On the hyperconvexity of holomorphically convex domains in the space $\mathbf{C}^n$

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§1. Preliminaries

In 1974, Jean-Luc Stehlé has given in his paper [4], such a conjecture[1] that holomorphically convex domain $D=\mathring{\bar{D}}$ in $\mathbf{C}^n$ is hyperconvex. In 1976, Jean-Louis Ermine has shown in his paper [1] that this conjecture is positive in case of holomorphically convex Reinhardt domains[2]. But, in general case, it is as yet unknown that this conjecture is positive or not. Evidently, holomorphically convex domain in $\mathbf{C}^n$ can be approximated by an increasing sequence of analytic polyhedra and analytic polyhedra are hyperconvex.

The purpose of this papeı is to give such a proof that this conjecture is positive in case of holomorphically convex domains of some type by means of the above approximation.

Definition 1.[3] *Let D be a relatively compact open set in $\mathbf{C}^n$. D is said to be hyperconvex if and only if there exists a plurisubharmonic function $p(z)$ defined on a neighbourhood of $\bar{D}$ and negative on D, such that*

$$\{z \in D \mid p(z) \leqq c\}$$

is a relatively compact set in D for any $c<0$.

The following lemma is easily shown from Definition 1.

[1]) Cf. [4], pp. 167, 177 in which D is relatively compact in $\mathbf{C}^n$.
[2]) Cf. [1], pp. 131—133.
[3]) Cf. [4], p. 163.

Lemma 1.[4] *Let D be a relatively compact open set in $\mathbf{C}^n$. D is hyperconvex if and only if there exists a plurisubharmonic function $p(z)$ defined on a neighbourhood of $\bar{D}$ and negative on D such that for $z \in D$*

$$\lim_{z \to \partial D} p(z) = 0.$$

In case of $D = \mathring{\bar{D}}$, Lemma 1 is modified by J.-L. Ermine as follows:

Lemma 2. *Let $D = \mathring{\bar{D}}$ be a connected and relatively compact open set in $\mathbf{C}^n$. Suppose that for any sequence $S = \{z_\nu\}$, $\nu \in \mathbf{N}$ which has no accumulating point in D, there exists a plurisubharmonic function $p_S(z)$ defined on D, such that $p_S(z) < 0$ on D and*

$$\lim_{\nu \to +\infty} p_s(z_\nu) = 0.\text{[5]}$$

Then, D is hyperconvex.

Proof. The proof can be seen in [1].

§2. Indicatrices of Finite Order

Let D_1 and D_2 be domains in $\mathbf{C}^n$, such that $\mathbf{C}^n - D_2 \subset\subset D_1$. Let $f_1(z)$ and $f_2(z)$ be holomorphic on D_1 and on D_2 resp., and both of f_1/f_2 and f_2/f_1 be holomorphic on $D_1 \cap D_2$.

We consider a *current* (in the sense of G. de Rham) on $\mathbf{C}^n$, defined by

$$\Theta_k = 2i\, d_z d_{\bar{z}} \log |f_k(z)| \quad (k = 1, 2).$$

Because of the pluriharmonicity of $\log |f_1/f_2|$, we have, on $D_1 \cap D_2$,

$$\Theta_1 - \Theta_2 = 2i\, d_z d_{\bar{z}} \log \left|\frac{f_1}{f_2}\right| = 0,$$

and then $\Theta_1 = \Theta_2$. Let us denote $f = f_1$ and $D = D_1$ and give the following definition.

Definition 2.[6] *The current on $\mathbf{C}^n$:*

$$\Theta = 2i\, d_z d_{\bar{z}} \log |f(z)|$$

is said to be a current associated to the hypersurface $V^{n-1} = \{f(z) = 0\} \subset\subset D$, where $f(z)$ is a holomorphic function defined on a bounded domain D in $\mathbf{C}^n$.

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[4]) Cf. [4], p. 163. The limit $\lim_{z \to \partial D} p(z)$ means that for any $\varepsilon > 0$, there exists a neighbourhood $U(\partial D)$ of ∂D, such that $|p(z)| < \varepsilon$, for every $z \in U(\partial D)$.

[5]) Cf. [1], p. 136, where the property is called "HC-convex".

[6]) Cf. [2], pp. 368—369.

Since this current Θ is positive, closed and of type (1, 1),[7] we construct a form

$$\tilde{\nu} = \frac{\Theta}{\pi^{n-1}} \wedge \tilde{\alpha}^{n-1}, \quad \tilde{\alpha} = \frac{i}{2} d_z d_{\bar{z}} \log \sum_p z_p \bar{z}_p$$

and give the following definition.

Definition 3.[8] *The function*

$$\tilde{\nu}(t) = \int_{\|z\| < t} \tilde{\nu} \quad \text{defined on} \quad t \geqq 0$$

is said to be a projective indicatrix (of the current Θ) whose centre is the origin or simply an indicatrix (of Θ) of centre O.

It is shown in the paper of P. Lelong [9] that there exists a limit $\tilde{\nu}(0) = \lim_{t \downarrow 0} \tilde{\nu}(t) > 0$, and the function $\tilde{\nu}(t)$ is increasing and positive for $t \geqq 0$.

Definition 4.[10] *Indicatrix $\tilde{\nu}(t)$ defined for $t \geqq t_0 \geqq 0$ is said to be of finite order λ, if and only if*

$$\limsup_{t \to +\infty} \frac{\log \tilde{\nu}(t)}{\log t} = \lambda < +\infty.$$

Definition 5. *A current Θ which is positive and closed is said to be of finite order, if its indicatrix is of finite order.*

Indicatrices of currents satisfy the following lemma.

Lemma 3.[11] *Let $\tilde{\nu}(t)$ be an indicatrix of a current Θ which is positive and closed on $\mathbf{C}^n$. Then, the following two conditions are equivalent for $s > 0$, $a \geqq 0$*

(i) $$\int_a^{+\infty} t^{-s} d\tilde{\nu}(t) < +\infty$$

(ii) $$\lim_{t \to +\infty} \tilde{\nu}(t) t^{-s} = 0 \quad \textit{and} \quad \int_a^{+\infty} \tilde{\nu}(t) t^{-s-1} dt < +\infty.$$

Proof. The proof is easy.

Lemma 4. *Let λ be an order of $\tilde{\nu}(t)$ and $\lambda_0 = \inf \{s | \int_a^{+\infty} t^{-s} d\tilde{\nu}(t) < +\infty\}$. Then, $\lambda \leqq \lambda_0$.*

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7) Cf. [2], pp. 365—369 & [3], pp. 244—245, pp. 247—250.
8) Cf. [2], pp. 371—373.
9) Cf. [2], pp. 371—372 & [3], pp. 259—261.
10) Cf. [2], p. 373.
11) Cf. [2], pp. 373—374, Proposition 2.

Proof.

$$\lambda_0 \geqq \inf\{s \mid \lim_{t\to+\infty} \tilde{\nu}(t)t^{-s}=0\}$$

$$=\sup\{s \mid \lim_{t\to+\infty} \tilde{\nu}(t)t^{-s}=C_s>0\}$$

$$=\sup\left\{s \,\middle|\, \lim_{t\to+\infty} \log\frac{\tilde{\nu}(t)}{t^s}=\log C_s\right\}$$

$$=\limsup_{t\to+\infty}\frac{\log\tilde{\nu}(t)}{\log t}=\lambda.$$

This shows that $\lambda \leqq \lambda_0$.

Lemma 5. *If an inequality*

$$\int_a^{+\infty} t^{-\mu-1}d\tilde{\nu}(t)<+\infty$$

holds for an integer μ *and* $a \geqq 0$, *the order of* $\tilde{\nu}(t)$ *is finite.*

Proof. The proof is easy from Lemma 4.

By means of the current of finite order, the following important properties of hypersurfaces have been obtained by P. Lelong and H. Skoda.

Theorem 1.[12] *Let* $f(z)$ *be a holomorphic function on a domain* D *in* $\mathbf{C}^n$ *and* Θ *be a positive and closed current on* $\mathbf{C}^n$ *associated to the hypersurface* $V^{n-1}=\{f(z)=0\}$ *containing no origin. If* Θ *is of finite order, there exists an entire function* $F(z)$ *on* $\mathbf{C}^n$, *such that*

$$V^{n-1}=\{F(z)=0\}.$$

Proof. The proof can be seen in [2].

In the paper of H. Skoda, a part of conditions in Theorem 1 is somewhat modified. It is as follows:

Corollary[13]**.** *Suppose that with the same hypothesis as Theorem* 1,

$$\int_a^{+\infty} t^{-\mu-1}d\tilde{\nu}(t)<+\infty,$$

where $\tilde{\nu}(t)$ *is an indicatrix of* Θ, μ *is an integer and* $a \geqq 0$. *Then, there exists an entire function* $F(z)$ *on* $\mathbf{C}^n$, *such that*

$$V^{n-1}=\{F(z)=0\}.$$

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[12]) Cf. [2], pp. 394—397, Theorem 5.

[13]) Cf. [5], p. 138, Theorem 7.2. The hypothesis that Θ is of finite order is replaced with the finiteness of integral.

§3. A Class of Holomorphic Functions

Definition 6. *A set* $\mathfrak{K}_D$ *of functions that are holomorphic on a domain* $D \subset \mathbf{C}^n$ *is said to constitute a class, if the relation* $f \in \mathfrak{K}_D$ *implies that* $cf \in \mathfrak{K}_D$, *where* c *is an arbitrary complex number.*

For example, the set $\mathfrak{P}_D$ of all polynomials defined on D, or the set $\mathfrak{G}_D$ of all holomorphic functions defined on D constitute a class resp., and $\mathfrak{G}_D$ contains every class $\mathfrak{K}_D$. Let us consider a particular class as follows:

Lemma 6. *Let* $\Theta_{f|\gamma}$ *be a current associated to the hypersurface* $V_f^{n-1} = \{f(z) = \gamma | f \in \mathfrak{G}_D, \gamma$ *be a complex const.*$\}$ *containing no origin. Let us define* $\mathfrak{F}_D = \{f(z) \in \mathfrak{G}_D |$ *the order of* $\Theta_{f|\gamma}$ *be finite for a complex const.* $\gamma\}$. *Then,* $\mathfrak{F}_D$ *constitutes a class.*

Proof. Suppose that $f(z) \in \mathfrak{F}_D$. Then, $g = cf$ (c: complex number) is also holomorphic on D. The currents $\Theta_{f|\gamma}$ and $\Theta_{g|c\gamma}$ that are associated to the hypersurfaces $V_f^{n-1} = \{f(z) = \gamma\}$ and $V_g^{n-1} = \{g(z) = c\gamma\}$ resp. have the following relations:

$$\Theta_{g|c\gamma} = 2i\, d_z\, d_{\bar{z}} \log |g - c\gamma| = 2i\, d_z\, d_{\bar{z}} \log |c(f - \gamma)| = \Theta_{f|\gamma}.$$

Let $\tilde{\nu}_f(t)$ and $\tilde{\nu}_g(t)$ be indicatrices of $\Theta_{f|\gamma}$ and $\Theta_{g|c\gamma}$ on $\|z\| < t$ resp., and λ_f and λ_g be orders of $\tilde{\nu}_f$ and $\tilde{\nu}_g$ resp. Evidently, we have

$$\lambda_g = \lambda_f.$$

This shows that

$$g = cf \in \mathfrak{F}_D.$$

Hence, $\mathfrak{F}_D$ constitutes a class.

Definition 7. *Let* D *be a relatively compact open set in* $\mathbf{C}^n$. D *is said to be* $\mathfrak{K}$*-convex, if and only if for any compact set* $K \subset D$, *the set*

$$\hat{K} = \bigcap_{f \in \mathfrak{K}_D} \{z \in D \,|\, |f(z)| \leqq \sup_{\zeta \in K} |f(\zeta)|\}$$

is also compact, where $\mathfrak{K}_D$ *is a class of functions defined in Definition* 6.

Remark. As a particular case of Definition 7, $\mathfrak{F}$-convexity, $\mathfrak{P}$-convexity and $\mathfrak{G}$-convexity can be defined corresponding to classes $\mathfrak{F}_D$, $\mathfrak{P}_D$ and $\mathfrak{G}_D$ resp. Especially, $\mathfrak{G}$-convexity is also called holomorphic convexity.

In Main Theorem, we are to give a proof that the Stehlé's conjecture is positive in case of $\mathfrak{F}$-convex domains in $\mathbf{C}^n$, and for that purpose we prepare for a definition of analytic polyhedra on class $\mathfrak{F}_D$ and a lemma.

Let us define an analytic polyhedron on class $\mathfrak{F}_D$ as follows:

Definition 8. *Let D be a relatively compact open set in $\mathbf{C}^n$. An open set $P(\subset\subset D)$ is said to be an analytic polyhedron on class $\mathfrak{F}_D$, if and only if there exist k holomorphic functions $f_\alpha(z)\in\mathfrak{F}_D$, for $\alpha=1, 2, \ldots, k$ defined on a neighbourhood $U(\bar{P})(\subset\subset D)$, such that*

$$P = \{z\in U(\bar{P}) \mid |f_\alpha(z)| < 1, \ \alpha = 1, 2, \ldots, k\}.$$

An approximation to the $\mathfrak{F}$-convex domain D in $\mathbf{C}^n$ by the sequence of analytic polyhedra on class $\mathfrak{F}_D$ is given by the following lemma.

Lemma 7.[14] *Let D be a relatively compact open and $\mathfrak{F}$-convex set of $\mathbf{C}^n$. Then, it is the union of an increasing sequence of bounded analytic polyhedra P_ν, $\nu\in\mathbf{N}$ on $\mathfrak{F}_D$, such that*

$$P_\nu \subset\subset P_{\nu+1} \subset\subset D, \quad D = \bigcup_\nu P_\nu.$$

Proof. Since D is the union of an increasing sequence of compact sets K_ν, $\nu\in\mathbf{N}$, it is sufficient to construct an analytic polyhedron P for an compact set K of this sequence, such that

$$K\subset \bar{P}, \quad P \subset\subset D,$$

and the functions $f_\alpha(z)$ $(\alpha=1, 2, \ldots, k)$ defining P, belong to $\mathfrak{F}_D$.

Since D is a relatively compact $\mathfrak{F}$-convex domain and $K\subset\subset D$, there exists a compact set $\hat{K}$, such that

$$K\subset \hat{K} \subset\subset D,$$

and for an arbitrary point $z_0\in\partial D$, there exists a neighbourhood $U(z_0)$ satisfying the relation

$$U(z_0)\cap D \subset D-\hat{K}$$

(see Definition 7). For a point $z_0^*\in U(z_0)\cap D$, there exists a function $f_{z_0^*}(z)\in\mathfrak{F}_D$ satisfying the following properties:

$$\sup_{z\in\hat{K}} |f_{z_0^*}(z)| = 1, \quad |f_{z_0^*}(z_0^*)| > 1.$$

Because of the construction of $\hat{K}$, it is evident that

$$|f_{z_0^*}(z)| > 1$$

for every point $z\in U(z_0)\cap D$.

Since ∂D is compact, ∂D can be covered by a finite number of neighbourhoods $U(z_\alpha)$ of $z_\alpha\in\partial D$ $(\alpha=1, 2, \ldots, k)$ which are constructed as the above $U(z_0)$, and the corresponding functions

$$f_\alpha(z) = f_{z_\alpha^*}(z)\in\mathfrak{F}_D$$

[14]) Cf. [6], pp. 140—141.

(z_α^* is an arbitrary fixed point in $U(z_\alpha)$) satisfy the following properties:

$$\sup_{z\in\hat{K}} |f_\alpha(z)| = 1$$

and

$$|f_\alpha(z)| > 1 \quad \text{on} \quad U(z_\alpha)\cap D.$$

Therefore, we have obtained the analytic polyhedron

$$P = \{z\in D \,|\, |f_\alpha(z)| < 1,\ f_\alpha\in\mathfrak{F}_D,\ \alpha = 1, 2, \dots, k\}$$

which possesses the following properties:

$$K\subset \hat{K}\subset \bar{P}, \quad P \subset\subset D.$$

Thus, our proof of Lemma 7 is completed.

Remark. As we see in the proof of Lemma 7, we can choose a neighbourhood $U(\partial D)$ of ∂D in $\mathbf{C}^n$, such that the relation

$$U(\partial D)\cap D\subset D-P$$

holds and every function $f_\alpha(z)$ ($\alpha=1, 2, \dots, k$) defining P satisfies the following inequality

$$|f_\alpha(z)| > 1 \quad \text{on} \quad U(\partial D)\cap D.$$

§4. Main Theorem

Main Theorem. *Let* $D=\mathring{\bar{D}}$ *be a connected, relatively compact and* $\mathfrak{F}$*-convex domain in* $\mathbf{C}^n$. *Then,* D *is hyperconvex.*

Proof. Let $S=\{z_\nu\}$, $\nu\in\mathbf{N}$ be a sequence of points in D, such that S has no accumulating point in D. To prove our theorem, it is sufficient to construct a plurisubharmonic function $p_S(z)$ defined on a neighbourhood of $\bar{D}$ and negative on D, such that for $S=\{z_\nu\}$, $\nu\in\mathbf{N}$,

$$\lim_{\nu\to+\infty} p_s(z_\nu) = 0 \quad \text{(by Lemma 2).}$$

There exists a set $E_S(\subset\partial D)$ of accumulating points of $\{z_\nu\}$, such that for any $z_0\in E_S$, $\{z_\nu\}$ has a subsequence $\{z_{\nu_k}\}$ converging to z_0. Then, it is sufficient to prove

$$\lim_{k\to+\infty} p_S(z_{\nu_k}) = 0$$

for the above $\{z_{\nu_k}\}$.

Since D is $\mathfrak{F}$-convex, there exists an increasing sequence of bounded analytic polyhedra P_ν, $\nu\in\mathbf{N}$ on class $\mathfrak{F}_D$, such that

$$z_\mu\in P_\nu \quad (\mu = 1, 2, \dots, \nu)$$

and

$$P_\nu \subset\subset P_{\nu+1} \subset\subset D, \quad D = \bigcup_\nu P_\nu \quad \text{(by Lemma 7).}$$

Corresponding to $\{P_\nu\}$, $\nu \in \mathbf{N}$, let us consider a sequence $\{\varepsilon_\nu\}$, $\nu \in \mathbf{N}$ such that $\varepsilon_\nu > 0$, $\varepsilon_\nu \to 0$ (as $\nu \to +\infty$).

For each P_ν, we can choose k_ν holomorphic functions $f_{\nu,\alpha}(z)$ ($\alpha = 1, 2, \ldots, k_\nu$) defined on a neighbourhood $U(\bar{P}_\nu)(\subset D)$ of P_ν, such that

$$P_\nu = \{z \in U(\bar{P}_\nu) \mid |f_{\nu,\alpha}(z)| < 1, \quad \alpha = 1, 2, \ldots, k_\nu\}.$$

Since $f_{\nu,\alpha} \in \mathfrak{F}_D$, the current $\Theta_{f_{\nu,\alpha}|\gamma}$ associated to the (complex) hypersurface

$$\overset{(\gamma)}{V}{}^{n-1}_{\nu,\alpha} = \{f_{\nu,\alpha}(z) = \gamma \mid \gamma \text{ be a complex const. and } |\gamma| = 1\}$$

is of finite order. Then, there exists an entire function $F^{(\gamma)}_{\nu,\alpha}(z)$ on $\mathbf{C}^n$, such that

$$\overset{(\gamma)}{V}{}^{n-1}_{\nu,\alpha} = \{F^{(\gamma)}_{\nu,\alpha}(z) = 0\} \quad \text{(by Theorem 1).}$$

Let us consider a fixed point $\tilde{z} \in \partial D - E_S$. For any neighbourhood $V(\tilde{z})$ of $\tilde{z}$, there exists a number ν_0, such that for any $\nu \geqq \nu_0$

$$\partial P_\nu \cap V(\tilde{z}) \neq \emptyset,$$

and for a point $z^*_\nu \in \partial P_\nu \cap V(\tilde{z})$, there exists at least a function $f_{\nu,\alpha}(z)$ defining P_ν, such that

$$f_{\nu,\alpha}(z^*_\nu) = \gamma^*_\nu, \quad |\gamma^*_\nu| = 1.$$

We can assume without loss of generality $\alpha = 1$, $\gamma^*_\nu = 1$ and consider an entire function $F^{(1)}_{\nu,1}(z)$ corresponding to the hypersurface $\{f_{\nu,1}(z) = 1\}$. Since $\log |F^{(1)}_{\nu,1}(z)| \geqq -\beta_\nu > -\infty$ ($\beta_\nu > 0$: const.) on a neighbourhood $U_\nu(\partial D)(\subset U(\bar{D}))$ of ∂D as shown in Lemma 7, Remark, $\log |F^{(1)}_{\nu,1}(z)|$ is continuous on $U_\nu(\partial D)$ and from the compactness of ∂D, there exist

$$\max_{z \in \partial D} \log |F^{(1)}_{\nu,1}(z)| = M_\nu, \quad \min_{z \in \partial D} \log |F^{(1)}_{\nu,1}(z)| = m_\nu$$

and

$$\tilde{m}_\nu = m_\nu - \frac{1}{\varepsilon_\nu}(M_\nu - m_\nu),$$

(in case of $M_\nu = m_\nu$, $\tilde{m}_\nu = m_\nu - 1 - 1/\varepsilon_\nu$).

The real hypersurface $W_{\nu,\alpha} = \{|f_{\nu,\alpha}(z)| = 1 \mid f_{\nu,\alpha}(z) \in \mathfrak{F}_D\}$ is expressed as a union of complex ones, such that

$$W_{\nu,\alpha} = \bigcup_\gamma \{f_{\nu,\alpha}(z) = \gamma \mid f_{\nu,\alpha}(z) \in \mathfrak{F}_D, \ \gamma \text{ be a const. and } |\gamma| = 1\}.$$

Therefore, for each α ($\alpha = 1, 2, \ldots, k_\nu$) and each γ ($|\gamma| = 1$), we can choose a number

$c_{\nu,\alpha}>0$, such that

$$\sup_{1\leq\alpha\leq k_\nu,|\gamma|=1}(\log|F^{(\gamma)}_{\nu,\alpha}(z)|-c_{\nu,\alpha})<\tilde{m}_\nu \quad \text{on } \bar{D},$$

where $F^{(\gamma)}_{\nu,\alpha}(z)$ is an entire function corresponding to the hypersurface $\{f_{\nu,\alpha}(z)=\gamma\}$. Because the family of entire functions $F^{(\gamma)}_{\nu,\alpha}(z)$ are uniformly upper bounded on compact set $\bar{D}\times\{|\gamma|=1\}$.[15] Then, the function

$$\varphi_\nu(z)=\sup\{\log|F^{(1)}_{\nu,1}(z)|, \sup_{1\leq\alpha\leq k_\nu,|\gamma|=1}(\log|F^{(\gamma)}_{\nu,\alpha}(z)|-c_{\nu,\alpha})\}$$

is evidently continuous plurisubharmonic on a neighbourhood $U(\bar{D})$ of D and satisfy the following relations:

$$\max_{z\in\partial D}\varphi_\nu(z)=\max_{z\in\bar{D}}\varphi_\nu(z)=M_\nu \quad \text{(by maximum principle)}$$

and

$$\min_{z\in\partial D}\varphi_\nu(z)=m_\nu.$$

Let us define a function

$$\psi_\nu(z)=\frac{\varepsilon_\nu\{\varphi_\nu(z)-M_\nu\}}{M_\nu-m_\nu},$$

(in case that $M_\nu=m_\nu$, $\psi_\nu(z)=\varepsilon_\nu\{\varphi_\nu(z)-M_\nu\}$). Then, $\psi_\nu(z)$ is continuous plurisubharmonic on $U(\bar{D})$ satisfying the following inequalities:

$$\psi_\nu(z)<0 \quad \text{on} \quad D,$$
$$-\varepsilon_\nu\leqq\psi_\nu(z)\leqq 0 \quad \text{on} \quad \partial D,$$

and especially on the point z^*_ν

$$\psi_\nu(z^*_\nu)=\frac{\varepsilon_\nu\{\sup_{1\leq\alpha\leq k_\nu,|\gamma|=1}(\log|F^{(\gamma)}_{\nu,\alpha}(z^*_\nu)|-c_{\nu,\alpha})-M_\nu\}}{M_\nu-m_\nu}<\frac{\varepsilon_\nu(\tilde{m}_\nu-M_\nu)}{M_\nu-m_\nu}<-1-\varepsilon_\nu,$$

(in case that $M_\nu=m_\nu$, $\psi_\nu(z^*_\nu)<\varepsilon_\nu(\tilde{m}_\nu-M_\nu)<-1-\varepsilon_\nu$).

Furthermore, let us construct a function $p_S(z)$ in the form of the upper envelope, such that

$$p_S(z)=\limsup_{\zeta\to z}q_S(\zeta), \quad q_S(\zeta)=\sup_{\nu\in\mathbf{N}}\psi_\nu(\zeta).$$

Obviously, $p_S(z)$ is continuous plurisubharmonic on $U(\bar{D})$, and $p_S(z)\leqq 0$ on D. But, we can not accept the equal sign, because of $\psi_\nu(z^*_\nu)<-1$, $z^*_\nu\in D$, $\nu\in\mathbf{N}$ (by maximum principle). Then, we have

$$p_S(z)<0 \quad \text{on} \quad D.$$

At last, we are to show that

$$\lim_{\nu\to+\infty}p_S(z_\nu)=0.$$

[15]) Cf. [2], pp. 376—379 Proposition 5 in which there exists a number $\delta>0$ such that $||a||>\delta>0$ and [2], pp. 394—397, Theorem 5.

From the construction of $p_S(z)$, it is shown that for any $z_0 \in E_S$, there exists a neighbourhood $U(z_0)$ which is independent of P_ν, such that

$$\tilde{z} \in U(z_0)$$

and $p_S(z)$ is continuous on $U(z_0)$. Therefore, for any $\varepsilon > 0$,

$$|p_S(z) - p_S(z_0)| < \frac{\varepsilon}{2} \quad \text{on} \quad U(z_0).$$

For this ε, we can choose a sufficiently laige integer k_0, such that

$$0 < \varepsilon_{\nu_k} < \frac{\varepsilon}{2} \quad \text{for} \quad k \geqq k_0.$$

Since $p_S(z_0) \geqq \sup_{k \in \mathbf{N}} \{\psi_{\nu_k}(z_0)\}$, $-\varepsilon_{\nu_k} \leqq \psi_{\nu_k}(z_0) \leqq 0$, it follows that

$$-\frac{\varepsilon}{2} < p_S(z_0) \leqq 0.$$

Hence, we can conclude that

$$|p_S(z_{\nu_k})| < \varepsilon$$

for $z_{\nu_k} \in S \cap U(z_0)$, $k \geqq k_0$. This shows in general that for $S = \{z_\nu\}$, $\nu \in \mathbf{N}$,

$$\lim_{\nu \to +\infty} p_S(z_\nu) = 0.$$

Thus, our proof of Main Theorem is completely finished.

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