1 of L^1).

Corollary 1. *Let Δ be a triangular matrix of nodes in $(-1, 1)$. There exists an $f\in CL^1(-1, 1)$ such that the Lagrange interpolation polynomials $L_n(\Delta,f)$ do not converge to f in L^1 norm as $n\to\infty$.*

From the proof of Theorem 1 and (2.8) below one also obtains estimates for the relative projection constant $\lambda(\mathcal{P}_n, L^1(-1, 1))$. Generally, for a normed linear space X and a subspace Y of X the relative projection constant of Y in X is defined by

$$\lambda(Y, X)=\inf\{\|P\|_{[X]};\ P \text{ is a linear projection of } X \text{ onto } Y\}.$$

Corollary 2. *For each $n\in\mathbf{N}$,*

$$\tfrac{1}{2}\lambda_n \leqq \lambda(\mathcal{P}_n, L^1(-1, 1)) \leqq \lambda_n, \tag{1.6}$$

where

$$\lambda_n = \|S_n^{1/2,1/2}\|_{[L^1]} = \frac{4}{\pi^2}\log n+\mathcal{O}(1), \quad n\to\infty. \tag{1.7}$$

Projections with norm divergence of order $\log n$ are, for example, the $S_n^{\alpha,\alpha}$ with $1/2\leqq\alpha\leqq 3/2$ [19]. For $n=1$ the minimal projection as well as the relative projection constant $\lambda(\mathcal{P}_1, L^1(-1, 1))$ can be determined explicitly. See Franchetti and Cheney [16]. Estimate (1.6) is to be compared with the corresponding result in case

$X=C[-1, 1]$ or $L^1_\varrho(-1, 1)$,

(1.8) $$\tfrac{1}{2}(\tilde{\lambda}_n-1) \leqq \lambda(\mathscr{P}_n, X) \leqq \tilde{\lambda}_n,$$

where

(1.9) $$\tilde{\lambda}_n = \|S_n^{-1/2, -1/2}\|_{[X]} = \frac{4}{\pi^2}\log n+\mathcal{O}(1), \quad n \to \infty.$$

In both spaces it is an open problem to determine the exact value of the projection constants and to find projection operators for which it is attained. As it is well-known in the trigonometric case the projection constants coincide with the Lebesgue constants of the Fourier partial sums, and it is a deep result that the minimal projection is unique, see [8] for $X=C_{2\pi}$ and [22] for $X=L^1_{2\pi}$ and generalizations. For a survey on projection operators compare also [7].

We further mention a remarkable parallelicity of the above results with the situation for minimal *polynomials* in $L^p(-1, 1)$ spaces. It is well-known that in the set of monic algebraic polynomials, the polynomials of minimal $L^p(-1, 1)$ norm for $p=2, \infty$, and 1 are given by the monic Legendre polynomials and the monic Chebyshev polynomials of the first and second kind, respectively. For $p=1$, this is the Korkin and Solotareff problem (see [6, p. 222]). On the level of linear projection *operators* on $L^p(-1, 1)$, the situation is quite similar. For $p=2$ the Legendre partial sums are minimal, for $p=\infty$ (1.8) and (1.9) imply that the Chebyshev partial sums of the first kind are nearly minimal, and for $p=1$ the near — minimality of the Chebyshev partial sums of the second kind now follows by Corollary 2. This parallelicity holds also in the spaces treated in Theorem 2.

In view of Theorem 2, Corollaries 1 and 2 also hold with L^1 replaced by $L^1_{w(0,-1/2)}$ or $L^1_{w(-1/2,0)}$.

2. Proofs

Let $\alpha, \beta > -1$ and let $\tilde{P}_n^{\alpha,\beta}(x)$ denote the Jacobi polynomial of degree $n\in\mathbf{P}$, normalized such that these form an orthonormal system on the interval $[-1, 1]$ with respect to the weight function $w_{\alpha,\beta}$ [33]. Particular cases to be used later are

(2.1) $$\tilde{P}_n^{-1/2,-1/2}(x) = \begin{cases}\sqrt{2/\pi}\cos(n \arccos x), & n\in\mathbf{N}\\ 1/\sqrt{\pi}, & n = 0,\end{cases}$$

(2.2) $$\tilde{P}_n^{1/2,1/2}(x) = \sqrt{\frac{2}{\pi}}\sin((n+1)\arccos x)(1-x^2)^{-1/2},$$

(2.3) $$\tilde{P}_n^{1/2,-1/2}(x) = \sqrt{\frac{2}{\pi}}\sin\left(\left(n+\frac{1}{2}\right)\arccos x\right)(1-x)^{-1/2},$$

(2.4) $$P_n^{-1/2,1/2}(x) = \sqrt{\frac{2}{\pi}}\cos\left(\left(n+\frac{1}{2}\right)\arccos x\right)(1+x)^{-1/2}.$$

The Fourier—Jacobi coefficients and the Jacobi partial sums of a function $f\in L^1_{w(\alpha,\beta)}$ are denoted by

$$f^\wedge(k;\ \alpha,\beta) = \int_{-1}^{1} f(t)\tilde{P}_k^{\alpha,\beta}(t)\, w_{\alpha,\beta}(t)\,dt,$$

(2.5)
$$S_n^{\alpha,\beta}(f;\ x) = \sum_{k=0}^{n} f^\wedge(k;\ \alpha,\beta)\tilde{P}_k^{\alpha,\beta}(x),$$

respectively. A lower bound for the norm of the operator $S_n^{a,b}$ on the space $L^1_{w(\alpha,\beta)}$ will be needed for particular values of $a\geqq\alpha>-1$, $b\geqq\beta>-1$:

(2.6)
$$\|S_n^{a,a}\|_{[L^1_{w(\alpha,\alpha)}]} \geqq \|S_{[n/2]}^{a,-1/2}\|_{[L^1_{w(\alpha,-1/2)}]},$$

(2.7)
$$\|S_n^{a,-1/2}\|_{[L^1_{w(\alpha,-1/2)}]} = \|S_n^{-1/2,a}\|_{[L^1_{w(-1/2,\alpha)}]}$$

$$\geqq 2^{-a-1/2}\frac{\Gamma(n+a+3/2)}{\Gamma(1/2)\Gamma(n+a+1)}\|P_n^{1/2,a}\|_{[L^1_{w(-1/2,a)}]}$$

$$= (4/\pi^2)\log n + C_a + \mathcal{O}\left(\frac{\log n}{n}\right) + \mathcal{O}(n^{-a-3/2}), \quad n\to\infty.$$

Here an asymptotic expansion due to Lorch [24] has been employed, see [19]. The results of [19] also imply that the rate $\log n$ is sharp, provided $2\alpha+1/2\leqq a\leqq 2\alpha+3/2$. In the particular cases needed in connection with Corollary 2 the constant $4/\pi^2$ is sharp, too, since it can be shown that

(2.8)
$$\|S_n^{1/2,1/2}\|_{[L^1]} \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+3/2)\theta}{\sin(\theta/2)}\right| d\theta = \frac{4}{\pi^2}\log n + \mathcal{O}(1), \quad n\to\infty.$$

Let $\Phi=\Phi_\varrho$ denote the set of all orthonormal systems $\varphi=\{\varphi_k\}_{k\in\mathbf{P}}$ with respect to the Chebyshev weight $\varrho(x)=w_{-1/2,-1/2}(x)$, i.e., each φ consists of functions $\varphi_k\in L^2_\varrho$, not necessarily polynomials, for which

$$\int_{-1}^{1}\varphi_j(x)\,\varphi_k(x)(1-x^2)^{-1/2}\,dx = \delta_{jk} \quad (j,k\in\mathbf{P}).$$

For $n\in\mathbf{P}$ and $\varphi\in\Phi$, let

$$\mathcal{P}_n^\varphi = \left\{\sum_{k=0}^{n} a_k\varphi_k;\ a_k\in\mathbf{R},\quad 0\leqq k\leqq n\right\}$$

denote the corresponding sets of abstract polynomials and, assuming further that $\varphi_k\in L^\infty(-1,1)$ for each $k\in\mathbf{P}$, let

$$S_n^\varphi g = \sum_{k=0}^{n}\int_{-1}^{1} g(t)\varphi_k(t)\varrho(t)\,dt \quad \varphi_k$$

be the nth partial sum with respect to φ of a function $g\in L^1_\varrho$.

Proposition 1. *Let $\varphi\in\Phi$ be a fundamental system in L^1_ϱ consisting of $L^\infty(-1, 1)$ functions, and suppose that there exists another system $\psi\in\Phi$ with the property*

$$\tau_t(\varphi_k; x) = c\varphi_k(x)\psi_k(t) \tag{2.9}$$

for $x, t\in[-1, 1]$, $k\in\mathbf{P}$, where τ_t is given by (1.5) and c is a constant, independent of k. For an arbitrary bounded linear projection P^φ_n of L^1_ϱ onto $\mathcal{P}^\varphi_n$ the Berman—Marcinkiewicz type identity

$$S^\varphi_n(g; x) = c^{-2}\int_{-1}^{1} (\tau_t P^\varphi_n \tau_t g)(x)(1-t^2)^{-1/2}\,dt \quad (g\in L^1_\varrho) \tag{2.10}$$

holds. Moreover,

$$\|S^\varphi_n\|_{[L^1_\varrho]} \leqq \pi c^{-2}\,\|P^\varphi_n\|_{[L^1_\varrho]}. \tag{2.11}$$

Proof. For each $n\in\mathbf{P}$, S^φ_n belongs to $[L^1_\varrho]$ and, denoting the right-hand side of (2.10) by $N^\varphi_n g$, the operator N^φ_n is well-defined on L^1_ϱ since $\tau_t P^\varphi_n \tau_t g$ is strongly continuous in t and so the Bochner integral exists. Moreover, N^φ_n is in $[L^1_\varrho]$ since P^φ_n is, and $\|\tau_t\|=1$ for each t. Thus it suffices to prove (2.10) for g in a dense subset, i.e. to show that $S^\varphi_n\varphi_k=N^\varphi_n\varphi_k$ for each $k\in\mathbf{P}$. Indeed, for $0\leqq k\leqq n$ one has, by (2.9),

$$N^\varphi_n(\varphi_k;\ x) = \varphi_k(x)\int_{-1}^{1} (\psi_k(t))^2(1-t^2)^{-1/2}\,dt = \varphi_k(x) = S^\varphi_n(\varphi_k;\ x),$$

and for $k>n$, setting $P^\varphi_n\varphi_k=\sum_{j=0}^n a_{jk}\varphi_j$, say,

$$N^\varphi_n(\varphi_k;\ x) = \sum_{j=0}^n a_{jk}\varphi_j(x)\int_{-1}^{1} \psi_j(t)\psi_k(t)(1-t^2)^{-1/2}\,dt = 0 = S^\varphi_n(\varphi_k;\ x).$$

This proves (2.10), and (2.11) is an immediate consequence.

Let us consider a subset of the class Φ which consists of those systems $\varphi(\alpha,\beta)=\{\varphi^{\alpha,\beta}_k\}_{k\in\mathbf{P}}$, $\alpha,\beta\geqq-1/2$, whose elements are in L^∞ and can be represented in terms of Jacobi polynomials and weights as

$$\varphi^{\alpha,\beta}_k(x) = \tilde{P}^{2\alpha+1/2,\,2\beta+1/2}_k(x)\,w_{\alpha+1/2,\,\beta+1/2}(x) \quad (k\in\mathbf{P}). \tag{2.12}$$

Proposition 2. *Let $\alpha,\beta\geqq-1/2$ and set $\omega(x)=w_{\alpha+1/2,\beta+1/2}(x)$.*

(i) Given a bounded linear projection P_n of $L^1_{w(\alpha,\beta)}$ onto $\mathcal{P}_n$, $P^{\varphi(\alpha,\beta)}_n(g;x)=P_n(g/\omega;\ x)\omega(x)$ defines a bounded linear projection of L^1_ϱ onto $\mathcal{P}^{\varphi(\alpha,\beta)}_n$, and

$$\|P^{\varphi(\alpha,\beta)}_n\|_{[L^1_\varrho]} = \|P_n\|_{[L^1_{w(\alpha,\beta)}]}. \tag{2.13}$$

(ii) In particular, choosing the Jacobi partial sum $S^{2\alpha+1/2,\,2\beta+1/2}_n$ for P_n, one has $P^{\varphi(\alpha,\beta)}_n=S^{\varphi(\alpha,\beta)}_n$ and hence

$$\|S^{\varphi(\alpha,\beta)}_n\|_{[L^1_\varrho]} = \|S^{2\alpha+1/2,\,2\beta+1/2}_n\|_{[L^1_{w(\alpha,\beta)}]}. \tag{2.14}$$

Proof. Obviously $P^{\varphi(\alpha,\beta)}_n$ is well-defined on L^1_ϱ and a projection onto $\mathcal{P}^{\varphi(\alpha,\beta)}_n$. The identity (2.13) is true since $\|P^{\varphi(\alpha,\beta)}_n g\|_{L^1_\varrho}=\|P_n f\|_{L^1_{w(\alpha,\beta)}}$ holds with $f=g/\omega$.

Concerning (ii), the choice $P_n=S_n^{2\alpha+1,2\beta+1}$ is clearly admissible in part (i). Since $P_n^{\varphi(\alpha,\beta)}(g;x)=S_n^{\varphi(\alpha,\beta)}(g;x)$ for each $g\in L_\varrho^1$, the proof is complete.

Proof of Theorem 1. Given P_n, Proposition 2 (i) furnishes a bounded linear projection $P_n^{\varphi(0,0)}$ of L_ϱ^1 onto $\mathcal{P}_n^{\varphi(0,0)}$ with

$$\|P_n^{\varphi(0,0)}\|_{[L_\varrho^1]}=\|P_n\|_{[L^1]}. \tag{2.15}$$

In view of (2.12), (2.2) one has

$$\varphi_k^{0,0}(x)=\tilde{P}_k^{1/2,1/2}(x)(1-x^2)^{1/2}=\sqrt{2/\pi}\sin((k+1)\arccos x). \tag{2.16}$$

In order to apply Proposition 1 it remains to show that the $\varphi_k^{0,0}$ are fundamental in L_ϱ^1 and that they satisfy (2.9). The first of these assertions holds since, e.g., the $(C,1)$ means of the Legendre expansion converge in L^1 and each Legendre polynomial has a unique representation in terms of Chebyshev polynomials of the second kind, so $\{\tilde{P}_k^{1/2,1/2}\}_{k\in\mathbf{P}}$ is fundamental in L^1, too. Moreover,

$$\tau_t(\varphi_k^{0,0};x)=\sqrt{\pi/2}\,\varphi_k^{0,0}(x)\tilde{P}_{k+1}^{-1/2,-1/2}(t), \tag{2.17}$$

thus (2.9) is satisfied with $\psi=\{\tilde{P}_{k+1}^{-1/2,-1/2}\}_{k\in\mathbf{P}}\in\Phi$ and $c=\sqrt{\pi/2}$. Hence Proposition 1 implies that

$$\|S_n^{\varphi(0,0)}\|_{[L_\varrho^1]}\leqq 2\|P_n^{\varphi(0,0)}\|_{[L_\varrho^1]},$$

i.e., in view of Proposition 2 (ii) and (2.15),

$$\|S_n^{1/2,1/2}\|_{[L^1]}\leqq 2\|P_n\|_{[L^1]}. \tag{2.18}$$

Applying (2.6), (2.7) with $a=1/2$, $\alpha=0$, the assertion of Theorem 1 follows.

Proof of Theorem 2. Since P_n need not be a symmetric operator, the two cases $(\alpha,\beta)=(0,-1/2)$ and $(\alpha,\beta)=(-1/2,0)$ have to be proved separately. Proceeding as above, we apply Propositions 2 and 1. The corresponding systems $\varphi(\alpha,\beta)$ are (cf. (2.3), (2.4), (2.12))

$$\begin{aligned}\varphi_k^{0,-1/2}(x)&=\sqrt{2/\pi}\sin((k+1/2)\arccos x),\\ \varphi_k^{-1/2,0}(x)&=\sqrt{2/\pi}\cos((k+1/2)\arccos x),\end{aligned} \tag{2.19}$$

respectively. These belong to $L^\infty(-1,1)$ and they are fundamental in $L^1_{w(0,-1/2)}$ and in $L^1_{w(-1/2,0)}$, respectively, which can be seen as in the proof of Theorem 1. Moreover, condition (2.9) of Proposition 1 is satisfied in view of the identities

$$\tau_t(\varphi_k^{0,-1/2};x)=\sqrt{\pi/2}\,\varphi_k^{0,-1/2}(x)\,\varphi_k^{-1/2,0}(t), \tag{2.20}$$

$$\tau_t(\varphi_k^{-1/2,0};x)=\sqrt{\pi/2}\,\varphi_k^{-1/2,0}(x)\,\varphi_k^{-1/2,0}(t), \tag{2.21}$$

so that we have $c=\sqrt{\pi/2}$ and $\psi=\varphi(-1/2,0)$ in both cases. Combining the assertions of Propositions 1 and 2, one has

$$\|P_n\|_{[L^1_{w(0,-1/2)}]} \geqq \frac{1}{2} \|S_n^{1/2,-1/2}\|_{[L^1_{w(0,-1/2)}]},$$

$$\|P_n\|_{[L^1_{w(-1/2,0)}]} \geqq \frac{1}{2} \|S_n^{-1/2,1/2}\|_{[L^1_{w(-1/2,0)}]},$$

respectively. By an application of (2.7) with $a=1/2$, $\alpha=0$ the proof is complete.

3. Remarks

It is also possible to deduce a Kharshiladze—Lozinski type theorem in connection with the Fourier—Bessel expansion of order $\alpha=-1/2$ (cf. [35], [27; §3]). Denoting the corresponding partial sum operator by

$$B_n(f;\ x) = \sum_{k=0}^n \int_0^1 f(t)\, J_k(t)\, dt J_k(x) \quad (n\in\mathbf{P})$$

for $f\in L^1(0, 1)$, where $J_k(x)=\sqrt{2}\cos(k+1/2)\pi x$, one has the following.

Theorem 3. *For each $n\in\mathbf{P}$ let P_n be a bounded linear projection of $L^1(0, 1)$ onto the set of abstract polynomials of the form $\sum_{k=0}^n a_k J_k$, $a_k\in\mathbf{R}$. Then*

$$\|P_n\|_{[L^1(0,1)]} \geqq \frac{1}{2} \|B_n\|_{[L^1(0,1)]} \geqq \frac{2}{\pi^2}\log n+\mathcal{O}(1) \quad (n\to\infty). \tag{3.1}$$

Proof. In order to set up a Berman—Marcinkiewicz type identity, the Chebyshev translation has to be replaced by the "Fourier—Bessel" translation $T_{FB,t}$, $0\leqq t\leqq 1$, which is defined by

$$T_{FB,t}(f;\ x) = \begin{cases} \frac{1}{2}\{f(|x-t|)+f(x+t)\}, & 0\leqq x+t\leqq 1 \\ \frac{1}{2}\{f(|x-t|)-f(2-x-t)\}, & 1< x+t\leqq 2 \end{cases}$$

for $f\in L^1(0, 1)$. Obviously the translation has the properties

$$T_{FB,t}(J_k;\ x) = \frac{1}{\sqrt{2}} J_k(x) J_k(t) \quad (0\leqq x,\ t\leqq 1),$$

$$\|T_{FB,t}f\|_{L^1(0,1)} \leqq \|f\|_{L^1(0,1)} \quad (f\in L^1(0, 1)),$$

which are required to deduce

$$B_n(f;\ x) = 2\int_0^1 (T_{FB,t}P_nT_{FB,t}f)(x)\, dt \quad (f\in L^1(0, 1)),$$

as well as (3.1). In view of the identity

$$\|B_n\|_{[L^1(0,1)]} = \|S_n^{-1/2,1/2}\|_{[L^1_{w(-1/2,0)}(-1,1)]}$$

the divergence order of the Lebesgue constant on the right-hand side of (3.1) is the same as in Theorem 2.

The algebraic version of the Berman—Marcinkiewicz identity (1.4) and of Theorem A can be subsumed under Proposition 1, after making a slight modification in order to take care of the zero degree term. Indeed, for the two systems φ and ψ one can choose the Chebyshev polynomials $\{\tilde{P}_k^{-1/2,-1/2}\}_{k\in\mathbf{P}}$, so that $\mathscr{P}_n^\varphi$ turns into $\mathscr{P}_n$. The crucial point in Proposition 1, however, is that different systems φ and ψ are admitted (cf. (2.20)) and that ψ even must not be fundamental in L_ϱ^1 (cf. (2.17)).

The fact that Proposition 1 applies in the four cases $\varphi=\varphi(\alpha,\beta)$ with $\alpha,\beta\in\{-1/2,0\}$, can be traced back to our requirement that the Chebyshev translation τ_t appears in equation (2.9). Hence, similar results for other values of α and β may be expected if an appropriate translation operator T_t can be found which satisfies (2.9) and has its $[L_\varrho^1]$ norm uniformly bounded with respect to t. One approach to determine such a T_t is to solve an initial boundary value problem for an associated hyperbolic partial differential equation (see e.g., Levitan [23]). If the system φ consists of eigenfunctions of a Sturm—Liouville differential equation [5, Chapter X]

$$L_x\varphi_k(x)+\mu_k\varphi_k(x)=0,\quad L_x=w^{-1}(x)\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]-q(x),$$

the appropriate translation T_t can be considered as the solution $u(x,t)=T_t(f;x)$ of the partial differential equation $(L_x-L_t)u(x,t)=0$ under initial conditions depending on f and additional boundary conditions.

For the orthonormal systems $\varphi(\alpha,\beta)=\{\varphi^{\alpha,\beta}\}_{k\in\mathbf{P}}$ the Sturm—Liouville operators are given by

$$L_x^{\alpha,\beta}=L_x^{-1/2,-1/2}-q^{\alpha,\beta}(x),\ q^{\alpha,\beta}(x)=\frac{\alpha(2\alpha+1)}{1-x}+\frac{\beta(2\beta+1)}{1+x},$$

where

$$L_x^{-1/2,-1/2}=(1-x^2)^{1/2}\frac{d}{dx}(1-x^2)^{1/2}\frac{d}{dx}$$

denotes the differential operator in the Chebyshev differential equation. If now $\alpha,\beta\in\{-1/2,0\}$, the potential function $q^{\alpha,\beta}$ vanishes, and the Chebyshev translation is a solution of $(L_x^{\alpha,\beta}-L_t^{\alpha,\beta})u(x,t)=0$. In these instances the final result says that the partial sum operators $S_n^{2\alpha+1/2,2\beta+1/2}$ are the appropriate reference operators for projections P_n of $L_{w(\alpha,\beta)}^1$ onto $\mathscr{P}_n$ in the sense that

$$\|S_n^{2\alpha+1/2,2\beta+1/2}\|_{[L_{w(\alpha,\beta)}^1]}\leqq C\|P_n\|_{[L_{w(\alpha,\beta)}^1]}.$$

It may be conjectured that this inequality holds true for all $\alpha,\beta\geqq-1/2$.

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