The smoothness of random Besov functions

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0. Introduction

Let $\{\varepsilon_n: n\in\mathbf{Z}\}$ be a sequence of independent random variables such that $P(\varepsilon_n=1)=P(\varepsilon_n=-1)=1/2$. If f is a function (or distribution) on the circle with

(0) $$f(t) \sim \sum_{n=-\infty}^{\infty} a_n e^{int},$$

we define $f^{\pm}$ by

(1) $$f^{\pm}(t) \sim \sum_{n=-\infty}^{\infty} \varepsilon_n a_n e^{int}.$$

The notation $\Lambda_\alpha^{p,r}$ will have essentially the same meaning as in Chapter 5 of [ST] (with the circle in place of $\mathbf{R}^n$); a definition is given at the beginning of Section 1 below. We shall be considering the following question: Given $f\in\Lambda_\alpha^{p,r}$, does it follow that $f^{\pm}\in\Lambda_\beta^{q,s}$ almost surely? Of course this depends on the values of the parameters; Theorems 0 through 3 below suffice for all $\alpha, \beta\in\mathbf{R}$ and $p, q, r, s\in[0, 1]$ except for the case $1<p<2$, $\beta=\alpha+1/2-1/p$, $p'\leqq q<\infty$ and $r=s=\infty$; here we do not know the answer (see the note following the statement of Theorem 3).

It may be appropriate to point out that many extremely familiar objects are included among the Besov spaces $\Lambda_\alpha^{p,r}$. In particular, they are generalizations of Lipschitz spaces: If $0<\alpha<1$ then $\Lambda_\alpha^{\infty,\infty}=\mathrm{Lip}_\alpha$, while $\Lambda_\alpha^{p,\infty}$ is the "L^p Lipschitz space" sometimes denoted Lip_α^p. The class of holomorphic functions in $\Lambda_{1/2}^{2,2}$ is the classical Dirichlet space ($\Lambda_{1/2}^{2,r}$ gives a weighted Dirichlet space), while the holomorphic functions in $\Lambda_0^{2,2}$ form the Hardy space H^2, and the holomorphic functions in $\Lambda_0^{\infty,\infty}$ give the Bloch space. (In some contexts the term "Besov space" has a more special meaning: The space denoted B_p in [AFP] and elsewhere consists of the holomorphic functions in $\Lambda_{1/p}^{p,p}$; these are precisely the $\Lambda_\alpha^{p,r}$ which are (uniformly) invariant under composition with the holomorphic automorphisms of the disc.)

Indeed, our interest in the present question began with [CSU]. There it is shown that the randomization of a Dirichlet function is almost surely a (pointwise) multiplier for the Dirichlet space: First one notes that if f is a Dirichlet function ($f\in\Lambda^{2,2}_{1/2}$) then $f^{\pm}\in\Lambda^{p,2}_{1/2}$ almost surely for $p<\infty$ (cf. [BS], Prop. 19, p. 300); then one shows that an element of $\Lambda^{p,2}_{1/2}$ must be a multiplier if $p>2$. Of course, various similar results may be found in [KA]; in particular Theorem 3 on page 89 of [KA] states that if $f\in\Lambda^{2,\infty}_{\alpha}$ then $f^{\pm}\in\Lambda^{\infty,\infty}_{\beta}$ almost surely for $\beta<\alpha$.

It will be convenient to break the argument into four pieces, depending on the methods needed: In Theorem 0 we see that the case $\beta>\alpha$ is trivial. The case $\beta=\alpha$ is dealt with in Theorem 1; here the positive results follow from Khinchine's inequality and the negative results from fairly explicit counterexamples involving the Rudin—Shapiro polynomials and the Borel—Cantelli lemma.

The case $\beta<\alpha$ is treated in Theorems 2 and 3: Much of Theorem 2 follows from Theorem 1 together with the inclusions among the various Besov spaces; the rest of Theorem 2 is proved using arguments similar to those used in the proof of Theorem 1. One boundary case which does not appear to be susceptible to arguments of this sort is dealt with in Theorem 3, which involves estimates on the L^p norm of a randomized Dirichlet kernel (Theorem 4, inspired by [SZ] and [HA]).

The proof of Theorem 3 has very little to do with the techniques used in the rest of the paper, because the corresponding Khinchine's-inequality argument fails. In fact we conjectured at first that Theorem 3 was false, because of our experience to the effect that positive results here should be controlled by L^2 estimates, while Theorem 1 shows that the best possible L^2 estimates under the hypotheses of Theorem 3 do not imply the conclusion.

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1. Definitions and statements of results

Let $\mu^0_n=1$ if $|n|\leqq 1$, $\mu^0_n=0$ if $|n|>1$. For $j=1,2,\dots$ let μ^j_n be the piecewise linear function of n which vanishes for $|n|\geqq 2^{j+1}$ and interpolates the values 0, 1, 0, 0, 1, and 0 for n equal to -2^{j+1}, -2^j, -2^{j-1}, 2^{j-1}, 2^j, and 2^{j+1} respectively. In other words, μ^j_n is the n-th Fourier coefficient of

$$(2)\qquad (2K_{(2^{j+1})}-K_{(2^j)})-(2K_{(2^j)}-K_{(2^{j-1})}),$$

where K_m is the Fejér kernel.

If $f(t)\sim\sum_{n=-\infty}^{\infty}a_ne^{int}$ we define

$$(3)\qquad S_jf(t)=\sum_{n=-\infty}^{\infty}\mu^j_na_ne^{int};$$

S_jf is simply a smooth version of "the sum of the terms in the Fourier series for f with $|n|\approx 2^j$".

The notation $\|f\|_p$ will refer to the norm of f in the space $L^p(\mathbf{T})$. For $1\leqq p\leqq\infty$, $1\leqq r\leqq\infty$, and $\alpha\in\mathbf{R}$ we define $\Lambda_\alpha^{p,r}$ to be the space of all f such that $\|f\|_{p,r,\alpha}<\infty$, where

$$(4)\qquad \|f\|_{p,r,\alpha}=\Big\{\sum_{j=0}^{\infty}(2^{\alpha j}\|S_jf\|_p)^r\Big\}^{1/r}\quad(1\leqq r<\infty),$$

$$(4')\qquad \|f\|_{p,\infty,\alpha}=\sup_j 2^{\alpha j}\|S_jf\|_p.$$

See [PT] or [ST], Chapter 5 for background. Because we are working on the circle, it is clear that $\Lambda_\alpha^{p,r}\subset\Lambda_\alpha^{q,s}$ if $q\leqq p$ and $s\geqq r$.

We will save a good deal of ink by adopting the following notation:

Definition. Suppose X and Y are two spaces of distributions on the circle. We will write $X\Rightarrow Y$ if $f^{\pm}\in Y$ almost surely whenever $f\in X$, and $X\not\Rightarrow Y$ otherwise.

Note that $X\subset Y$ does not imply that $X\Rightarrow Y$ (for example, it is well known that $L^1\not\Rightarrow L^1$). However, it is clear that either of the two conditions $X\subset Z\Rightarrow Y$ or $X\Rightarrow Z\subset Y$ is sufficient to imply that $X\Rightarrow Y$; note as well that $\Lambda_\alpha^{2,r}\Rightarrow\Lambda_\alpha^{2,r}$, so that in fact $X\subset Y$ does imply $X\Rightarrow Y$ if either of X and Y is of the form $\Lambda_\alpha^{2,r}$. In these terms our question becomes "Given $p,q,r,s\in[1,\infty]$ and $\alpha,\beta\in\mathbf{R}$ is it true that $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$?" The following four theorems give an answer in "almost every" case (see the note following Theorem 3):

Theorem 0. *If* $\beta>\alpha$ *then* $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\beta^{q,s}$, *regardless of the values of* p, q, r, *and* s.

Theorem 1. *Suppose* $p,q,r,s\in[1,\infty]$. *Then* $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\alpha^{q,s}$ *if and only if one of the following holds:*

(i) $p\geqq 2$, $r<\infty$, $q<\infty$, *and* $s\geqq r$,
(ii) $p\geqq 2$, $r=\infty$, $q\leqq 2$, *and* $s=\infty$.

Theorem 2. *Suppose* $\beta<\alpha$.
(i) *If* $\beta<\alpha+1/2-1/p$ *then* $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$, *regardless of the values of* r, q, *and* s.
(ii) *If* $\beta>\alpha+1/2-1/p$ *then* $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\beta^{q,s}$, *regardless of the values of* r, q, *and* s.
(iii) *Suppose* $\beta=\alpha+1/2-1/p$.
(a) *If either* $r<\infty$, $q<\infty$, *and* $s\geqq r$ or $q\leqq 2$ *and* $r=s=\infty$ *then* $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$.
(b) *If* $q=\infty$ *or* $s<r$ *then* $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\beta^{q,s}$.

Theorem 3. *If* $q<\infty$ *and* $\beta=\alpha-1/2$ *then* $\Lambda_\alpha^{1,\infty}\Rightarrow\Lambda_\beta^{q,\infty}$.

Note. One may verify that Theorems 0 through 3 answer our question except for $1<p<2$, $\beta=\alpha+1/2-1/p$, $q<\infty$, $r=s=\infty$. If one supposes in addition that $1\leqq q<p'$ then an interpolation argument shows that $\Lambda_\alpha^{p,\infty}\Rightarrow\Lambda_\beta^{q,\infty}$; we leave the details to the interested reader. (Here p' denotes the exponent conjugate to p; our positive results lead to interpolable inequalities by Lemma 1 below.) We do not know how to settle the case $p'\leqq q<\infty$ here.

The proof of Theorem 3 is the hardest in the paper, which is to say both "most difficult" and "least soft". We begin by showing that Theorem 0 is quite easy:

Proof of Theorem 0. Suppose $\beta>\alpha$. We may choose a sequence $c_j\geqq 0$ such that $\sum_{j=1}^{\infty} 2^{\alpha j}c_j<\infty$ while $2^{\beta j}c_j$ is unbounded. Let $f(t)=\sum_{j=1}^{\infty} c_j e^{i2^j t}$. Then (surely) $\|S_j f^{\pm}\|_p=c_j$ for all p, since $S_j f(t)=c_j e^{i2^j t}$. Thus $\sum_{j=1}^{\infty} 2^{\alpha j}\|S_j f\|_p<\infty$, so that $f\in \Lambda_\alpha^{p,1}\subset\Lambda_\alpha^{p,r}$, although $2^{\beta j}\|S_j f^{\pm}\|_q$ is surely unbounded, which is to say that $f^{\pm}\notin\Lambda_\beta^{q,\infty}$, so that $f^{\pm}\notin\Lambda_\beta^{q,s}$. ∎

Similarly, one needs nothing more than a lacunary series to prove the following:

Lemma 0. *If* $s<r$ *then* $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\alpha^{q,s}$.

It is known that the Besov spaces are independent of the particular choice of the multipliers used in their definition, as long as the multipliers are sufficiently smooth (e.g., [PT], Ch. 8). The following proposition is somewhat analogous; it will serve to simplify various calculations later.

Proposition 0. (i) *Suppose* $f\sim\sum_{j=1}^{\infty} f_j$ *with* $f_j(t)=\sum_{n=2^j}^{2^{j+1}-1} a_n e^{int}$. *Then*

$$\|f\|_{p,r,\alpha}\leqq c\left\{\sum_{j=1}^{\infty}(2^{\alpha j}\|f_j\|_p)^r\right\}^{1/r}$$

for $1\leqq r<\infty$; *similarly for* $r=\infty$.

(ii) *Suppose in addition that* $f_j=0$ *for* $j=2k+1$, $k=0, 1, \dots$. *Then*

$$\|f\|_{p,r,\alpha}\geqq c\left\{\sum_{j=1}^{\infty}(2^{\alpha j}\|f_j\|_p)^r\right\}^{1/r}$$

for $1\leqq r<\infty$; *similarly for* $r=\infty$.

Proof. (i) It is clear that $S_j f=S_j f_{j-1}+S_j f_j$, so the expression (2) for μ_n^j in terms of the Fejér kernels shows that

$$\|S_j f\|_p\leqq\|S_j f_{j-1}\|_p+\|S_j f_j\|_p\leqq 6(\|f_{j-1}\|_p+\|f_j\|_p).$$

(ii) The additional hypothesis implies that $f_{2k}=S_{2k-1}f+S_{2k}f$, so that

$$\|f_{2k}\|_p\leqq\|S_{2k-1}f\|_p+\|S_{2k}f\|_p. \quad ∎$$

2. Theorem 1

In general, the letter c will denote a finite positive constant the value of which may vary from occurrence to occurrence. If A and B are positive quantities such that $A\leqq cB$ and $B\leqq cA$ we will write $A\approx B$. We will be making a great deal of use of Kahane's Banach space version of Khinchine's inequality ([AG], p. 176 or [LT], p. 74):

If X is a Banach space, $x_1, x_2, \ldots, x_N \in X$ and $1 \leqq p < \infty$ then

$$E\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_X^p \approx \left\{E\left\|\sum_{n=1}^N \varepsilon_n x_n\right\|_X^2\right\}^{p/2}. \tag{5}$$

(Of course when X is one dimensional this becomes the more familiar statement

$$E\left|\sum_{n=1}^N \varepsilon_n x_n\right|^p \approx \left\{\sum_{n=1}^N |x_n|^2\right\}^{p/2}.) \tag{5'}$$

The following lemma is immediate from two basic results in [KA]:

Lemma 1. *Suppose f is a distribution on the circle such that $f^{\pm} \in \Lambda_\beta^{q,s}$ almost surely. Then*

$$E\|f^{\pm}\|_{q,s,\beta}^p < \infty, \quad 1 \leqq p < \infty. \tag{6}$$

(In fact much stronger results are known ([KA] p. 23), but (6) will suffice for our purposes.)

Proof. Let us define $\sum_N(g) = \sum_{j=0}^N S_j g$. It is easy to see that in general we have

$$\|g\|_{q,s,\beta} \approx \sup_N \left\|\sum_N(g)\right\|_{q,s,\beta}.$$

Now suppose that $f^{\pm} \in \Lambda_\beta^{q,s}$ almost surely; this says that the Fourier series for $f^{\pm}$ is almost surely "S-bounded" in $\Lambda_\beta^{q,s}$, if S is the summation matrix defined by the operators $\sum_N$. This implies that the partial sums for the Fourier series for $f^{\pm}$ are almost surely bounded in $\Lambda_\beta^{q,s}$, by [KA], Thm. 2.1 (p. 13), and this implies (6), by [KA], Thm. 2.4 (p. 20). ∎

The next lemma is a simple consequence of Khinchine's inequality:

Lemma 2. *If f is a trigonometric polynomial and $1 \leqq p < \infty$, $1 \leqq r < \infty$, then*

$$E\|f^{\pm}\|_p^r \approx \|f\|_2^r.$$

Proof. We apply Khinchine's inequality in the Banach space $L^p(\mathbf{T})$, then Fubini's theorem and the scalar version of Khinchine's inequality:

$$E(\|f^{\pm}\|_p^r) \approx \{E(\|f^{\pm}\|_p^p)\}^{r/p} = \left\{E(2\pi)^{-1}\int_0^{2\pi} |f^{\pm}(t)|^p\, dt\right\}^{r/p}$$

$$= \left\{(2\pi)^{-1}\int_0^{2\pi} E\,|f^{\pm}(t)|^p\, dt\right\}^{r/p} \approx \left\{(2\pi)^{-1}\int_0^{2\pi} \|f\|_2^p\, dt\right\}^{r/p} = \|f\|_2^r. \quad ∎$$

The following explains why $p \geqq 2$ in Theorem 1:

Proposition 1. *If $X \Rightarrow \Lambda_\beta^{q,s}$ then $X \subset \Lambda_\beta^{2,s}$.*

Proof. We need only consider the case $q < \infty$, because if $q \geqq 2$ then $X \Rightarrow \Lambda_\beta^{q,s} \subset \Lambda_\beta^{2,s}$ so that $X \Rightarrow \Lambda_\beta^{2,s}$, and it is clear that this implies that $X \subset \Lambda_\beta^{2,s}$.

Suppose first that $s<\infty$ and $f\in X$. The hypothesis says that $\|f^{\pm}\|_{q,s,\beta}<\infty$ a.s.; now (6) shows that $E\|f^{\pm}\|_{q,s,\beta}^{s}<\infty$. It follows from Lemma 2 that

$$\|f\|_{2,s,\beta}^{s} = \sum_{j=0}^{\infty} 2^{\beta js}\|S_j f\|_2^s$$

$$\approx \sum_{j=0}^{\infty} 2^{\beta js} E\|S_j f^{\pm}\|_q^s = E\|f^{\pm}\|_{q,s,\beta}^s < \infty,$$

so that $f\in\Lambda_{\beta}^{2,s}$.

The case $s=\infty$ is similar: Suppose $X\Rightarrow\Lambda_{\beta}^{q,\infty}$ and $f\in X$. Then

$$\|f\|_{2,\infty,\beta} = \sup_j 2^{\beta j}\|S_j f\|_2 \approx \sup_j 2^{\beta j} E\|S_j f^{\pm}\|_q$$

$$\leqq E\sup_j 2^{\beta j}\|S_j f^{\pm}\|_q = E\|f^{\pm}\|_{q,\infty,\beta} < \infty,$$

again by (6); thus $f\in\Lambda_{\beta}^{2,\infty}$. ∎

Lemma 3. *If* $\Lambda_{\alpha}^{p,1}\Rightarrow\Lambda_{\alpha}^{2,\infty}$ *then* $p\geqq 2$.

Proof. The hypothesis implies that $\Lambda_{\alpha}^{p,1}\subset\Lambda_{\alpha}^{2,\infty}$, by Proposition 1, so that $\|f\|_{2,\infty,\alpha}\leqq c\|f\|_{p,1,\alpha}$. Now suppose that g is a trigonometric polynomial and let $f(t)=e^{i2^k t}g(t)$. If k is large enough then

$$2^{\alpha k}\|g\|_2 \approx \|f\|_{2,\infty,\alpha} \leqq c\|f\|_{p,1,\alpha} \approx c2^{\alpha k}\|g\|_p.$$

That is, $\|g\|_2\leqq c\|g\|_p$ for any trigonometric polynomial g, which shows that $p\geqq 2$. ∎

We have done everything we need to prove Theorem 1, except for the construction of counterexamples in two "endpoint" cases, which we postpone until after the proof:

Proof of Theorem 1. The easy part is to show that either of (i) or (ii) implies that $\Lambda_{\alpha}^{p,r}\Rightarrow\Lambda_{\alpha}^{q,s}$.

Suppose (i): $p\geqq 2$, $r<\infty$, $q<\infty$, and $s\geqq r$. Let f be an element of $\Lambda_{\alpha}^{p,r}$. Then, as in the proof of Proposition 1,

$$E\|f^{\pm}\|_{q,s,\alpha}^{r} \leqq E\|f^{\pm}\|_{q,r,\alpha}^{r}$$

$$= \sum_{j=0}^{\infty} 2^{\alpha jr} E\|S_j f^{\pm}\|_q^r \approx \sum_{j=0}^{\infty} 2^{\alpha jr}\|S_j f\|_2^r = \|f\|_{2,r,\alpha}^r \leqq \|f\|_{p,r,\alpha}^r < \infty,$$

by Khinchine's inequality. This certainly implies that $\|f^{\pm}\|_{q,s,\alpha}<\infty$ almost surely.

To show that (ii) implies that $\Lambda_{\alpha}^{p,r}\Rightarrow\Lambda_{\alpha}^{q,s}$ is even easier. Suppose that $p\geqq 2$, $q\leqq 2$, $s=r=\infty$. Then

$$\Lambda_{\alpha}^{p,r} = \Lambda_{\alpha}^{p,\infty} \subset \Lambda_{\alpha}^{2,\infty} \Rightarrow \Lambda_{\alpha}^{2,\infty} \subset \Lambda_{\alpha}^{q,\infty} = \Lambda_{\alpha}^{q,s},$$

so that $\Lambda_{\alpha}^{p,r}\Rightarrow\Lambda_{\alpha}^{q,s}$.

Now we suppose that $\Lambda_\alpha^{p,r} \Rightarrow \Lambda_\alpha^{q,s}$ and we shall show that one of (i) or (ii) in the statement of the theorem must hold. Proposition 1 shows that $\Lambda_\alpha^{p,r} \Rightarrow \Lambda_\alpha^{2,s}$; now $\Lambda_\alpha^{p,1} \subset \Lambda_\beta^{p,r} \Rightarrow \Lambda_\alpha^{2,s} \subset \Lambda_\alpha^{2,\infty}$, so that $\Lambda_\alpha^{p,1} \Rightarrow \Lambda_\alpha^{2,\infty}$, which implies that $p \geqq 2$, by Lemma 3. Now Lemma 0 shows that $s \geqq r$.

All that remains is to show that $q<\infty$ if $r<\infty$ and that $q \leqq 2$ if $r=\infty$. In other words, we must show that $\Lambda_\alpha^{p,r} \not\Rightarrow \Lambda_\alpha^{\infty,s}$ and that $\Lambda_\alpha^{p,\infty} \not\Rightarrow \Lambda_\alpha^{q,\infty}$ if $q>2$. But $\Lambda_\alpha^{p,r} \Rightarrow \Lambda_\alpha^{\infty,s}$ would imply $\Lambda_\alpha^{\infty,1} \Rightarrow \Lambda_\alpha^{\infty,\infty}$, contradicting Proposition 3 below, and similarly $\Lambda_\alpha^{p,\infty} \Rightarrow \Lambda_\alpha^{q,\infty}$, $q>2$ would imply $\Lambda_\alpha^{\infty,\infty} \Rightarrow \Lambda_\alpha^{q,\infty}$, contradicting Proposition 2. ∎

Thus the proof of Theorem 1 will be complete when we have proved Propositions 2 and 3 below.

Lemma 4. *Suppose $q>2$ and $k<\infty$. There exists a trigonometric polynomial g such that $\|g\|_\infty=1$ while $P(\|g^\pm\|_q>k)>0$. (Here "P" denotes "probability".)*

Proof. The construction of the Rudin—Shapiro polynomials (e.g., [KZ] p. 33) shows that we may choose a sequence $a_n=\pm 1$ in such a way that $\|f\|_\infty \leqq c_1$, if $f(t)=2^{-j/2}\sum_{n=1}^{2^j} a_n e^{int}$. On the other hand, if $h(t)=2^{-j/2}\sum_{n=1}^{2^j} e^{int}$, then one easily calculates that $\|h\|_q \geqq c_2 2^{j(1/2-1/q)}$, so that $\|h\|_q>c_1 k$ if j is large enough. Let $g=c_3 f$, choosing c_3 so that $\|g\|_\infty=1$. Note $\|c_3 h\|_q>c_3 c_1 k \geqq k$. Thus

$$P(\|g^\pm\|_q>k) \geqq P(g^\pm=c_3 h)=2^{-2^j}>0. \quad ∎$$

Proposition 2. *If $q>2$ then $\Lambda_\alpha^{\infty,\infty} \not\Rightarrow \Lambda_\alpha^{q,\infty}$.*

Proof. For $k=1, 2, \ldots$ let g_k be a holomorphic trigonometric polynomial with $\|g_k\|_\infty=1$ and $P(\|g_k^\pm\|_q>k)=\delta_k>0$ (Lemma 4). We will choose two sequences of positive integers $j_1<j_1'<j_2<j_2' \ldots$ and then define

$$f(t)=\sum_{k=1}^{\infty}\sum_{\nu=j_k}^{j_k'} 2^{-2\alpha\nu} e^{i2^{2\nu}t} g_k(t). \tag{7}$$

Let us set $f_j(t)=\sum_{n=2^j}^{2^{j+1}-1}\hat{f}(n)e^{int}$, as in Proposition 0. If we choose the j_k in such a way that $2^{(2j_k)}>\deg(g_k)$ then we will have $f_j=0$ except for $j=2\nu, j_k \leqq \nu \leqq j_k'$, in which case $f_j(t)=2^{-\alpha j}e^{i2^j t}g_k(t)$. Now Proposition 0 shows that $f \in \Lambda_\alpha^{\infty,\infty}$, because $\|g_k\|_\infty$ is bounded.

On the other hand, suppose we choose j_k and j_k' in such a way that $\sum_k \delta_k(j_k'-j_k+1)=\infty$. It will follow that $\sum_j P(2^{\alpha j}\|f_j^\pm\|_q>A)=\infty$ for any $A<\infty$, since $P(2^{\alpha j}\|f_j^\pm\|_q>k)=P(\|g_k^\pm\|_q>k)=\delta_k$ for $j=2\nu$, $j_k \leqq \nu \leqq j_k'$. This shows that $2^{\alpha j}\|f_j^\pm\|_q$ is almost surely unbounded, by the Borel—Cantelli lemma, and now Proposition 0 shows that $f^\pm \notin \Lambda_\alpha^{q,\infty}$ almost surely. ∎

Proposition 3. $\Lambda_\alpha^{\infty,1} \not\Rightarrow \Lambda_\alpha^{\infty,\infty}$.

Proof. This is somewhat simpler than Proposition 2, in part because we are taking $q=\infty$ here, but largely because much of the work has already been done by Salem, Zygmund, Rudin and Shapiro:

As in the proof of Lemma 4 we may choose a sequence $a_n=\pm 1$ in such a way that if $g_k(t)=2^{-k/2}\sum_{n=1}^{2^k} a_n e^{int}$ then $\|g_k\|_\infty \leqq c$. Now choose $b_k \geqq 0$ so that $\sum_{k=1}^{\infty} b_k<\infty$ but $k^{1/2}b_k$ is unbounded. Choose a strictly increasing sequence (j_k) with $2j_k>k$ and define

$$f(t) = \sum_{k=1}^{\infty} b_k 2^{-\alpha(2j_k)} e^{i2^{(2j_k)}t} g_k(t). \tag{8}$$

Let $f_j(t)=\sum_{n=2^j}^{2^{j+1}-1}\hat{f}(n)e^{int}$, as before. This time we have $f_j=0$ except for $j=2j_k$, in which case $f_j(t)=b_k 2^{-\alpha j}e^{-i2^j t}g_k(t)$. It follows that $\sum_j 2^{\alpha j}\|f_j\|_\infty \leqq c\sum_k b_k<\infty$, so that $f\in \Lambda_\alpha^{\infty,1}$, by Proposition 0.

However a classical result of Salem and Zygmund ([SZ] p. 278) shows that almost certainly $\|g_k^\pm\|_\infty \geqq ck^{1/2}$ except for finitely many values of k, so that $2^{\alpha(2j_k)}\|f_{2j_k}^\pm\|_\infty$ is almost surely greater than $ck^{1/2}b_k$ except for finitely many values of k. In particular $2^{\alpha j}\|f_j^\pm\|_\infty$ is almost surely unbounded, so that $f^\pm \notin \Lambda_\alpha^{\infty,\infty}$, by Proposition 0. ∎

(The result of Salem—Zygmund cited here was refined considerably by Halász [HA]; we shall have more to say about this in Section 4.)

3. Theorem 2

Much of Theorem 2 will follow from Theorem 1 together with the Besov embedding theorem ([PT] p. 63):

Suppose $q\geqq p$, $\beta\leqq\alpha$, *and* $\beta-1/q=\alpha-1/p$. *Then*

$$\Lambda_\alpha^{p,r} \subset \Lambda_\beta^{q,r}. \tag{9}$$

The reader will have no difficulty verifying the following lemma:

Lemma 5. *If* $\beta<\alpha$ *then* $\Lambda_\alpha^{p,\infty}\subset\Lambda_\beta^{p,1}$, *so that* $\Lambda_\alpha^{p,r}\subset\Lambda_\beta^{p,s}$ *for all* r, s.

Proof of Theorem 2. We begin with part (i): Suppose that $\beta<\alpha$ and $\beta<\alpha+1/2-1/p$. We will consider separately the cases $p\geqq 2$ and $p<2$; first suppose $p\geqq 2$. Choose γ so that $\beta<\gamma<\alpha$ and $\gamma-\beta<1$; now define $t=(\gamma-\beta)^{-1}$. We use a trivial inclusion, Lemma 5, Theorem 1, (9) above, and another trivial inclusion:

$$\Lambda_\alpha^{p,r}\subset\Lambda_\alpha^{2,\infty}\subset\Lambda_\gamma^{2,1}\Rightarrow\Lambda_\gamma^{t,1}\subset\Lambda_\beta^{\infty,1}\subset\Lambda_\beta^{q,s},$$

so $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$.

Now suppose $\beta<\alpha$, $\beta<\alpha+1/2-1/p$, and $p<2$. Let $\beta_1=\alpha+1/2-1/p$, choose γ such that $\beta<\gamma<\beta_1$ and $\gamma-\beta<1$, and let $t=(\gamma-\beta)^{-1}$. Applying the same facts as before but in a slightly different order,

$$\Lambda_\alpha^{p,r}\subset\Lambda_\alpha^{p,\infty}\subset\Lambda_{\beta_1}^{2,\infty}\subset\Lambda_\gamma^{2,1}\Rightarrow\Lambda_\gamma^{t,1}\subset\Lambda_\beta^{\infty,1}\subset\Lambda_\beta^{q,s},$$

so that $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$.

We turn to part (ii) of the theorem: We suppose that $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$, and we shall show that $\beta\geqq\alpha+1/2-1/p$. The obvious inclusions and Proposition 1 imply that $\Lambda_\alpha^{p,1}\subset\Lambda_\beta^{2,\infty}$, so that $\|f\|_{2,\infty,\beta}\leqq c\|f\|_{p,1,\alpha}$. But let $f(t)=\sum_{n=2^k}^{2^{k+1}}e^{int}$ and suppose $p>1$; then $\|f\|_{2,\infty,\beta}\approx2^{\beta k}2^{k/2}$ and $\|f\|_{p,1,\alpha}\approx2^{\alpha k}2^{k(1-1/p)}$, so that $\beta+1/2\leqq\alpha+1-1/p$. (Proposition 0 may be of some use in estimating the norms in question.) Similarly one sees that $\|f\|_{1,1,\alpha}\approx k2^{\alpha k}$, which shows that $\beta+1/2\leqq\alpha$ if $p=1$.

Now for Theorem 2 (iii): We suppose for the remainder of this section that $\beta<\alpha$ and $\beta=\alpha+1/2-1/p$ (so that $p<2$). The Besov theorem ((9) above) shows that $\Lambda_\alpha^{p,r}\subset\Lambda_\beta^{2,r}$; if we suppose either that $r<\infty$, $q<\infty$, and $s\geqq r$ or that $q\leqq2$ and $r=s=\infty$ then Theorem 1 shows that $\Lambda_\beta^{2,r}\Rightarrow\Lambda_\beta^{q,s}$, so that $\Lambda_\alpha^{p,r}\Rightarrow\Lambda_\beta^{q,s}$. This gives (iii) (a).

For (iii) (b) we need only show that $\Lambda_\alpha^{p,1}\not\Rightarrow\Lambda_\beta^{\infty,\infty}$ and that $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\beta^{q,s}$ if $s<r$. We begin with the case $p>1$ and the outline the modifications needed if $p=1$:

Choose a sequence c_j so that $c_j2^{j\alpha}\|f_j\|_p=1$, where $f_j(t)=\sum_{n=2^j}^{2^{j+1}}e^{int}$. It follows that $c_j2^{j\beta}\|f_j\|_2\approx1$, as in the proof of (ii). We will let

$$f(t)=\sum_{j=1}^\infty a_jc_jf_j(t) \tag{10}$$

for a suitable sequence $a_j\geqq0$.

If we choose a_j so that $\sum_{j=1}^\infty a_j<\infty$ but $j^{1/2}a_j$ is unbounded then $f\in\Lambda_\alpha^{p,1}$ but the argument used in the proof of Proposition 3 shows that a.s. $f^\pm\notin\Lambda_\beta^{\infty,\infty}$; thus $\Lambda_\alpha^{p,1}\not\Rightarrow\Lambda_\beta^{\infty,\infty}$.

On the other hand, suppose that $s<r$, and choose $a_j>0$ so that $\sum_{j=1}^\infty a_j^r<\infty$ but $\sum_{j=1}^\infty a_j^s=\infty$ (with the appropriate change if $r=\infty$). Then (10) gives a function f with $f\in\Lambda_\alpha^{p,r}$ but $f\notin\Lambda_\beta^{2,s}$. By Proposition 1 this shows that $\Lambda_\alpha^{p,r}\not\Rightarrow\Lambda_\beta^{q,s}$.

For $p=1$ one need merely take f to be a weighted sum of "translates" of Fejér kernels rather than Dirichlet kernels. (Note that Kahane's "contraction principle", [KA] p. 20, shows that the almost sure behavior of the L^∞ norm is at least as bad for a series of randomized Fejér kernels as for Dirichlet kernels, because the coefficients of the Fejér kernel of degree $2N$ dominate the coefficients of the Dirichlet kernel of degree N.) ∎

4. Theorem 3

We do not see how to prove Theorem 3 by a method anything like that used in Theorem 2: Suppose that $q>2$ and $\beta=\alpha-1/2$. Then (9) shows that $\Lambda_\alpha^{1,\infty}\subset\Lambda_\beta^{2,\infty}$, but Theorem 1 shows that $\Lambda_\beta^{2,\infty}\not\Rightarrow\Lambda_\beta^{q,\infty}$. Similar problems arise if one attempts this crude a proof of Theorem 2 (i), but in that case there is some "slack" in the values of the parameters, which saves the day.

Theorem 3 will follow from the following result, which will be established by the "method of subsequences".

Theorem 4. *Let* $D_N(t)=N^{-1/2}\sum_{n=1}^N e^{int}$, *and suppose* $q<\infty$. *Then*

$$\lim_{N\to\infty}\|D_N^\pm\|_q^q=\Gamma(1+q/2)$$

almost surely.

This is perhaps not surprising in view of the result of Salem and Zygmund ([SZ], Theorem 3.5.2) which states that the distribution (on the circle) of the function $D_N^\pm$ almost certainly approaches a (complex) normal distribution; note however that convergence in distribution certainly does not *a priori* imply convergence in mean.

Let us begin by showing how Theorem 4 implies Theorem 3:

Proof of Theorem 3. Suppose we know Theorem 4, and suppose $f\in\Lambda_\alpha^{1,\infty}$, $\beta=\alpha-1/2, q<\infty$. Let $d_j=D_{2^j}+2^{-j/2}+\bar{D}_{2^j}$. Theorem 4 shows that $\|d_j^\pm\|_q$ is almost surely bounded, so that

$$\|f^\pm\|_{q,\infty,\beta}=\sup_j 2^{\beta j}\|S_j f^\pm\|_q$$

$$=\sup_j 2^{(\beta+1/2)j}\|(S_j f^\pm)*d_{j+1}\|_q=\sup_j 2^{\alpha j}\|(S_j f)*(d_{j+1}^\pm)\|_q$$

$$\leqq\sup_j 2^{\alpha j}\|S_j f\|_1\|d_{j+1}^\pm\|_q\leqq\|f\|_{1,\infty,\alpha}\sup_j\|d_j^\pm\|_q<\infty$$

almost surely, which is to say $f^\pm\in\Lambda_\beta^{q,\infty}$ almost surely. ∎

Theorem 4 will follow from the following estimate on the variance of $\|D_N^\pm\|_q^q$:

Proposition 4. *Let* D_N *be as in Theorem* 4, *and suppose* q *is an even positive integer. Then*

$$\sigma^2(\|D_N^\pm\|_q^q)=E(\|D_N^\pm\|_q^q-E\|D_N^\pm\|_q^q)^2\leqq\frac{c\log(N)}{N}.$$

(Here c depends on q but is of course independent of N.)

Supposing this for now, we prove Theorem 4:

Proof of Theorem 4. If X_N is a sequence of functions defined on the circle, $X_N \to X$ in distribution, $q<r\leqq\infty$, and $\|X_N\|_r$ is bounded, then $\|X_N\|_q \to \|X\|_q$ (by uniform integrability of $|X_N|^q$). Hence the central limit theorem of Salem and Zygmund mentioned above ([SZ] Theorem 3.5.2) shows that we may assume that q is an even integer in Theorem 4, and that it is sufficient to show that $\|D_N^{\pm}\|_q$ is almost surely *bounded.*

Let N_j be an increasing sequence of positive integers such that

$$\sum_{j=1}^{\infty} \log(N_j)/N_j < \infty. \tag{11}$$

Proposition 4 shows that

$$\sum_{j=1}^{\infty} E(\|D_{N_j}^{\pm}\|_q^q - E\|D_{N_j}^{\pm}\|_q^q)^2 < \infty,$$

so that

$$\sum_{j=1}^{\infty} (\|D_{N_j}^{\pm}\|_q^q - E\|D_{N_j}^{\pm}\|_q^q)^2 < \infty$$

almost surely; thus

$$\lim_{j\to\infty} (\|D_{N_j}^{\pm}\|_q^q - E\|D_{N_j}^{\pm}\|_q^q) = 0 \quad \text{almost surely.} \tag{12}$$

But it is immediate from Khinchine's inequality that $E\|D_N^{\pm}\|_q^q$ is bounded, so (12) shows that

$$\|D_{N_j}^{\pm}\|_q \quad \text{is almost surely bounded.} \tag{13}$$

(In fact (13) would suffice to prove Theorem 3.)

Now let us suppose that the sequence (N_j) satisfies

$$\frac{N_{j+1}-N_j}{(N_j)^{1/2}} \leqq c, \tag{14}$$

in addition to (11). (One might take $N_j \cong j^{3/2}$.) Suppose that $N_j \leqq N < N_{j+1}$. It follows that

$$\big|\|D_{N_j}^{\pm}\|_q - \|D_N^{\pm}\|_q\big| \leqq \|(D_{N_j}-D_N)^{\pm}\|_q \tag{15}$$

$$= \Big\|(N_j^{-1/2}-N^{-1/2})\sum_{n=1}^{N_j}\varepsilon_n e^{int} - N^{-1/2}\sum_{n=N_j+1}^{N}\varepsilon_n e^{int}\Big\|_q$$

$$\leqq \|D_{N_j}^{\pm}\|_q + (N_j)^{-1/2}(N_{j+1}-N_j),$$

so that $\|D_N\|_q$ is almost surely bounded, by (13) and (14). ∎

We can postpone the proof of Proposition 4 no longer. The basic idea comes from [HA], where extremely precise estimates on the almost sure behavior of $\|D_N^{\pm}\|_\infty$ are obtained using the "characteristic function" (the Fourier transform of the distribution). The key to Halász' argument is the fact that the two random variables

$D_N^{\pm}(t)$ and $D_N^{\pm}(s)$ are close to independent unless t is close to $\pm s$ (modulo 2π). Here "close to independent" means one thing in Halász' paper; for us it shall mean the following:

Lemma 6. *Suppose q is a positive even integer and define $D_N(t)$ as above. Let $\varphi_N(t)=E|D_N^{\pm}(t)|^q$. Then*

(16)
$$\left|E|D_N^{\pm}(t)|^q|D_N^{\pm}(s)|^q-\varphi_N(t)\varphi_N(s)\right| \le c\min\left(1, N^{-1}(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1})\right)$$

for $s,\ t\in\mathbf{R}$.

Proof. Note first that the left side of (16) is bounded, by Khinchine's inequality (together with the Schwarz inequality). We need to show that the left side of (16) is no larger than

$$cN^{-1}(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1}).$$

We use a "characteristic function" argument. It seems to be necessary to regard e as an element of $\mathbf{R}^2$ rather than $\mathbf{C}$. Fix s, t, and N. For $\xi=(\xi_1,\xi_2)\in\mathbf{R}^2$ define

(17)
$$\chi_1(\xi)=E\exp\left(i[\xi_1\operatorname{Re}(D_N^{\pm}(s))+\xi_2\operatorname{Im}(D_N^{\pm}(s))]\right),$$

and define $\chi_2(\zeta)$ similarly, with t and ζ in place of s and ξ.

For $(\xi,\zeta)=(\xi_1,\xi_2,\zeta_1,\zeta_2)\in\mathbf{R}^4$ define

(18)
$$\nu(\xi,\zeta)=E\exp\left(i[\xi_1\operatorname{Re}(D_N^{\pm}(s))+\xi_2\operatorname{Im}(D_N^{\pm}(s))+\zeta_1\operatorname{Re}(D_N^{\pm}(t))+\zeta_2\operatorname{Im}(D_N^{\pm}(t))]\right).$$

If α_1 and α_2 are non-negative integers, it follows that

(19)
$$i^{-(\alpha_1+\alpha_2)}\frac{\partial^{(\alpha_1+\alpha_2)}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}}\chi_1(0,0)=E\left((\operatorname{Re}(D_N^{\pm}(s)))^{\alpha_1}(\operatorname{Im}(D_N^{\pm}(s)))^{\alpha_2}\right),$$

with similar formulas for χ_2 and ν.

Now let $q=2k$; we are supposing that k is a positive integer. It follows from (19) and the corresponding formulas for χ_2 and ν that

(20)
$$E|D_N^{\pm}(s)|^q|D_N^{\pm}(t)|^q-\varphi_N(s)\varphi_N(t)=\Delta_\xi^k\Delta_\zeta^k\Psi(0,0,0,0),$$

if we define

$$\Psi(\xi_1,\xi_2,\zeta_1,\zeta_2)=\nu(\xi_1,\xi_2,\zeta_1,\zeta_2)-\chi_1(\xi_1,\xi_2)\chi_2(\zeta_1,\zeta_2),$$

and Δ_ξ denotes the operator $(\partial/\partial\xi_1)^2+(\partial/\partial\xi_2)^2$, with Δ_ζ defined similarly.

We shall obtain an upper bound on the right-hand side of (20) by using "Cauchy's Estimates": Extend Ψ to an *entire* function in $\mathbf{C}^4$ using (18), (17), and the corresponding formula for $\chi_2(\zeta_1, \zeta_2)$. We shall show that

(21) $$|\Psi(\xi_1, \xi_2, \zeta_1, \zeta_2)| = O\big(N^{-1}(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1})\big)$$
$$(\xi_j, \zeta_j \in \mathbf{C},\ |\xi_j|, |\zeta_j| \leqq 1/2).$$

Cauchy's Estimates lead from (21) to

(22) $$|\Delta_\xi^k \Delta_\zeta^k \Psi(0, 0, 0, 0)| = O\big(N^{-1}(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1})\big),$$

and (22) proves the lemma, by (20). (In (22) the notation Δ_ξ refers not to the Laplacian in $\mathbf{C}^2$, but rather to the operator $\Delta_\xi = (\partial/\partial\xi_1)^2+(\partial/\partial\xi_2)^2$.) We begin the estimates involved in (21).

Note first that

(23) $$\Big|\sum_{n=1}^N T_1(jt)T_2(js)\Big| \leqq c(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1}),$$

where T_1 and T_2 each denote either of the functions sin or cos (so that (23) includes four inequalities.) One may prove (23) by expressing the trigonometric functions in terms of exponentials and adding several geometric series.

Second, note that if $z_n \in \mathbf{C}$ with $\sum_{n=1}^N |z_n|^2 \leqq c$ and $\big|\sum_{n=1}^N z_n\big| \leqq c$ then

(24) $$\prod_{n=1}^N (1-z_n) = 1-\sum_{n=1}^N z_n + O\Big(\Big|\sum_{n=1}^N z_n\Big|^2+\sum_{n=1}^N |z_n|^2\Big),$$

where the constant implicit in the notation "O" is independent of N. (Write "$\Pi = \exp(\Sigma \log)$" and do some calculus.)

Independence shows that

(25) $$\chi_1(\xi_1, \xi_2) = \prod_{n=1}^N \cos\big(N^{-1/2}[\xi_1 \cos(ns)+\xi_2 \sin(ns)]\big)$$

and similarly for χ_2, while

(26)
$$\nu(\xi_1, \xi_2, \zeta_1, \zeta_2) = \prod_{n=1}^N \cos\big(N^{-1/2}[\xi_1\cos(ns)+\xi_2\sin(ns)+\zeta_1\cos(nt)+\zeta_2\sin(nt)]\big).$$

We apply the "addition formula" for the cosine in (26), divide by (25) and the corresponding formula for χ_2, and then apply (24) to see that

(27) $$\frac{\nu(\xi_1, \xi_2, \zeta_1, \zeta_2)}{\chi_1(\xi_1, \xi_2)\chi_2(\zeta_1, \zeta_2)}$$
$$= \prod_{n=1}^N \big\{1-\tan\big(N^{-1/2}[\xi_1\cos(ns)+\xi_2\sin(ns)]\big)\tan\big(N^{-1/2}[\zeta_1\cos(nt)+\zeta_2\sin(nt)]\big)\big\}$$
$$= \prod_{n=1}^N (1-\tan\tan) = 1-\sum_{n=1}^N \tan\tan + O\Big(\Big|\sum_{n=1}^N \tan\tan\Big|^2+\sum_{n=1}^N |\tan\tan|^2\Big);$$

here we have omitted the arguments of the function "tan".

Now the fact that $\tan(z)=z+O(|z^3|)$ $(|z|\leqq 1)$ shows that

$$\sum_{n=1}^{N}\tan\tan = \sum_{n=1}^{N}\left[N^{-1/2}(\xi_1\cos(ns)+\xi_2\sin(ns))\right] \tag{28}$$
$$\times\left[N^{-1/2}(\zeta_1\cos(nt)+\zeta_2\sin(nt))\right]+O\left(\sum_{n=1}^{N}N^{-3/2}N^{-1/2}\right)$$

for the values of ξ_j and ζ_j under consideration; the error here is of course $O(N^{-1})$. If we apply (28) to (27), along with easier estimates on the term $\Sigma\,|\tan\tan|^2$ we obtain

$$\frac{\nu(\xi_1,\xi_2,\zeta_1,\zeta_2)}{\chi_1(\xi_1,\xi_2)\chi_2(\zeta_1,\zeta_2)} \tag{29}$$
$$= 1-N^{-1}\sum_{n=1}^{N}[\xi_1\cos(ns)+\xi_2\sin(ns)][\zeta_1\cos(nt)+\zeta_2\sin(nt)]+O(N^{-1}).$$

Now the sum in (29) is equal to the sum of four sums of the sort considered in (23) above; thus, for $|\xi_j|, |\zeta_j|\leqq 1/2$ we have

$$\frac{\nu(\xi_1,\xi_2,\zeta_1,\zeta_2)}{\chi_1(\xi_1,\xi_2)\chi_2(\zeta_1,\zeta_2)} \tag{30}$$
$$= 1+N^{-1}O(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1})+O(N^{-1}) = 1+\vartheta,$$

where the last line is an abbreviation for the preceding.

If we rearrange (30) and recall the definition of Ψ we obtain

$$\Psi(\xi_1,\xi_2,\zeta_1,\zeta_2) = \vartheta\chi_1(\xi_1,\xi_2)\chi_2(\zeta_1,\zeta_2). \tag{31}$$

But note that $\cos(z)=1+O(|z|^2)$ for $z=O(1)$; now (24) shows that $\chi_j=O(1)$, by arguments like those above. Hence (31) implies that $\Psi=O(\vartheta)$, which is (21). ∎

It is easy to see that Lemma 6 implies Proposition 4:

Proof of Proposition 4. A few standard manipulations and Fubini's theorem show that

$$\sigma^2(\|D_N^{\pm}\|_q^q) = (2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi}\left(E|D_N^{\pm}(s)|^q|D_N^{\pm}(t)|^q-\varphi_N(s)\varphi_N(t)\right)ds\,dt, \tag{32}$$

where $\varphi_N(r)=E|D_N^{\pm}(r)|^q$, as in Lemma 6. Now that lemma shows that

$$\sigma^2(\|D_N^{\pm}\|_q^q)\leqq c\int_0^{2\pi}\int_0^{2\pi}\min\left(1, N^{-1}(|1-e^{i(s-t)}|^{-1}+|1-e^{i(s+t)}|^{-1})\right)ds\,dt$$
$$\leqq c\int_0^{2\pi}\min\left(1, N^{-1}|1-e^{it}|^{-1}\right)dt\leqq c\log(N)/N. \quad ∎$$

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