On a differential equation arising in a Hele Shaw flow moving boundary problem

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1. Introduction

The present paper is mainly devoted to the following differential equation:

Given $f(\zeta)$, *analytic and univalent in a neighbourhood of* $|\zeta|\leqq 1$, *find* $f(\zeta, t)$, *analytic and univalent as a function of* ζ *in a neighbourhood of* $|\zeta|\leqq 1$, *continuously differentiable with respect to* t *for* $t\in\mathbf{R}$ *in an interval containing* $t=0$, *satisfying*

$$\mathrm{Re}[\dot{f}(\zeta, t)\cdot\overline{\zeta f'(\zeta, t)}] = 1 \quad \textit{for} \quad |\zeta| = 1 \tag{1}$$

and $f(\zeta, 0)=f(\zeta)$ (*for* $|\zeta|<1$). *In* (1) $\dot{f}$ *and* f' *denote derivatives with respect to* t *and* ζ *respectively.*

This differential equation arose in the paper [5] by S. Richardson as describing the solution of a two-dimensional moving boundary problem for so called Hele Shaw flows. The moving boundary in question then was the boundary of the domain $\Omega_t=f(\mathbf{D}, t)$, where $\mathbf{D}=\{\zeta\in\mathbf{C}\colon|\zeta|<1\}$ and t is time. Richardson did not prove existence or unicity for solutions of (1). However, this, essentially, was done in [10]. The existence of solutions was proved by using an iterative process, the proof of convergence of which was fairly complicated. Unicity was proved only with respect to solutions which depended analytically on t.

The aim of the present paper is primarily to give a more elementary proof of existence of solutions of (1) in the case that $f(\zeta)$ is a polynomial or a rational function. In that case (1) can be reduced to a finite system of ordinary differential equations (in t) and this system has a unique solution by standard theory. This solution is a polynomial or a rational function (as a function of ζ) of the same sort as $f(\zeta)$. (Theorem 4.)

We will also consider a generalization of the differential equation (1) in order to prove a result on the "moment map"

$$f \to (c_0, c_1, c_2, \dots), \tag{2}$$

where

$$c_n = \iint_\Omega z^n \, dx \, dy, \quad \Omega = f(\mathbf{D})$$

($\{c_n\}$ are the "analytic moments" of the domain $\Omega = f(\mathbf{D})$). We prove (Theorem 6) that when (2) is viewed as a mapping from the set of univalent polynomials of degree $\leqq r$, normalized by $f(0)=0$ and $f'(0)>0$, it is an immersion, i.e. its Fréchet derivative is one-to-one.

Treatments of the moving boundary problem lying behind (1) by other methods are found in [4], [5], [7] and [8]. For the hydrodynamical background and the derivation of (1), we refer to [5] and [3]. ([3] is an extended version of the present paper.)

2. Treatment of the differential equation

a) *Some notations used*

$$\mathbf{D}(a;\ r) = \{z \in \mathbf{C}:\ |z-a| < r\} \quad (\text{if } a \in \mathbf{C},\ r > 0),$$

$$\mathbf{D} = \mathbf{D}(0;\ 1),$$

$$\mathbf{P} = \mathbf{C} \cup \{\infty\} = \text{the Riemann sphere.}$$

If $\Omega \subset \mathbf{C}$ is an open set

$$H(\Omega) = \{\text{holomorphic functions on } \Omega\},$$

$$M(\Omega) = \{\text{meromorphic functions on } \Omega\}.$$

For an arbitrary set $E \subset \mathbf{C}$ let

$$H(E) = \{\text{functions, holomorphic in some open set containing } E\}.$$

In $H(E)$ two functions are identified if they agree on some neighbourhood of (i.e. open set containing) E.

$$\mathcal{O} = \{f \in H(\overline{\mathbf{D}}): f' \neq 0 \quad \text{on} \quad \overline{\mathbf{D}}\},$$

$$\mathcal{O}_0 = \{f \in \mathcal{O}: f(0) = 0\},$$

$$\mathcal{O}_1 = \{f \in \mathcal{O}: f(0) = 0 \quad \text{and} \quad f'(0) > 0\},$$

$$H(\overline{\mathbf{D}})_1 = \{f \in H(\overline{\mathbf{D}}): f(0) = 0 \quad \text{and} \quad \operatorname{Im} f'(0) = 0\}.$$

If f is a function meromorphic in an open set $U \subset \mathbf{P}$ we set

$\operatorname{Div}_U f$ = the divisor of f in U
= the formal sum of the zeroes of f (occurring with plus signs) and poles (with negative signs), both counted with multiplicities,

$\mathrm{P}\,\mathrm{div}_U f$ = the pole divisor of f in U
= the formal sum of the poles of f in U counted with multiplicities (and with plus signs),

$\mathrm{Z}\,\mathrm{div}_U f$ = the zero divisor of f in U
= the formal sum of the zeroes of f in U (counted with multiplicities).

Thus $\mathrm{Div}_U f = \mathrm{Z}\,\mathrm{div}_U f - \mathrm{P}\,\mathrm{div}_U f$.

When $U=\mathbf{P}$ we just write Div in place of $\mathrm{Div}_{\mathbf{P}}$ etc.

The set (or abelian group) of divisors is partially ordered in a natural way, namely so that a divisor $\sum_{j=1}^{r} n_j \cdot (\zeta_j)$ (n_j integers, $\zeta_j \in \mathbf{P}$) is non-negative, $\sum_{j=1}^{r} n_j \cdot (\zeta_j) \geqq 0$, if and only if $n_j \geqq 0$ for all j, assuming here that all the ζ_j are distinct. Then $D_1 \geqq D_2$ (D_1, D_2 divisors) means that $D_1 - D_2 \geqq 0$. With respect to this partial order the concepts max ($=$sup) and min ($=$inf) make sense and will be used.

$\mathscr{R}_n$ and $\mathscr{P}_r$ are defined in Section 2. c.

$*$ will denote the reflection map in $\partial\mathbf{D}$ and various associated maps, namely,

$$\zeta^* = 1/\bar{\zeta} \quad \text{for points} \quad \zeta \in \mathbf{P},$$

$$\Big(\sum_j n_j \cdot (\zeta_j)\Big)^* = \sum_j n_j \cdot (\zeta_j^*) \quad \text{for divisors } (n_j \text{ integers}),$$

$$E^* = \{\zeta^* \in \mathbf{P} : \zeta \in E\} \quad \text{for sets} \quad E \subset \mathbf{P},$$

$$F^*(\zeta) = \overline{F(\zeta^*)} \quad \text{for meromorphic functions } F.$$

b) *Reformulation of the equation*

In terms of the notations introduced above, we now reformulate our problem as follows.

Given $f_0 \in \mathcal{O}_1$, find an $\varepsilon > 0$ and a map

$$(-\varepsilon, \varepsilon) \ni t \to f_t \in \mathcal{O}_1$$

such that the function $f(\zeta, t) = f_t(\zeta)$ is continuously differentiable in a neighbourhood of $\overline{\mathbf{D}} \times (-\varepsilon, \varepsilon)$ and such that

$$\mathrm{Re}\,[\dot{f}(\zeta, t) \cdot \overline{\zeta f'(\zeta, t)}] = 1 \tag{3}$$

holds for $\zeta \in \partial\mathbf{D}$, $t \in (-\varepsilon, \varepsilon)$.

The requirement $f_t \in \mathcal{O}_1$ means that the mapping function f_t shall be normalized ($f_t(0)=0, f_t'(0)>0$), analytically extendible across $\partial\mathbf{D}$ and locally univalent on $\overline{\mathbf{D}}$ ($f_t' \neq 0$ on $\overline{\mathbf{D}}$). Since f_t originally appeared as a mapping function it is natural to require it to be univalent on $\mathbf{D}$ (or $\overline{\mathbf{D}}$). However, in the mathematical treatment of (3) it makes no difference whether f_t is univalent on $\overline{\mathbf{D}}$ or merely locally univalent and the latter condition being simpler to work with, we have preferred to use that one. Observe also that if f_0 actually is univalent on $\overline{\mathbf{D}}$ and $t \to f_t$ solves (3), then if $\varepsilon > 0$ is chosen small enough also all f_t are univalent on $\overline{\mathbf{D}}$.

The goal of Section 2 is to show first that (3) can be written in the form $\dot{f} = \mathscr{F}(f)$ for a suitable operator $\mathscr{F}$ (Proposition 1) and then that $\mathscr{F}$ can be viewed as a smooth vector field on certain spaces of rational functions (these spaces regarded as differential manifolds). The existence of a solution of (3) when f_0 is rational then follows immediately (Theorem 4).

Proposition 1. *Given* $f \in \mathcal{O}_1$ *the equation*

$$\operatorname{Re}[\overline{\zeta f'(\zeta)} \cdot g(\zeta)] = 1 \quad \textit{for} \quad \zeta \in \partial\mathbf{D} \tag{4}$$

has a unique solution g *in* $H(\overline{\mathbf{D}})_1$. *This solution is given by* $g = \mathscr{F}(f)$ *where* $\mathscr{F}\colon \mathcal{O}_1 \to H(\overline{\mathbf{D}})_1$ *is the operator defined by*

$$\mathscr{F}(f)(\zeta) = \zeta f'(\zeta) \cdot \frac{1}{2\pi i} \int_{\partial\mathbf{D}} |f'(z)|^{-2} \frac{z+\zeta}{z-\zeta} \frac{dz}{z}. \tag{5}$$

Proof. We transform the problem into a statement about two other analytic functions, F and G, related to f and g as follows.

$$\begin{cases} F(\zeta) = f'(\zeta) & (6) \\ G(\zeta) = \dfrac{g(\zeta)}{\zeta} & (7) \end{cases}$$

and, conversely,

$$\begin{cases} f(\zeta) = \int_0^\zeta F(z)\, dz & (8) \\ g(\zeta) = \zeta G(\zeta). & (9) \end{cases}$$

Then the first statement in the proposition transforms into:

Given $F \in H(\overline{\mathbf{D}})$, *non-vanishing on* $\overline{\mathbf{D}}$ *and with* $F(0) > 0$, *the equation*

$$\operatorname{Re}[\overline{F} \cdot G] = 1 \quad \textit{on} \quad \partial\mathbf{D} \tag{10}$$

has a unique solution G *in* $H(\overline{\mathbf{D}})$ *satisfying* $\operatorname{Im} G(0) = 0$.

On dividing by $|F|^2$ in (10) we get another equivalent formulation:

Given $F \in H(\overline{\mathbf{D}})$, *non-vanishing on* $\overline{\mathbf{D}}$ *and with* $F(0) > 0$, *the problem*

$$\operatorname{Re}[G/F] = |F|^{-2} \quad \textit{on} \quad \partial\mathbf{D} \tag{11}$$

$$G/F \in H(\overline{\mathbf{D}}), \quad \operatorname{Im} G/F(0) = 0 \tag{12}$$

has a unique solution for G *(or for* G/F*).*

Now in this last formulation the statement is directly seen to be true. Namely, the solution for G/F of (11), (12) is

$$G/F = \mathscr{P}[|F|^{-2}]$$

where $\mathscr{P}$ stands for "the Poisson integral of". Explicitly

$$\frac{G(\zeta)}{F(\zeta)} = \frac{1}{2\pi}\int_0^{2\pi} |F(e^{i\theta})|^{-2}\cdot\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}\,d\theta = \frac{1}{2\pi i}\int_{\partial\mathbf{D}} |F(z)|^{-2}\cdot\frac{z+\zeta}{z-\zeta}\frac{dz}{z}. \tag{13}$$

Actually (13) only gives G/F as an analytic function in $\mathbf{D}$, but it is easy to see that G/F extends analytically across $\partial\mathbf{D}$ as required in (12). In fact, the function $|F(z)|^{-2}$ in the last integral of (13) can be replaced by $\left(F(z)F^*(z)\right)^{-1}$, which is holomorphic in a neighbourhood of $\partial\mathbf{D}$, and then the path of integration for that integral can be moved slightly out from $\partial\mathbf{D}$ showing that the last member of (13) is an analytic function of ζ in some neighbourhood of $\overline{\mathbf{D}}$.

Thus the statement (11)—(12) is true and so also the statement (4) of the proposition is true. The expression (5) for the (hence well-defined) operator $\mathscr{F}$ follows from (6), (7) and (13). This proves the proposition.

Theorem 2. *Let* $f\in\mathcal{O}_1$.

(i) *Let* U *be any open connected set containing* $\overline{\mathbf{D}}$. *Then if* $f\in H(U)$ $\mathscr{F}(f)\in H(U)$. *Thus* $\mathscr{F}$ *is well-defined as an operator*

$$\mathscr{F}\colon \mathcal{O}_1\cap H(U)\to H(U).$$

Moreover, with $H(U)$ *provided with the topology of uniform convergence on compact sets, this operator is continuous.*

(ii) *If* f *is a polynomial of degree* $\leqq r$ *then so is* $\mathscr{F}(f)$.

(iii) *If* f *is a rational function with*

$$\operatorname{Pdiv} f \leqq \sum_{j=1}^{r} n_j\cdot(\zeta_j)+n_0\cdot(\infty) \tag{14}$$

where $\zeta_j\in\mathbf{C}\setminus\overline{\mathbf{D}}$, $n_j\geqq 0$ *and* $n_0\geqq 1$ *then* $\mathscr{F}(f)$ *is a rational function with*

$$\operatorname{Pdiv}\mathscr{F}(f)\leqq \sum_{j=1}^{r}(n_j+1)\cdot(\zeta_j)+n_0\cdot(\infty). \tag{15}$$

Proof. (ii) is a special case of (iii) so only (i) and (iii) need to be proven. Returning to the functions F and G used in the proof of Proposition 1, the relation

$$g=\mathscr{F}(f)\quad (\text{for}\quad f\in\mathcal{O}_1,\ g\in H(\overline{\mathbf{D}})_1)$$

is equivalent to

$$\operatorname{Re}[\overline{F}\cdot G]=1\quad\text{on}\quad \partial\mathbf{D}$$

for F and $G\in H(\overline{\mathbf{D}})$ satisfying $F(0)>0$, $\operatorname{Im} G(0)=0$, F non-vanishing on $\overline{\mathbf{D}}$ and related to f and g by (6) to (9). Apart from the continuity statement in (i), (i) and (iii) now follow from the following lemma.

Lemma 3. *Suppose* $F, G \in H(\overline{\mathbf{D}})$ *satisfy*

(16) $$\operatorname{Re}[\overline{F} \cdot G] = 1 \quad on \quad \partial\mathbf{D}$$

and that F *has no zero on* $\overline{\mathbf{D}}$. *Then*

(i) *if* $F \in H(U)$ *then* $G \in H(U)$ *(for* $U \supset \overline{\mathbf{D}}$ *open and connected)*,
(ii) *if* F *is a rational function then so is* G, *and*

(17) $$\mathrm{P} \operatorname{div} G \leqq \mathrm{P} \operatorname{div} F.$$

How the theorem (except for the continuity statement) follows from the lemma:

(i) If $f \in H(U)$ then $F \in H(U)$ and, by (i) of the lemma, $G \in H(U)$. Hence $g \in H(U)$ by (9).

(iii) Suppose f is rational with

$$\mathrm{P} \operatorname{div} f \leqq \textstyle\sum_{j=1}^{r} n_j \cdot (\zeta_j) + n_0 \cdot (\infty)$$

(where $n_0 \geqq 1$). Then $F = f'$ is rational with

$$\mathrm{P} \operatorname{div} F \leqq \textstyle\sum_{j=1}^{r} (n_j+1) \cdot (\zeta_j) + (n_0-1) \cdot (\infty)$$

and it follows from (9) and from (ii) of the lemma that

$$\mathrm{P} \operatorname{div} g \leqq \mathrm{P} \operatorname{div} G + 1 \cdot (\infty) \leqq \textstyle\sum_{j=1}^{r} (n_j+1) \cdot (\zeta_j) + n_0 \cdot (\infty)$$

as claimed.

It remains to prove the lemma and the continuity assertion in the theorem.

Proof of the lemma. Relation (16) can be written

(18) $$\operatorname{Re}[F^* \cdot G - 1] = 0 \quad \text{on} \quad \partial\mathbf{D}.$$

This shows that the function

(19) $$H = F^* \cdot G - 1$$

which is holomorphic in a neighbourhood of $\partial\mathbf{D}$ extends by reflection to be analytic in a domain which is symmetric with respect to $\partial\mathbf{D}$. In fact, (18) shows that

(20) $$H = -H^* \quad \text{on } \partial\mathbf{D}, \text{ and hence identically,}$$

so that if H is *a priori* analytic in (say) V (20) defines an analytic extension of it to $V \cup V^*$.

To prove (i) of the lemma we observe that *a priori* the function H defined by (19) will be holomorphic in $U^* \cap \overline{\mathbf{D}}$ $(F \in H(U), G \in H(\overline{\mathbf{D}}))$. Thus it extends analytically to $(U^* \cap \overline{\mathbf{D}}) \cup (U \cap \overline{\mathbf{D}}^*)$ $(= U \cap U^*$, in view of $U \supset \overline{\mathbf{D}})$, in particular to $U \cap \overline{\mathbf{D}}^*$. Since F^* is holomorphic and has no zeroes in $U \cap \overline{\mathbf{D}}^*$ it follows from (19) that G is holomorphic there. Thus G is holomorphic in $U = (U \cap \overline{\mathbf{D}}^*) \cup \overline{\mathbf{D}}$, and (i) is proven.

To prove (ii) of the lemma observe first that for F a rational function, H defined by (19) is meromorphic in $\overline{\mathbf{D}}$, hence by reflection is meromorphic on all $\mathbf{P}$. This means that H is a rational function. Hence also G is rational (by (19)).

Now (17) follows from the following computation in which the first inequality depends on $Z\operatorname{div}_{\mathbf{D}^*} F^*=0$, the second one on $\mathrm{P}\operatorname{div}_{\mathbf{D}} G=0$ and where also the symmetry (20) of $H=F\cdot G^*-1$ is used.

$$\mathrm{P}\operatorname{div} G=\mathrm{P}\operatorname{div}_{\mathbf{D}^*} G \leqq \mathrm{P}\operatorname{div}_{\mathbf{D}^*}(F^*\cdot G-1)=[\mathrm{P}\operatorname{div}_{\mathbf{D}}(F^*\cdot G-1)]^*$$
$$\leqq (\mathrm{P}\operatorname{div}_{\mathbf{D}} F^*)^*=\mathrm{P}\operatorname{div}_{\mathbf{D}^*} F=\mathrm{P}\operatorname{div} F.$$

This proves Lemma 3.

It remains to prove the continuity of $\mathscr{F}\colon \mathcal{O}_1\cap H(U)\to H(U)$ for $U\supset\overline{\mathbf{D}}$ open and connected. So suppose $f_n\to f$ uniformly on compact subsets of $U\,(f_n, f\in\mathcal{O}_1\cap H(U))$ and we shall prove that $\mathscr{F}(f_n)\to\mathscr{F}(f)$ in the same topology. It is clear (by the maximum principle) that it is enough to prove that $\mathscr{F}(f_n)\to\mathscr{F}(f)$ uniformly on every compact subset of U which does not contain any zero of f' (in $U\setminus\overline{\mathbf{D}}$).

Let K be such a compact subset. Then we can choose an open connected set V with nice boundary, such that $K\cup\overline{\mathbf{D}}\subset V\subset\overline{V}\subset U$, and such that also $\overline{V}$ avoids all zeroes of f'. Since the function $(f'(z)f'^*(z))^{-1}$ then is holomorphic in a neighbourhood of $\overline{V}\setminus\mathbf{D}$ and equals $|f'(z)|^{-2}$ on $\partial\mathbf{D}$ we have, for $\zeta\in\mathbf{D}$

$$\mathscr{F}(f)(\zeta)=\zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial\mathbf{D}}|f'(z)|^{-2}\cdot\frac{z+\zeta}{z-\zeta}\frac{dz}{z} \tag{21}$$
$$=\zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial\mathbf{D}}\frac{1}{f'(z)f'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\frac{dz}{z}$$
$$=\zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial V}\frac{1}{f'(z)f'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\frac{dz}{z}.$$

Both the first and the last members of this equation are functions (in ζ) holomorphic in V. Thus the equality between these is valid for all $\zeta\in V$.

Formula (21), with equality holding between the extreme members for all $\zeta\in V$, also is valid with f_n in place of f whenever n is large enough. For $f_n\to f$ on compacts implies that f_n' has no zeroes on $\overline{V}$ for n large (since f' has none), and so all that has been said about f above also applies to f_n (for large n).

Thus

$$\mathscr{F}(f_n)(\zeta)=\zeta f_n'(\zeta)\frac{1}{2\pi i}\int_{\partial V}\frac{1}{f_n'(z)f_n'^*(z)}\cdot\frac{z+\zeta}{z-\zeta}\cdot\frac{dz}{z} \tag{22}$$

for $\zeta\in V$, n large.

Now $f_n \to f$ uniformly on compacts implies $f_n' \to f'$ uniformly on K and on ∂V, and $f_n'^* \to f'^*$ uniformly on ∂V. Therefore, since $\left|\frac{z+\zeta}{z-\zeta}\right|$ is bounded above for $z \in \partial V$, $\zeta \in K$ and $f_n'(z) f_n'^*(z)$ is bounded away from zero for $z \in \partial V$ and n large, (22) and (21) show that $\mathscr{F}(f_n)(\zeta) \to \mathscr{F}(f)(\zeta)$ uniformly for $\zeta \in K$ as $n \to \infty$.

This proves the continuity of $\mathscr{F}$ and finishes the proof of Theorem 2.

c) Existence and uniqueness of rational solutions to the equation

We now apply Theorem 2 to the differential equation $\dot{f} = \mathscr{F}(f)$. Given integers $n_0, n_1, \ldots, n_r \geqq 1$ let

$$n = (n_0, n_1, \ldots, n_r) \in \mathbf{Z}^{r+1},$$

$$|n| = n_0 + n_1 + \ldots + n_r,$$

$\mathscr{R}_n =$ {rational functions f which have r distinct poles $\zeta_1, \ldots, \zeta_r$ (depending on f) in $\mathbf{C}$ of orders exactly $n_1, \ldots, n_r$ respectively, a pole of order at most n_0 at ∞ and no other poles}.

Thus $f \in \mathscr{R}_n$ means that there exist $\zeta_j = \zeta_j(f) \in \mathbf{C}$ $(j=1, \ldots, r)$, $a_{jk} = a_{jk}(f) \in \mathbf{C}$ $(k=1, \ldots, n_j,\ j=1, \ldots, r)$ and $a_k = a_k(f) \in \mathbf{C}$ $(k=0, \ldots, n_0)$ with $\zeta_i \neq \zeta_j$ for $i \neq j$ and with $a_{jn_j} \neq 0$ $(j=1, \ldots, r)$ such that

$$f(\zeta) = \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{a_{jk}}{(\zeta-\zeta_j)^k} + \sum_{k=0}^{n_0} a_k \zeta^k. \tag{23}$$

For $r=0$, $n=(n_0)$ $\mathscr{R}_n$ reduces to

$$\mathscr{R}_n = \mathscr{P}_{n_0} = \{\text{polynomials of degree} \leqq n_0\}.$$

With (ζ_j, a_{jk}, a_k) in (23) as local coordinates $\mathscr{R}_n$ is given the structure of a complex differentiable manifold of dimension $r + \sum_1^r n_j + n_0 + 1 = |n+1|$. We shall regard $\mathscr{R}_n$ as a real manifold, hence of dimension $2|n+1|$. Then $\mathscr{R}_n \cap \mathscr{O}_1$ and $\mathscr{R}_n \cap H(\overline{\mathbf{D}})_1$ are submanifolds of $\mathscr{R}_n$ of dimension $2|n+1|-3$.

Now we may consider the operator $\mathscr{F}\colon \mathscr{O}_1 \to H(\overline{\mathbf{D}})_1$ as a vector field on $\mathscr{O}_1$ (the tangent space of $\mathscr{O}_1$ at any point $f \in \mathscr{O}_1$ may be identified with $H(\overline{\mathbf{D}})_1$ in a natural way) and the content of part (iii) of Theorem 2 then is that the restriction of this vector field to the submanifold $\mathscr{O}_1 \cap \mathscr{R}_n$ is tangent to $\mathscr{O}_1 \cap \mathscr{R}_n$. (We shall motivate this in a moment.) Thus $\mathscr{F}|_{\mathscr{O}_1 \cap \mathscr{R}_n}$ may be considered as a vector field on $\mathscr{O}_1 \cap \mathscr{R}_n$.

Moreover, this vector field is smooth as can easily be seen from, say, (5). Now a smooth vector field on a finite dimensional manifold always admits a unique integral curve through any point on the manifold and so it follows that given $f_0 \in \mathscr{O}_1 \cap \mathscr{R}_n$ there is a unique smooth map $t \to f_t \in \mathscr{O}_1 \cap \mathscr{R}_n$ defined in some interval around $t=0$ and satisfying $\dot{f}_t=\mathcal{F}(f_t)$. This is roughly the proof of Theorem 4 below, asserting existence and uniqueness of rational solutions of $\dot{f}_t=\mathcal{F}(f_t)$, given f_0 rational.

In order to work out the details of the above discussion consider an arbitrary differentiable curve

$$t \to f_t \tag{24}$$

in $\mathcal{R}_n$. With

$$f_t(\zeta) = \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{a_{jk}(t)}{(\zeta-\zeta_j(t))^k} + \sum_{k=0}^{n_0} a_k(t)\zeta^k \tag{25}$$

the tangent vector of this curve at the point f_t becomes

(26)

$$\begin{aligned}\dot{f}_t(\zeta) &= \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{\dot{a}_{jk}(t)}{(\zeta-\zeta_j(t))^k} + \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{k a_{jk}(t)\dot{\zeta}_j(t)}{(\zeta-\zeta_j(t))^{k+1}} + \sum_{k=0}^{n_0} \dot{a}_k(t)\zeta^k \\ &= \sum_{j=1}^{r} \frac{n_j a_{jn_j}(t)\dot{\zeta}_j(t)}{(\zeta-\zeta_j(t))^{n_j+1}} + \sum_{j=1}^{r} \sum_{k=1}^{n_j} \frac{(k-1)a_{jk-1}(t)\dot{\zeta}_j(t)+\dot{a}_{jk}(t)}{(\zeta-\zeta_j(t))^k} \\ &\quad + \sum_{k=0}^{n_0} \dot{a}_k(t)\zeta^k.\end{aligned}$$

Now for any fixed $f\in\mathcal{R}_n$ consider all curves with $f_0=f$. As (24) varies over all such curves the derivatives $\dot{\zeta}_j(0)$, $\dot{a}_{jk}(0)$, $\dot{a}_k(0)$ range over all $\mathbf{C}^r\times\mathbf{C}^{\sum_1^r n_j}\times\mathbf{C}^{n_0+1}=\mathbf{C}^{|n+1|}$ and it follows from the last member of (26) (observing that $n_j a_{jn_j}(t)\neq 0$ there) that the tangent vector $\dot{f}_0$ then ranges over all

$$T_f(\mathcal{R}_n) = \{\text{rational functions } g \text{ with } \operatorname{P div} g \leqq \textstyle\sum_{j=1}^{r} (n_j+1)\cdot(\zeta_j)+n_0\cdot(\infty)\}.$$

($\zeta_j=\zeta_j(0)$ are the poles of $f=f_0$). This means that the linear space $T_f(\mathcal{R}_n)$ is the tangent space of $\mathcal{R}_n$ at $f\in\mathcal{R}_n$ (whence the notation for it).

Considering $\mathcal{R}_n\cap\mathcal{O}_1$ instead of $\mathcal{R}_n$ it is easy to see that the tangent space of $\mathcal{R}_n\cap\mathcal{O}_1$ at $f\in\mathcal{R}_n\cap\mathcal{O}_1$ is

$$\begin{aligned}T_f(\mathcal{R}_n\cap\mathcal{O}_1) &= \{\text{rational functions } g \text{ with } \operatorname{P div} g \leqq \textstyle\sum_{j=1}^{r}(n_j+1)\cdot(\zeta_j)+n_0\cdot(\infty) \\ &\qquad \text{and with } g(0)=0,\ \operatorname{Im} g(0)=0\} \\ &= T_f(\mathcal{R}_n)\cap H(\overline{\mathbf{D}})_1 \ (\zeta_1, \dots, \zeta_r, \infty \text{ are the poles of } f).\end{aligned}$$

In terms of the above notation, part (iii) of Theorem 2 (together with the fact that $\mathcal{F}(f)\in H(\overline{\mathbf{D}})_1$) say that

$$\mathcal{F}(f)\in T_f(\mathcal{R}_n\cap\mathcal{O}_1) \quad \text{for} \quad f\in\mathcal{R}_n\cap\mathcal{O}_1.$$

Thus $\mathscr{F}(f)$ is tangent to $\mathscr{R}_n \cap \mathcal{O}_1$ for $f \in \mathscr{R}_n \cap \mathcal{O}_1$, i.e. $\mathscr{F}|_{\mathscr{R}_n \cap \mathcal{O}_1}$ is a smooth vector field on $\mathscr{R}_n \cap \mathcal{O}_1$, and so the problem

$$\begin{cases} \dot{f}_t = \mathscr{F}(f_t) \\ f_0 \in \mathscr{R}_n \cap \mathcal{O}_1 \quad \text{given,} \end{cases}$$

has a unique smooth solution

$$t \to f_t \in \mathscr{R}_n \cap \mathcal{O}_1$$

defined in a neighbourhood of $t=0$.

We have now proved

Theorem 4. *Given any rational function f which is holomorphic and locally univalent on $\overline{\mathbf{D}}$ and satisfies $f(0)=0, f'(0)>0$, choose $n=(n_0, n_1, \ldots, n_r)$ with $n_j \geqq 1$ so that $f \in \mathscr{R}_n$ (*i.e.* $n_1, \ldots, n_r$ shall be some enumeration of the exact orders of the finite poles of f and n_0 shall be greater or equal to the order of the pole of f at infinity). Then the problem*

$$\begin{cases} \dot{f}_t = \mathscr{F}(f_t) \\ f_0 = f \end{cases}$$

or, equivalently, the problem (3) *has a unique solution*

$$t \to f_t \in \mathscr{R}_n \cap \mathcal{O}_1$$

defined in a neighbourhood of $t=0$.

d) *The moment property of solutions*

The next theorem shows the existence of an infinite number of simple constants of motion for a solution $t \to f_t$ of our differential equation, namely the analytic moments

$$c_n = \iint_{\Omega_t} z^n \, dx \, dy = \iint_{\mathbf{D}} f(\zeta)^n |f'(\zeta)|^2 \, d\xi \, d\eta \quad (\zeta = \xi + i\eta)$$

for $n=1, 2, \ldots$. Here $\Omega_t = f_t(\mathbf{D})$, which is regarded as a Riemann surface over $\mathbf{C}$ if f_t is not (globally) univalent on $\mathbf{D}$. The zeroth order moment

$$c_0 = \iint_{\Omega_t} dx \, dy = |\Omega_t|$$

will increase linearly with t. This moment property of solutions of (3) was discovered by Richardson ([5]).

Since the c_n are linear in z^n we also obtain constants of motion by taking linear combinations of the z^n. Thus define, for arbitrary polynomials $P(z)$ and for $f \in \mathcal{O}_1$

$$I_P(f) = \iint_{\Omega} P(z) \, dx \, dy = \iint_{\mathbf{D}} P(f(\zeta)) |f'(\zeta)|^2 \, d\xi \, d\eta \tag{27}$$

where $\Omega = f(\mathbf{D})$ (as a Riemann surface). Then we have

Theorem 5. *Suppose* $(-\varepsilon, \varepsilon)\ni t\to f_t\in\mathcal{O}_1$ *solves* (3) (*equivalently* $\dot{f}_t=\mathcal{F}(f_t)$). *Then*

$$\frac{d}{dt} I_P(f_t) = 2\pi P(0)$$

for each polynomial $P(z)$.

Proof. Let $Q'(z)=P(z)$. Then we have, using the formalism of differential forms

$$\begin{aligned}
\frac{d}{dt} 2iI_P(f_t) &= \frac{d}{dt} 2i\iint_{\Omega_t} P(z)\,dx\,dy = \frac{d}{dt}\iint_{\Omega_t} Q'(z)\,d\bar{z}\,dz \\
&= \frac{d}{dt}\iint_{\Omega_t} d\bar{z}\,dQ(z) = \frac{d}{dt}\iint_{\mathbf{D}} \overline{df_t(\zeta)}\,dQ(f_t(\zeta)) \\
&= \iint_{\mathbf{D}} \overline{d\dot{f}_t(\zeta)}\,dQ(f_t(\zeta)) + \iint_{\mathbf{D}} \overline{df_t(\zeta)}\,d\left(\frac{d}{dt}Q(f_t(\zeta))\right) \\
&= \int_{\partial\mathbf{D}} \overline{\dot{f}_t(\zeta)}\,dQ(f_t(\zeta)) - \int_{\partial\mathbf{D}} \frac{d}{dt}\left(Q(f_t(\zeta))\right)\overline{df_t(\zeta)} \\
&= \int_{\partial\mathbf{D}} Q'(f_t(\zeta))\left(\overline{\dot{f}_t(\zeta)}f_t'(\zeta)\,d\zeta - \dot{f}_t(\zeta)\overline{f_t'(\zeta)\,d\zeta}\right) \\
&= 2i\int_{\partial\mathbf{D}} P(f_t(\zeta))\,\mathrm{Im}\,[\overline{\dot{f}_t(\zeta)}f_t'(\zeta)\,d\zeta] \\
&= 2i\int_{\partial\mathbf{D}} P(f_t(\zeta))\,\mathrm{Re}\,[\overline{\dot{f}_t(\zeta)}\,\zeta f_t'(\zeta)]\,\frac{d\zeta}{i\zeta} \\
&= 2i\int_0^{2\pi} P(f_t(e^{i\theta}))\,d\theta = 2i\cdot 2\pi P(f_t(0)) = 2i\cdot 2\pi P(0).
\end{aligned} \tag{28}$$

This proves Theorem 5.

3. Generalizations and applications

a) *The generalized differential equation*

Now we are going to consider a generalization of equation (3), obtain similar results for this equation as for (3), and apply these results to prove that a certain mapping is non-singular (Theorem 6). The generalized equation is

$$\mathrm{Re}\,[\dot{f}(\zeta)\cdot\overline{\zeta f'(\zeta)}] = \begin{cases}\cos n\theta\\ \sin n\theta\end{cases} \tag{29}$$

where $\zeta=e^{i\theta}\in\partial\mathbf{D}$ and $n\in\mathbf{Z}$. For the case with $\cos n\theta$ in the right member and $n=0$ (29) is the old equation (3). Each $n>0$ in $\mathbf{Z}$ and each choice of $\cos n\theta$ and $\sin n\theta$ gives a new equation. For negative $n\in\mathbf{Z}$ the same set of equations appears again (essentially).

Let us therefore make the following *convention* in order to be able to speak of the equations (29) conveniently:

Choose the right member of (1) *to be*

$$\cos n\theta \quad \textit{for} \quad n = 0, 1, 2, \ldots,$$
$$\sin n\theta \quad \textit{for} \quad n = -1, -2, \ldots .$$

Thus there is precisely one equation (29) for each $n \in \mathbf{Z}$ and, aside from the zero function and up to a sign, all possible choices of right members in (29) are covered.

(29) can be rewritten as

$$\text{(30)} \qquad \operatorname{Im}[\overline{\dot{f}(\zeta)} f'(\zeta)\, d\zeta] = \operatorname{Re}[\dot{f}(\zeta)\overline{\zeta f'(\zeta)}]\, d\theta = \begin{cases} \cos n\theta\, d\theta & (n \geqq 0) \\ \sin n\theta\, d\theta & (n < 0) \end{cases}$$
$$= \begin{cases} \dfrac{1}{2}(\zeta^n + \zeta^{-n}) \dfrac{d\zeta}{i\zeta} & (n \geqq 0) \\[2ex] \dfrac{1}{2i}(\zeta^n - \zeta^{-n}) \dfrac{d\zeta}{i\zeta} & (n < 0) \end{cases}$$

for $\zeta = e^{i\theta} \in \partial\mathbf{D}$.

Using (28) this shows that if $t \to f = f_t \in \mathcal{O}_1$ is a solution of (29) $n \geqq 0$ and $P(z)$ is a polynomial, then

$$\frac{d}{dt} I_P(f) = \int_{\partial\mathbf{D}} P(f(\zeta)) \operatorname{Im}[\overline{\dot{f}(\zeta)} f'(\zeta)\, d\zeta] = \frac{1}{2i} \int_{\partial\mathbf{D}} P(f(\zeta))(\zeta^n + \zeta^{-n}) \frac{d\zeta}{\zeta}$$
$$= \begin{cases} 2\pi P(f(0)) & (n = 0) \\ \pi \operatorname*{Res}_{\zeta=0} \dfrac{P(f(\zeta))}{\zeta^{n+1}} & (n > 0) \end{cases}$$
$$= \begin{cases} 2\pi P(0) \\ \pi[A_1(f) P'(0) + \ldots + A_n(f) P^{(n)}(0)] & (n > 0) \end{cases}$$

where $A_j(f)\, j = 1, \ldots, n$ are complex numbers that depend on f and, in particular,

$$A_n(f) = \frac{1}{n!} f'(0)^n \neq 0.$$

Similarly, for $n < 0$

$$\frac{d}{dt} I_P(f) = i\pi \operatorname*{Res}_{\zeta=0} P(f(\zeta)) \cdot \zeta^{n-1} = i\pi[A_1(f) P'(0) + \ldots + A_{-n}(f) P^{(-n)}(0)].$$

Just as for (3), equation (29) can be solved for $\dot{f}$ (uniquely with the requirement $\dot{f} \in H(\overline{\mathbf{D}})_1$) whenever $f \in \mathcal{O}_1$. Namely,

$$\text{(31)} \qquad \operatorname{Re}[\dot{f}(\zeta) \cdot \overline{\zeta f'(\zeta)}] = \begin{cases} \cos n\theta & n = 0, 1, 2, \ldots \\ \sin n\theta & n = -1, -1, \ldots \end{cases}$$

for $\zeta=e^{i\theta}\in\partial\mathbf{D}$ is equivalent to

$$f=\mathscr{F}_n(f)$$

where $\mathscr{F}_n\colon \mathcal{O}_1\to H(\overline{\mathbf{D}})_1$ are the operators defined by

$$\mathscr{F}_n(f)(\zeta)=\zeta f'(\zeta)\cdot\frac{1}{2\pi i}\int_{\partial\mathbf{D}}|f'(z)|^{-2}\cdot\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\cdot\frac{z+\zeta}{z-\zeta}\frac{dz}{z} \tag{32}$$

$$=\zeta f'(\zeta)\mathscr{P}\left[|f'(e^{i\theta})|^{-2}\cdot\begin{Bmatrix}\cos n\theta\\ \sin n\theta\end{Bmatrix}\right](\zeta)$$

($\cos n\theta$ for $n\geqq 0$, $\sin n\theta$ for $n<0$).

Thus $\mathscr{F}_0=\mathscr{F}$.

Also Theorem 2, Lemma 3 and Theorem 4 have their generalizations to the equation (29) for arbitrary $n\in\mathbf{Z}$. Theorem 2 generalizes to

Theorem 2′. *Theorem 2 holds true with any $\mathscr{F}_n(n\in\mathbf{Z})$ in place of $\mathscr{F}$ under the following assumptions:*

For (ii) $r>|n|$ $(|n|=\pm n\geqq 0)$, *and for* (iii) $n_0>|n|$.

Proof. The proof of Theorem 2′ is essentially the same as that of Theorem 2 with the role of Lemma 3 now played by the more general Lemma 3′ below.

Lemma 3′. *Suppose $F, G\in H(\overline{\mathbf{D}})$ satisfy*

$$\mathrm{Re}\,[\bar{F}\cdot G]=H \quad \text{on} \quad \partial\mathbf{D} \tag{33}$$

where H is rational function which is real on $\partial\mathbf{D}$, and suppose that F has no zero on $\overline{\mathbf{D}}$. Then

(i) *if $U\supset\overline{\mathbf{D}}$ is open and connected and $U\setminus\mathbf{D}$ does not contain any pole of H, $F\in H(U)$ implies $G\in H(U)$*

(ii) *if F is rational then also G is rational, and*

$$\mathrm{P\,div}_{\mathbf{D}^*}\,G\leqq\max\{\mathrm{P\,div}_{\mathbf{D}^*}\,F,\ \mathrm{P\,div}_{\mathbf{D}^*}\,H\}. \tag{34}$$

How the theorem (except for the continuity statement in (i)) follows from the lemma:

We only treat the case $n\geqq 0$, the case $n<0$ being similar. With

$$F(\zeta)=f'(\zeta)$$

$$G(\zeta)=\frac{\dot{f}(\zeta)}{\zeta}=\frac{\mathscr{F}_n(f)(\zeta)}{\zeta}$$

$$H(\zeta)=\tfrac{1}{2}(\zeta^n+\zeta^{-n})$$

(31) reads

$$\operatorname{Re}[\bar{F}\cdot G] = H \quad \text{on} \quad \partial\mathbf{D}.$$

Now (i) of Theorem 2′ is proved in exactly the same way as (i) of Theorem 2, with Lemma 3′ in place of Lemma 3. (ii) of the theorem is a special case of (iii).

To prove (iii) let f be rational with

$$\mathrm{P\,div} f \leqq \sum_{j=1}^{r} n_j\cdot(\zeta_j)+n_0\cdot(\infty)$$

where $n_0>|n|$. Then $F=f'$ is rational with

$$\mathrm{P\,div}\, F \leqq \sum_{j=1}^{r}(n_j+1)\cdot(\zeta_j)+(n_0-1)\cdot(\infty)$$

and since

$$\mathrm{P\,div}_{\mathbf{D}^*} H = n\cdot(\infty) \leqq (n_0-1)\cdot(\infty)$$

(34) shows that

$$\begin{aligned}\mathrm{P\,div}\,\mathscr{F}_n(f) &\leqq \mathrm{P\,div}\, G+1\cdot(\infty) \leqq \sum_{j=1}^{r}(n_j+1)\cdot(\zeta_j)+(n_0-1)\cdot(\infty)+1\cdot(\infty)\\ &= \sum_{j=1}^{r}(n_j+1)\cdot(\zeta_j)+n_0\cdot(\infty).\end{aligned}$$

This proves (iii) of Theorem 2′, and also finishes the proof of that theorem.

Proof of Lemma 3′. The relation (33) shows that the function $F^*\cdot G-H$ is purely imaginary on $\partial\mathbf{D}$ and hence extends by reflection to be holomorphic in some region which is symmetric with respect to $\partial\mathbf{D}$. This gives (i) of the lemma exactly as in the proof of Lemma 3 (i).

Moreover, it is clear (by a reflection argument) that if F is rational then so is G. Now the rest of (ii) follows from the following series of inequalities.

$$\begin{aligned}\mathrm{P\,div}\, G &= \mathrm{P\,div}_{\mathbf{D}^*} G \leqq \mathrm{P\,div}_{\mathbf{D}^*} F^*\cdot G \leqq \max\{\mathrm{P\,div}_{\mathbf{D}^*}(F^*\cdot G-H),\ \mathrm{P\,div}_{\mathbf{D}^*} H\}\\ &= \max\{[\mathrm{P\,div}_{\mathbf{D}}(F^*\cdot G-H)]^*,\ \mathrm{P\,div}_{\mathbf{D}^*} H\}\\ &\leqq \max\{[\mathrm{P\,div}_{\mathbf{D}} F^*\cdot G]^*,\ [\mathrm{P\,div}_{\mathbf{D}} H]^*,\ \mathrm{P\,div}_{\mathbf{D}^*} H\}\\ &\leqq \max\{[\mathrm{P\,div}_{\mathbf{D}} F^*]^*,\ \mathrm{P\,div}_{\mathbf{D}^*} H\} = \max\{\mathrm{P\,div}_{\mathbf{D}^*} F,\ \mathrm{P\,div}_{\mathbf{D}^*} H\}.\end{aligned}$$

b) *Non-singularity of the moment mapping*

Now we want to apply Theorem 2′ to prove the non-singularity of a certain mapping. For $f\in\mathcal{O}_1$, $n\geqq 0$ define

$$c_n(f) = \iint_\Omega z^n\,dx\,dy = \frac{1}{2i}\iint_\Omega z^n\,d\bar{z}\,dz = \iint_{\mathbf{D}} f^n\cdot|f'|^2\,d\xi\,d\eta$$

where $\Omega=f(\mathbf{D})$ and where Ω is regarded as a Riemann surface over $\mathbf{C}$ in the first two integrals above if f is not globally univalent. The numbers $c_0, c_1, c_2, \ldots$

are called the analytic moments of Ω or of f. The map

$$f \to (c_0, c_1, c_2, \dots) \tag{35}$$

has attracted some attention in recent years. For example, H. S. Shapiro raised the question ([2], Problem 1, p. 193) whether the map (35), defined on the set of univalent functions mapping $\mathbf{D}$ onto Jordan domains, was one-to-one. Shapiro conjectured that the answer was "no", and this was confirmed in 1978 by M. Sakai ([6]), who constructed two different Jordan domains having the same set of moments c_n.

Here we shall prove a modest result in the other direction, namely that when restricted to the set of locally univalent polynomials of any given degree the map (35) is at least locally one-to-one (Theorem 6 below).

Recall that $\mathcal{P}_N$ denotes the set (or linear space) of polynomials of degree $\leqq N$. It is easy to see (by a computation) that for $f \in \mathcal{O}_1 \cap \mathcal{P}_N$

$$c_n(f) = 0 \quad \text{for} \quad n \geqq N. \tag{36}$$

Conversely (but somewhat deeper), if (36) holds for some $f \in \mathcal{O}_1$ which is univalent on $\overline{\mathbf{D}}$ then $f \in \mathcal{P}_N$. (See [1] where the result on p. 16 easily implies the assertion above.)

By (36) only the moments $c_0(f), \dots, c_{N-1}(f)$ are of interest for $f \in \mathcal{O}_1 \cap \mathcal{P}_N$. Thus we consider the map

$$J\colon f \to (c_0(f), \dots, c_{N-1}(f))$$

for $f \in \mathcal{O}_1 \cap \mathcal{P}_N$. Since $c_0(f) = |\Omega| \geqq 0$, hence is real, we may consider J as a map

$$J\colon\ \mathcal{O}_1 \cap \mathcal{P}_N \to V_N$$

where $V_N = \mathbf{R} \times \mathbf{C}^{N-1}$. Notice that

$$\dim_{\mathbf{R}} V_N = \dim_{\mathbf{R}} \mathcal{O}_1 \cap \mathcal{P}_N = 2N-1$$

(V_N is a linear space, $\mathcal{O}_1 \cap \mathcal{P}_N$ is an open subset of a linear space).

Clearly J is smooth, even real analytic. Now we have

Theorem 6. *The Fréchet derivative of J is everywhere non-singular. Hence J is a local diffeomorphism.*

Remark. C. Ullemar has proved special cases of Theorem 6. Namely, when $N=3, 4$ or 5 she proves that the restriction of J to polynomials $f \in \mathcal{O}_1 \cap \mathcal{P}_N$ with real coefficients is locally one-to-one ([9] p. 14—16). She also conjectures an expression for the Jacobian of J for arbitrary N (p. 16 in [9]), and for $N=3$ she proves that J is globally one-to-one on the set of those $f \in \mathcal{P}_N$ which are univalent on $\mathbf{D}$ and have real coefficients (p. 17—23 in [9]).

Proof. Observe first that $c_n(f)=I_P(f)$ with $P(z)=z^n$, where I_P is defined by (27). If $t\to f_t\in\mathcal{O}_1\cap\mathcal{P}_N$ is any differentiable curve with $f_0=f$ given, then

$$\frac{d}{dt}I_P(f_t)=\int_{\partial\mathbf{D}}P(f_t(\zeta))\,\mathrm{Im}\,[\overline{\dot{f}_t(\zeta)}f_t'(\zeta)\,d\zeta]$$

according to (28). Thus

$$\frac{d}{dt}c_n(f_t)=\int_{\partial\mathbf{D}}f_t(\zeta)^n\cdot\mathrm{Im}\,[\overline{\dot{f}_t(\zeta)}f_t'(\zeta)\,d\zeta]\quad(n=0,1,\ldots,N-1),$$

and

$$\frac{d}{dt}J(f_t)=\Big(\int_{\partial\mathbf{D}}f_t(\zeta)^n\cdot\mathrm{Im}\,[\overline{\dot{f}_t(\zeta)}f_t'(\zeta)\,d\zeta]\Big)_{n=0}^{N-1}\tag{37}$$

$$=\Big(\int_{\partial\mathbf{D}}\mathrm{Im}\,[\bar{\dot{f}}_t\,df_t],\ldots,\int_{\partial\mathbf{D}}f_t^{N-1}\,\mathrm{Im}\,[\bar{\dot{f}}_t\,df_t]\Big).$$

As $t\to f_t\in\mathcal{O}_1\cap\mathcal{P}_N$ traces through all curves with $f_0=f$ the tangent vector $\dot{f}_0$ at f traces through all of

$$(\mathcal{P}_N)_1=\{h\in\mathcal{P}_N\colon\ h(0)=0,\ \mathrm{Im}\,h'(0)=0\}.$$

In other words $(\mathcal{P}_N)_1$ is the tangent space of $\mathcal{O}_1\cap\mathcal{P}_N$ at the point f.

Now (37) shows that the Fréchet derivative of J at $f\in\mathcal{O}_1\cap\mathcal{P}_N$ is the linear (over $\mathbf{R}$) map

$$dJ_f\colon\ (\mathcal{P}_N)_1\to V_N$$

defined by

$$dJ_f(h)=\Big(\int_{\partial\mathbf{D}}f(\zeta)^n\cdot\mathrm{Im}\,[\overline{h(\zeta)}f'(\zeta)\,d\zeta]\Big)_{n=0}^{N-1}\tag{38}$$

$$=\Big(\int_{\partial\mathbf{D}}\mathrm{Im}\,[\bar{h}df],\ldots,\int_{\partial\mathbf{D}}f^{N-1}\,\mathrm{Im}\,[\bar{h}df]\Big).$$

We have to show that this linear map is non-singular. Since the domain and range spaces have the same dimension $(=2N-1$ over $\mathbf{R})$ is enough to show that dJ_f is surjective. For that purpose we shall make use of the operators $\mathcal{F}_n$ defined by (32).

Namely, for $m=-N+1,\ldots,0,1,\ldots,N-1$ choose $h=h_m=\mathcal{F}_m(f)$ in (38). (Observe that $\mathcal{F}_m(f)\in(\mathcal{P}_N)_1$ for $|m|<N$ by Theorem 2′.) Then, by the definition of $\mathcal{F}_m$,

$$\mathrm{Im}\,[\overline{h_m(\zeta)}f'(\zeta)\,d\zeta]=\begin{cases}\cos m\theta\,d\theta & (m\geqq 0)\\ \sin m\theta\,d\theta & (m<0)\end{cases}$$

$$=\begin{cases}\dfrac{1}{2}(\zeta^m+\zeta^{-m})\dfrac{d\zeta}{i\zeta} & (m\geqq 0)\\[2ex] \dfrac{1}{2i}(\zeta^m-\zeta^{-m})\dfrac{d\zeta}{i\zeta} & (m<0)\end{cases}$$

as in (30). This shows that the n:th component ($n=0, \ldots, N-1$) of $dJ_f(h_m)$ for $m>0$ is

$$(dJ_f(h_m))_n = \int_{\partial \mathbf{D}} f(\zeta)^n \operatorname{Im}[\overline{h_m(\zeta)} f'(\zeta)\, d\zeta] = \frac{1}{2i} \int_{\partial \mathbf{D}} f(\zeta)^n (\zeta^m + \zeta^{-m}) \frac{d\zeta}{\zeta} =$$

$$= \pi \operatorname*{Res}_{\zeta=0} \frac{f(\zeta)^n}{\zeta^m} \frac{d\zeta}{\zeta} = \begin{cases} * & \text{for} \quad n < m \\ \pi f'(0)^m & \qquad\ \ n = m \\ 0 & \qquad\ \ n > m. \end{cases}$$

Here, and in the sequel, * stands for complex numbers whose values are unimportant for us. For $m=0$ we obtain

$$(dJ_f(h_m))_n = \begin{cases} 2\pi & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n > 0 \end{cases}$$

and for $m<0$

$$(dJ_f(h_m))_n = \frac{1}{2i} \int_{\partial \mathbf{D}} f(\zeta)^n (\zeta^m - \zeta^{-m}) \frac{d\zeta}{i\zeta} = -i\pi \operatorname*{Res}_{\zeta=0} \frac{f(\zeta)^n}{\zeta^{|m|}} \cdot \frac{d\zeta}{\zeta}$$

$$= \begin{cases} * & \text{for} \quad n < |m| \\ -i\pi f'(0)^{|m|} & \qquad\ \ n = |m| \\ 0 & \qquad\ \ n > |m|. \end{cases}$$

In summary, the range of dJ_f contains the vectors

$$\begin{array}{lll} (2\pi, 0, \ldots & 0) & (m = 0) \\ (*, \ldots, *, \pi f'(0)^m, & 0, \ldots, 0) & (m = 1, \ldots, N-1) \\ (*, \ldots, *, -i\pi f'(0)^{|m|}, & 0, \ldots, 0) & (m = -1, \ldots, -N+1) \\ \uparrow \qquad\qquad \uparrow & \quad\ \uparrow & \\ \text{component } 0 \quad |m| & \ N-1 & \end{array}$$

Since $f'(0) \neq 0$ these vectors span $V_N = \mathbf{R} \times \mathbf{C}^{N-1}$ over $\mathbf{R}$. Thus dJ_f is surjective, hence non-singular and so Theorem 6 is proven.

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