Noncommutative classical invariant theory

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Abstract. In this thesis, we consider some aspects of *noncommutative classical invariant theory*, i.e., noncommutative invariants of *the classical group* $SL(2, k)$. We develop a *symbolic method* for invariants and covariants, and we use the method to compute some invariant algebras. The subspace $\tilde{I}_d^m$ of the noncommutative invariant algebra $\tilde{I}_d$ consisting of homogeneous elements of degree m has the structure of a module over the *symmetric group* S_m. We find the explicit decomposition into irreducible modules. As a consequence, we obtain the *Hilbert series* of the commutative classical invariant algebras. The *Cayley—Sylvester theorem* and the *Hermite reciprocity law* are studied in some detail. We consider a new power series $\tilde{H}(\tilde{I}_d, t)$ whose coefficients are the number of irreducible S_m-modules in the decomposition of $\tilde{I}_d^m$, and show that it is rational. Finally, we develop some analogues of all this for covariants.

Abstract

In this thesis, we consider noncommutative invariants of the classical group $SL(2, k)$. We develop a symbolic method, and with the help of this method we compute some invariant algebras. The invariant algebras are stable under permutations of the factors in homogeneous elements, and we decompose the homogeneous subspaces into irreducible modules over the symmetric group. We study the Cayley—Sylvester theorem and the Hermite reciprocity law in some detail, and we introduce a "false" Hilbert series, whose coefficients are not dimensions, but the number of irreducible components in the decomposition into irreducible modules over the symmetric groups. Finally, we consider classical covariants.

Foreword

In this thesis, I will discuss some aspects of the noncommutative invariant theory of the classical group $SL(2, k)$. This subject was suggested to me by my teacher, Dr. Gert Almkvist, during a series of seminars on invariant theory held by him. I would like to thank him for many stimulating discussions and much invaluable advice.

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Introduction and preliminaries

Classical invariant theory is concerned with the invariants of the group $SL(2, k)$, where k is an algebraically closed field of characteristic 0. This group will always be denoted by G in the sequel.

The most classical part of the subject treats commutative invariants. The foundations of this theory were laid by Cayley and Sylvester in the 1840's, and it was further developed by, among others, Aronhold, Clebsch, Gordan, and Hilbert. In later years, noncommutative invariants have attracted some interest, see, e.g., [3], [4], [9], [12], [13], [15], [24].

In this thesis, we will discuss some aspects of the theory of noncommutative invariants and covariants. We will develop a symbolic method for noncommutative invariants, and it will be seen that this method is not essentially different from its commutative counterpart. In fact, had the 19th century invariant theorists considered noncommutative invariants, they would have developed the method in this case, too. In the commutative case, Gordan proved that the algebra of invariants if finitely generated (the famous Endlichkeitssatz, which was extended to $SL(n, k)$ by Hilbert). Unfortunately, this is not true in the noncommutative case. But we have something that is almost as good: the algebra of noncommutative invariants is finitely generated if we allow permutations of the factors in homogeneous polynomials. This has been proved by Koryukin [14]. Hence, it should be interesting to study the invariant algebras taking into account this new structure (which is degenerate in the commutative case). We will consider some aspects of this after we have developed the symbolic method.

Let us start by reviewing the representation theory of the group $SL(2, k)$ and of the symmetric groups S_m, since this theory and the theory of symmetric functions will be extensively used throughout our discussion.

Fundamentals on the representation theory of $SL(2, k)$

The group $G=SL(2, k)$ is reductive, hence every finite-dimensional, rational G-module is completely reducible. There is precisely one irreducible G-module R_d of dimension $d+1$ for every integer $d\geqq 0$. This module can be described as follows: let V be the standard G-module with basis e_1, e_2, and let $e_1^*=X$, $e_2^*=Y$ be the dual basis in V^* (the dual space). On V^* G acts by

$$\begin{cases} g^{-1}\cdot X = aX+bY \\ g^{-1}\cdot Y = cX+dY \end{cases}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

Then

$$R_d = S^d(V^*),$$

the dth symmetric power of V^*, i.e.,

$$R_d = \left\{a_0X^d + a_1\binom{d}{1}X^{d-1}Y + \ldots + a_dY^d;\ a_i \in k\right\}.$$

Hence

$$S(V^*) = \bigoplus_{d\geqq 0} S^d(V^*) = \bigoplus_{d\geqq 0} R_d \cong k[X, Y],$$

the polynomial algebra in X, Y. An expression of the type

$$a_0X^d + a_1\binom{d}{1}X^{d-1}Y + \ldots + a_dY^d$$

is called a *binary form of degree d*. For the details and the proofs of all this, we refer to [21]. We denote the G-character of R_d by χ_d. The subgroup

$$T = \left\{\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix};\ \xi \in k^*\right\}$$

plays an important role in the theory, and to simplify notation, we write

$$\chi_d\left(\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}\right) = \chi_d(\xi).$$

It is easily seen that

$$\chi_d(\xi) = \xi^d + \xi^{d-2} + \ldots + \xi^{-d} = \frac{\xi^{d+1} - \xi^{-(d+1)}}{\xi - \xi^{-1}}.$$

The algebra of noncommutative polynomials in the coefficients a_i will be identified with the tensor algebra

$$T(R_d^*) = \bigoplus_{m\geqq 0} T^m(R_d^*),$$

and the commutative algebra is identified with

$$S(R_d^*) = \bigoplus_{m\geqq 0} S^m(R_d^*).$$

It is convenient to regard $S^m(R_d^*)$ as the subspace of symmetric tensors in $T^m(R_d^*)$. The group G acts on these algebras, and we denote the invariant algebras by

$$T(R_d^*)^G = \tilde{I}_d = \bigoplus_{m\geqq 0} \tilde{I}_d^m, \quad \text{and} \quad S(R_d^*)^G = I_d = \bigoplus_{m\geqq 0} I_d^m.$$

Another object that we are going to study is the algebra of (noncommutative) covariants $\tilde{C}_d$. It is defined by

$$\tilde{C}_d = (T(R_d^*) \otimes_k R)^G,$$

where $R = k[X, Y]$. Its commutative counterpart is

$$C_d = (S(R_d^*) \otimes_k R)^G.$$

These two algebras are bi-graded,

$$\tilde{C}_d = \bigoplus_{m,e \geqq 0} \tilde{C}_{dme}, \quad \text{and} \quad C_d = \bigoplus_{m,e \geqq 0} C_{dme},$$

where

$$\tilde{C}_{dme} = (T^m(R_d^*) \otimes_k R_e)^G, \quad \text{and} \quad C_{dme} = (S^m(R_d^*) \otimes_k R_e)^G,$$

respectively.

When $A = \bigoplus_{m \geqq 0} A_m$ is a graded k-algebra, we denote its Hilbert series (sometimes called Poincaré series) by $H(A, t)$, i.e.

$$H(A, t) = \sum_{m \geqq 0} (\dim_k A_m) t^m$$

(provided that $\dim_k A_m < \infty$, of course). This series is an element of the formal power series ring $Z[[t]]$, but we will sometimes treat t as a real or complex variable.

A useful device when dealing with Hilbert series is the *Reynolds operator:* consider the field extension $\mathbf{C}(t^n) \to \mathbf{C}(t)$, which is Galois with Galois group generated by $t \mapsto \exp(2\pi i/n)t$. If $f \in \mathbf{C}(t)$, we define the Reynolds operator φ_n by

$$(\varphi_n f)(t^n) = \frac{1}{n} \sum_{k=1}^{n} f(\exp(2k\pi i/n)\, t).$$

Since the right-hand side is fixed by the Galois group, it is clear that it lies in $\mathbf{C}(t^n)$. If f is represented by a power series $\sum a_k t^k$, φ_n has the effect of killing all terms $a_k t^k$ such that $n \nmid k$, whence

$$(\varphi_n f)(t) = \sum_{k \geqq 0} a_{nk} t^k.$$

When defining the Hilbert series of the covariant algebras, we use the grading in the first component, i.e.,

$$H(\tilde{C}_d, t) = \sum_{m \geqq 0} (\dim_k \tilde{C}_{dm}) t^m,$$

where $\tilde{C}_{dm} = \bigoplus_{e \geqq 0} \tilde{C}_{dme}$ (it will later be seen that $\dim_k \tilde{C}_{dm} < \infty$). The Hilbert series of I_d and C_d were studied in the 19th century, and it is well-known that they are rational. Suppose M is a finite-dimensional, rational G-module with character χ_M. Write

$$\chi_M\left(\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}\right) = \chi_M(\xi).$$

We can write

$$\chi_M(\xi) = \sum_{l \geqq 0} \alpha_l \frac{\xi^{l+1} - \xi^{-(l+1)}}{\xi - \xi^{-1}},$$

where the α_l are non-negative integers, and only finitely many are non-zero.

The set of G-invariants of M is the set

$$M^G = \{m \in M;\ g \cdot m = m \text{ for all } g \in G\},$$

and we have

$$\dim_k M^G = \alpha_0,$$

since $R_0^G = R_0$ and $R_d^G = 0$ for $d \geqq 1$.

Writing

$$\frac{\xi^{l+1} - \xi^{-(l+1)}}{\xi - \xi^{-1}} = \xi^l + \xi^{l-2} + \ldots + \xi^{-l},$$

we see that α_0 is the difference between the coefficients of 1 and ξ^2 (or ξ^{-2}) in $\chi_M(\xi)$. It is convenient to let

$$\int : Z[\xi, \xi^{-1}] \to Z$$

denote the "coefficient of 1" map (see [4]). In particular, we have

$$\alpha_0 = \int (1 - \xi^{-2}) \chi_M(\xi) = \int (1 - \xi^2) \chi_M(\xi) = \tfrac{1}{2} \int (2 - \xi^2 - \xi^{-2}) \chi_M(\xi).$$

If we put $\xi = e^{ix}$, then

$$\frac{\xi^{l+1} - \xi^{-(l+1)}}{\xi - \xi^{-1}} = \frac{\sin(l+1)x}{\sin x},$$

whence

$$\alpha_0 = \frac{1}{\pi} \int_0^{2\pi} \sin^2 x \, \chi_M(e^{ix}) \, dx.$$

Hence $\int$ is an integral in the usual sense. We will use $\int$ and the analytical counterpart interchangeably.

Symmetric functions and symmetric groups

Here we will only give the most basic definitions, and we refer to Macdonald's book [17] for a full treatment of this very useful theory.

The symmetric group on n letters will be denoted by S_n and the ring of symmetric functions in n variables $Z[x_1, \ldots, x_n]^{S_n}$ by Λ_n. By Λ we denote the ring of symmetric functions in countably many variables (see [17] for the definition of Λ). A Z-basis for Λ is usually indexed by partitions λ. The bases that will appear here are: the monomial symmetric functions m_λ, the complete symmetric functions h_λ, the elementary symmetric functions e_λ, and the Schur functions s_λ.

We denote the transition matrix between the bases s_λ and m_λ by K, and this matrix is called the Kostka matrix. Its elements are also indexed by partitions, $K = (K_{\lambda\mu})$, and $K_{\lambda\mu}$ is the number of tableaux of shape λ and weight μ. The transition matrices between the other bases can be found in [17], p. 56. There is an involution ω on the ring Λ given by

$$\omega(e_r) = h_r.$$

Its effect on the Schur functions is especially important; it is given by

$$\omega(s_\lambda) = s_{\lambda'},$$

where λ' is the conjugate partition of λ.

We denote by R_n (not to be confused with R_d, the irreducible G-module of dimension $d+1$; we still insist on using Macdonald's notation) the Z-module of generalized characters on S_n, and we let

$$R = \bigoplus_{n \geq 0} R_n.$$

The module R has a ring structure, where the multiplication is defined by the induction product: if $f \in R_n$, $g \in R_m$, then their induction product is

$$f \cdot g = \mathrm{ind}_{S_n \times S_m}^{S_{n+m}} (f \times g).$$

The rings Λ and R are isomorphic, and the isomorphism is given by the characteristic map ch: $R \to \Lambda$. The elements χ^λ of R_n defined by $\mathrm{ch}(\chi^\lambda) = s_\lambda$ (where $|\lambda| = n$) are the irreducible characters of S_n. Then $\chi^{(n)}$ is the trivial character, and $\chi^{(1^n)}$ is the sign character. The involution ω on Λ corresponds to multiplication by $\chi^{(1^n)}$ on R_n, i.e.,

$$\chi^{\lambda'} = \chi^{(1^n)} \chi^\lambda.$$

We let M^λ be the irreducible S_n-module with character χ^λ.

Gaussian polynomials

The Gaussian polynomials (or q-binomial coefficients) $\begin{bmatrix} n \\ r \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}(q) = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-r+1})}{(1-q)(1-q^2)\ldots(1-q^r)}.$$

Obviously

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}.$$

There are two generating functions:

$$\Pi_{i=0}^{n-1} (1+q^i t) = \Sigma_{r=0}^{n} q^{1/2 r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix}(q)\, t^r,$$

and

$$\Pi_{i=0}^{n-1} (1-q^i t)^{-1} = \Sigma_{r=0}^{\infty} \begin{bmatrix} n+r-1 \\ r \end{bmatrix}(q)\, t^r.$$

The Gaussian polynomials are related to the symmetric functions by

$$e_r(1, q, \ldots, q^{n-1}) = q^{1/2r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix},$$

and

$$h_r(1, q, \ldots, q^{n-1}) = \begin{bmatrix} n+r-1 \\ r \end{bmatrix},$$

as can be seen from the generating functions.

For more information on these polynomials, see [17], and [1] for more about their use in invariant theory.

S-Algebras

Consider the free associative algebra

$$A = k\langle x_1, \ldots, x_n \rangle = \bigoplus_{m \geqq 0} A_m,$$

where A_m is the subspace consisting of homogeneous polynomials of degree m. The symmetric group S_m acts on A_m by permutation of the factors.

A subalgebra or ideal may or may not be closed under this action, e.g., the subalgebra $k\langle x_1 x_2 \rangle$ is not closed, since it does not contain $x_2 x_1$. Let us call a closed subalgebra (or ideal) an S-subalgebra (S-ideal). Often we will simply write S-algebra, when it is clear what the "big" algebra is.

Let us also say that an S-subalgebra B of $k\langle x_1, \ldots, x_n \rangle$ is finitely generated as an S-subalgebra if there is a finite set $\{f_1, \ldots, f_s\} \subseteq B$ such that B is the smallest S-algebra containing $\{f_1, \ldots, f_s\}$. If B is finitely generated as S-algebra, it does not have to be finitely generated as an algebra. For more information on S-algebras, see Koryukin's paper [14]. We now concentrate on the tensor algebra $T(R_d^*) \cong k\langle a_0, \ldots, a_d \rangle$. By the definition of the G-action, it is clear that the actions of G and the symmetric groups commute. Hence the invariant algebras $\tilde{I}_d$ are S-subalgebras, and the $\tilde{I}_d^m$:s are S_m-modules. Furthermore, if we let the symmetric group S_m act only on the first factor in $T^m(R_d^*) \otimes_k R_e$, it is clear that the same holds for $\tilde{C}_d$ and $\tilde{C}_{dme}$.

Finally, let us note that I_d^m is the maximal trivial sub-S_m-module of $\tilde{I}_d^m$.

The symbolic method

The symbolic method in the commutative classical invariant theory was developed by Aronhold, Clebsch, and Gordan in the 1860's. In [24], Teranishi describes a symbolic method for non-commutative invariants. Here we will develop the symbolic method for non-commutative classical invariants, and also for non-commutative classical covariants along the lines of Dieudonné—Carrell in [8]. Our description of the method will show that there is not really any difference between the commutative and the non-commutative cases.

1. The Method

Let V be the standard $SL(n, k)$-module with basis $e_1, \ldots, e_n$.

Definition. We define a multilinear function $V^n \to k$, denoted by $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n]$ by

$$[x_1, \ldots, x_n] = \det(\xi_{ij}),$$

where $x_i = \sum_{j=1}^n \xi_{ij} e_j$. We define a function $(y_1, \ldots, y_n) \mapsto [y_1, \ldots, y_n]$ from $(V^*)^n$ to k analogously. (These are sometimes called brackets.) Finally we define the scalar product $\langle x, y\rangle$ of x and y, where $x \in V$, $y \in V^*$, by $\langle x, y\rangle = y(x)$. This is a function $V \times V^* \to k$. Clearly the bracket functions and the scalar product function are $SL(n, k)$-invariant. In fact, if $g \in GL(n, k)$, then, informally,

$$g \cdot [x_1, \ldots, x_n] = (\det g)[x_1, \ldots, x_n],$$
$$g \cdot [y_1, \ldots, y_n] = (\det g)^{-1}[y_1, \ldots, y_n],$$
$$g \cdot \langle x, y\rangle = \langle x, y\rangle.$$

One of the cornerstones of classical invariant theory is the

Fundamental Theorem. Let $f\colon V^p \times (V^*)^q \to k$ be a multilinear form invariant under $SL(n, k)$. Then f is a linear combination of products of factors of the types

i) functions $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n]$ from V^n to k,
ii) functions $(y_1, \ldots, y_n) \mapsto [y_1, \ldots, y_n]$ from $(V^*)^n$ to k,
ii) functions $(x, y) \mapsto \langle x, y\rangle$ from $V \times V^*$ to k.

For the proof, we refer to [8].

By the formula

$$(y_1 \otimes \ldots \otimes y_m)(x_1, \ldots, x_m) = \langle x_1, y_1\rangle \ldots \langle x_m, y_m\rangle,$$

$x_i \in V$, $y_i \in V^*$, we identify the tensor space $T^m(V^*)$ with the space of m-linear forms on V. The with symmetric power $S^m(V^*)$ then corresponds to the subspace consisting of symmetric m-linear forms.

We now restrict our attention to the classical case $n=2$. We have earlier defined

$$R_d = S^d(V^*),$$

$$\tilde{C}_{dme} = (T^m(R_d^*)\otimes_k R_e)^G,$$

and

$$\tilde{I}_d^m = \tilde{C}_{dm0}.$$

If $g\in T^m(R_d^*)$, then g is an m-linear form on R_d. Let

$$\varphi\colon T^d(V^*) \to S^d(V^*)$$

be the projection. Then we get an m-linear form $\varphi^* g$ on $T^d(V^*)$ by

$$(\varphi^* g)(z_1, \dots, z_m) = g(\varphi z_1, \dots, \varphi z_m).$$

If we only consider decomposable tensors $z_i = y_{i1}\otimes \dots \otimes y_{id}$, we get an md-linear form ω_g on V^* by

$$\omega_g(y_{11}, \dots, y_{1d}, y_{21}, \dots, y_{md}) = (\varphi^* g)(z_1, \dots, z_m).$$

If $h\in R_e$, we interpret h as a symmetric e-linear form on V. Hence an element $f=\sum (g_i\otimes h_i)$ of $T^m(R_d^*)\otimes_k R_e$ gives rise to a form

$$\omega_f\colon (V^*)^{md}\times V^e \to k,$$

and it is obvious that f is invariant under G if and only if ω_f is invariant. By the fundamental theorem ω_f is a linear combination of products of factors $[yy']$, $\langle x, y\rangle$. The form ω_f is called the *symbolic expression of f*.

Now we must describe how to get the invariant f from its symbolic expression ω_f. This process is known as *restitution*. Denote for the moment the basis in V^* by

$$e_1^* = X, \quad e_2^* = Y,$$

and write

$$y_{ij} = \eta_{ij1}e_1^* + \eta_{ij2}e_2^*.$$

Then

$$z_i = \sum\nolimits_{1\leqq k_1, \dots, k_d \leqq 2} \eta_{i1k_1}\cdots\eta_{idk_d}(e_{k_1}^*\otimes \dots \otimes e_{k_d}^*),$$

and the first step must be to replace every product $\eta_{i1k_1}\cdots\eta_{idk_d}$ by one sole coefficient $\eta_{ik_1\dots k_d}$. If we write the elements of R_d as

$$\sum\nolimits_{\nu=1}^{d} \binom{d}{\nu} a_\nu X^{d-\nu}Y^\nu = \sum\nolimits_{\nu=1}^{d} \binom{d}{\nu} a_\nu e_1^{*d-\nu} e_2^{*\nu},$$

and note that we are only interested in symmetric tensors z_i, we see that the next step is to replace $\eta_{ik_1\dots k_d}$ by $a_{i\nu}$, where ν is the number of k_j's equal to 2. Let e_1, e_2 be the basis in V to which e_1^*, e_2^* is the dual basis. Write

$$x_i = \xi_{i1}e_1 + \xi_{i2}e_2.$$

In the expression for $\omega_f(y_{11}, \ldots, y_{md}, x_1, \ldots, x_e)$ we finally replace every product $\xi_{1i_1} \cdots \xi_{ei_e}$ by $X^{e-\mu} Y^{\mu}$, where μ is the number of i_j's equal to 2.

We can simplify the restitution process if we already from the beginning consider symmetric tensors of the form

$$z_i = y_i \otimes \ldots \otimes y_i$$

with d factors. Similarly, we note that instead of $x_1, \ldots, x_e$ we can consider only one x, which appears e times in ω_f. By abuse of notation, we write $\omega_f(y_1, \ldots, y_m, x)$ for

$$\omega_f(y_1, \ldots, y_1, y_2, \ldots, y_m, x, \ldots, x)$$

with each y_i appearing d times and x appearing e times. Since each bracket [ ] and $\langle , \rangle$ contains two symbols (we call the x:s and y:s *symbols*), it is clear that $md-e$ must be even for any covariants to exist. Consequently, $\tilde{I}_d^m = 0$ if md is odd.

2. Some Examples

Example 1. Let $f \in \tilde{I}_d^2$. To get the symbolic expression for f we have to put d y_1's and d y_2's into $\frac{1}{2} \cdot 2d = d$ brackets [ ] (there are no x:s involved here). Since $[y, y] = 0$, the only case we need consider is

$$\omega_f = [y_1, y_2]^d.$$

Hence

$$\omega_f = (\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^d = \sum_{i=0}^d \binom{d}{i} (-1)^i \eta_{11}^{d-i} \eta_{12}^i \eta_{21}^i \eta_{22}^{d-i},$$

and the restitution consists in replacing $\eta_{11}^{d-i}\eta_{12}^i$ by a_{1i} and $\eta_{21}^i \eta_{22}^{d-i}$ by a_{2d-i}. We then obtain

$$f\left(\sum \binom{d}{i} a_{1i} X^{d-i} Y^i, \sum \binom{d}{i} a_{2i} X^{d-i} Y^i\right) = \sum_{i=0}^d \binom{d}{i} (-1)^i a_{1i} a_{2d-i}.$$

As an element of $T^2(R_d^*)$,

$$f = \sum_{i=0}^d \binom{d}{i} (-1)^i a_i a_{d-i}.$$

In particular, we have $\dim_k \tilde{I}_d^2 = 1$.

Example 2. Let d be even, $d = 2q$, and let $f \in \tilde{I}_d^3$. We obtain the symbolic expression ω_f by putting d y_1's, d y_2's, and dy_3's into $\frac{1}{2} \cdot 3 \cdot d = 3q$ brackets [ ]. We need only consider the case

$$\omega_f = [y_1 y_2]^q [y_1 y_3]^q [y_2 y_3]^q =$$
$$= (\eta_{11}\eta_{22} - \eta_{12}\eta_{21})^q (\eta_{11}\eta_{32} - \eta_{12}\eta_{31})^q (\eta_{21}\eta_{32} - \eta_{22}\eta_{31})^q$$
$$= (-1)^q \sum_{i,j,k=0}^{q} \binom{q}{i}\binom{q}{j}\binom{q}{k} (-1)^{i+j+k} \eta_{11}^{q-i+j} \eta_{12}^{q+i-j} \eta_{21}^{q+i-k} \eta_{22}^{q-i+k} \eta_{31}^{q-j+k} \eta_{32}^{q+j-k}.$$

In the restitution we replace $\eta_{11}^{q-i+j}\eta_{12}^{q+i-j}$ by a_{1q+i-j}, etc., whence

$$f = \sum_{i,j,k=0}^{q} \binom{q}{i}\binom{q}{j}\binom{q}{k} (-1)^{i+j+k} a_{q+i-j}\, a_{q-i+k}\, a_{q+j-k},$$

as an element of $T^3(R_{2q}^*)$. In particular, $\dim_k \tilde{I}_d^3 = 1$ if d is even, and 0 if d is odd.

Example 3. To obtain ω_f when $f \in \tilde{I}_d^{d+1}$, we must put $d\, y_1$'s, ..., $d\, y_{d+1}$'s into $\frac{1}{2} d(d+1)$ brackets. One possibility is

$$\omega_f = \prod_{1 \leq i < j \leq d+1} [y_i y_j] = \prod_{1 \leq i < j \leq d+1} (\eta_{i1}\eta_{j2} - \eta_{i2}\eta_{j1}),$$

which is the expansion of the Vandermonde determinant

$$\det (\eta_{i1}^{d+1-j} \eta_{i2}^{j-1})_{1 \leq i, j \leq d+1}$$

whence

$$\omega_f = \sum_{\sigma \in S_{d+1}} (\operatorname{sgn} \sigma) \eta_{11}^{d+1-\sigma(1)} \eta_{12}^{\sigma(1)-1} \ldots \eta_{d+11}^{d+1-\sigma(d+1)} \eta_{d+12}^{\sigma(d+1)-1},$$

which restitutes to the *standard polynomial*

$$s_n = \sum_{\sigma \in S_{d+1}} (\operatorname{sgn} \sigma)\, a_{\sigma(0)}\, a_{\sigma(1)} \ldots a_{\sigma(d)}.$$

In the last sum S_{d+1} acts on the set $\{0, 1, ..., d\}$. The invariants in the above examples are also discussed in [4], p. 207—208, and in [24], p. 9.

Example 4. Consider $f \in \tilde{C}_{d1d}$. To obtain ω_f we must put $d\, y_1$'s and $d\, x$'s into $d\langle\, ,\rangle$:s and $\frac{1}{2}(1 \cdot d - d) = 0$ brackets [ ]. Hence

$$\omega_f = \langle x, y_1 \rangle^d = (\eta_{11}\xi_1 + \eta_{12}\xi_2)^d = \sum_{i=0}^{d} \binom{d}{i} \eta_{11}^i \eta_{12}^{d-i} \xi_1^i \xi_2^{d-i},$$

wherefore

$$f = \sum_{i=0}^{d} \binom{d}{i} a_{d-i} X^i Y^{d-i},$$

i.e., the binary form itself. This element will play an important role later, and we will denote it by γ (this element appears in the commutative case too, see [21], p. 55). In fact, we will show later that the covariant algebra $\tilde{C}_d$ in a certain sense is generated by γ, a theorem that was proved by Gordan in the commutative case (see [10], p. 48 and p. 110).

Example 5. If $f \in \tilde{C}_{122}$, then

$$\omega_f = \langle x, y_1\rangle\langle x, y_2\rangle = (\eta_{11}\xi_1+\eta_{12}\xi_2)(\eta_{21}\xi_1+\eta_{22}\xi_2)$$
$$= \eta_{11}\eta_{21}\xi_1^2+(\eta_{11}\eta_{22}+\eta_{12}\eta_{21})\xi_1\xi_2+\eta_{12}\eta_{22}\xi_2^2.$$

Thus

$$f = a_0^2X^2+(a_0a_1+a_1a_0)XY+a_1^2Y^2.$$

Example 6. If $f \in \tilde{C}_{222}$, then

$$\omega_f = [y_1y_2]\langle x, y_1\rangle\langle x, y_2\rangle$$
$$= (\eta_{11}^2\eta_{21}\eta_{22}-\eta_{11}\eta_{12}\eta_{21}^2)\xi_1^2+(\eta_{11}^2\eta_{22}^2-\eta_{12}^2\eta_{21}^2)\xi_1\xi_2+(\eta_{11}\eta_{12}\eta_{22}^2-\eta_{12}^2\eta_{21}\eta_{22})\xi_2^2$$

and

$$f = (a_0a_1-a_1a_0)X^2+(a_0a_2-a_2a_0)XY+(a_1a_2-a_2a_1)Y^2.$$

Example 7. If $f \in \tilde{C}_{422}$, we get ω_f by putting $4\, y_1$:*s*, $4\, y_2$:*s*, and $2\, x$:*s* into $2\ \langle , \rangle$:*s*, and $\frac{1}{2}(4\cdot 2-2)=3$ brackets [ ]. Hence

$$\omega_f = [y_1y_2]^3\langle x, y_1\rangle\langle x, y_2\rangle$$
$$= (\eta_{11}^4\eta_{21}\eta_{22}^3-3\eta_{11}^3\eta_{12}\eta_{21}^2\eta_{22}^2+3\eta_{11}^2\eta_{12}^2\eta_{21}^3\eta_{22}-\eta_{11}\eta_{12}^3\eta_{21}^4)\xi_1^2$$
$$+(\eta_{11}^4\eta_{22}^4-2\eta_{11}^3\eta_{12}\eta_{21}\eta_{22}^3+2\eta_{11}\eta_{12}^3\eta_{21}^3\eta_{22}-\eta_{12}^4\eta_{21}^4)\xi_1\xi_2$$
$$+(\eta_{11}^3\eta_{12}\eta_{22}^4-3\eta_{11}^2\eta_{12}^2\eta_{21}\eta_{22}^3+3\eta_{11}\eta_{12}^3\eta_{21}^2\eta_{22}^2-\eta_{12}^4\eta_{21}^3\eta_{22})\xi_2^2,$$

which after restitution gives

$$f = (a_0a_3-3a_1a_2+3a_2a_1-a_3a_0)X^2+(a_0a_4-2a_1a_3+2a_3a_1-a_4a_0)XY$$
$$+(a_1a_4-3a_2a_3+3a_3a_2-a_4a_1)Y^2.$$

One may note that the covariants in the last two examples abelianize to 0.

3. Two Remarks

As was noted earlier, the symmetric group S_m acts on $\tilde{C}_{dme}$ by permutation of the a_i's. Let $\sigma \in S_m$ and $f \in \tilde{C}_{dme}$. It is clear that if

$$\omega_f = [y_1y_2]^{p_{12}}[y_1y_3]^{p_{13}}\dots,$$

then

$$\omega_{\sigma f} = [y_{\sigma(1)}y_{\sigma(2)}]^{p_{12}}[y_{\sigma(1)}y_{\sigma(3)}]^{p_{13}}\dots .$$

Hence, if we consider $S^m(R_d^*)$ as the subspace of symmetric tensors in $T^m(R_d^*)$, we see that the symbolic expressions for the elements of C_{dme} are precisely those which are symmetric in the symbols y_i.

Let us finally record two identities which will be very useful later:

i) $[y_1y_2]\langle x, y_3\rangle+[y_3y_1]\langle x, y_2\rangle+[y_2y_3]\langle x, y_1\rangle = 0$

ii) $[y_1y_4][y_2y_3]+[y_2y_4][y_3y_1]+[y_3y_4][y_1y_2] = 0.$

The former is proved by direct computation, and then the latter follows by letting $\xi_1\mapsto\eta_{42}$ and $\xi_2\mapsto-\eta_{41}$.

Some applications of the symbolic method

1. Some Results on the S-Algebra Structure of $\tilde{I}_d$ and $\tilde{C}_d$, $d=1,2$

Since $G=SL(2,k)$ is reductive, it follows from [14] that the algebras $\tilde{I}_d$ and $\tilde{C}_d$ are finitely generated as S-subalgebras of $T(R_d^*)$ and $T(R_d^*)\otimes_k R$, respectively. In this section we are going to determine S-algebra generators of $\tilde{I}_d$ and $\tilde{C}_d$ for $d=1$ and $d=2$, i.e., (finitely many) invariants and covariants which together with the ordinary algebra operations and permutations generate these algebras, and thereby we will show the power of the symbolic method.

Proposition 1.1. *$\tilde{I}_1$ is generated by $a_0a_1-a_1a_0$ as an S-algebra.*

Proof. Let $f\in\tilde{I}_1^m$. For any invariants to exist, m must be even, $m=2q$, say. To obtain the symbolic expression for f we must put one $y_1, \dots,$ one y_{2q} into $\frac{1}{2}\cdot 1\cdot 2q=q$ brackets [ ]. One possibility is

$$\omega_f = [y_1y_2][y_3y_4]\dots[y_{2q-1}y_{2q}],$$

and it is clear that all other possibilities are permutations of this one. Now

$$[y_1y_2] = \eta_{11}\eta_{22}-\eta_{12}\eta_{21},$$

which restitutes to $a_0a_1-a_1a_0$, whence ω_f restitutes to

$$f = (a_0a_1-a_1a_0)^q. \qquad \text{Q.E.D.}$$

Later we will prove more on the S-structure of $\tilde{I}_1$ (Proposition 2.1 below).

Proposition 1.2. *$\tilde{C}_1$ is generated by $a_0a_1-a_1a_0$ and $\gamma=a_0X+a_1Y$ as an S-algebra.*

Proof. Let $f\in\tilde{C}_{1me}$. To obtain ω_f we must put one $y_1, \dots,$ one y_m, and e x's into $\frac{1}{2}(m-e)$ brackets [ ] and $e\langle\,,\rangle$'s. Hence $m-e$ must be even, $m-e=2q$, say. One possibility is

$$\omega_f = [y_1y_2]\dots[y_{m-e-1}y_{m-e}]\langle x, y_{m-e+1}\rangle\dots\langle x, y_m\rangle,$$

and this is obviously the only possibility modulo permutations. Noting that ω_f restitutes to

$$(a_0 a_1 - a_1 a_0)^q (a_0 X + a_1 Y)^e,$$

the proposition is proved. Q.E.D.

Proposition 1.3. *$\tilde{I}_2$ is generated by the noncommutative discriminant*

$$\Delta = a_0 a_2 - 2a_1^2 + a_2 a_0$$

and the standard polynomial

$$s_3 = a_0 a_1 a_2 - a_0 a_2 a_1 + a_1 a_2 a_0 - a_1 a_0 a_2 + a_2 a_0 a_1 - a_2 a_1 a_0$$

as an S-algebra.

Proposition 1.4. *$\tilde{C}_2$ is generated by Δ, s_3,*

$$\gamma = a_0 X^2 + 2a_1 XY + a_2 Y^2,$$

and

$$\delta = (a_0 a_1 - a_1 a_0) X^2 + (a_0 a_2 - a_2 a_0) XY + (a_1 a_2 - a_2 a_1) Y^2$$

as an S-algebra.

Before the proofs of these propositions, we need a lemma on symbolic expressions.

Lemma. *Assume $m \geqq 5$. Let*

$$\omega = [y_1 y_2][y_2 y_3] \cdots [y_{m-1} y_m] \langle x, y_1 \rangle \langle x, y_m \rangle,$$

$$\omega_1 = [y_1 y_2]^2 [y_3 y_4][y_4 y_5] \cdots [y_{m-1} y_m] \langle x, y_3 \rangle \langle x, y_m \rangle,$$

$$\omega_2 = [y_3 y_4]^2 [y_1 y_2][y_2 y_5] \cdots [y_{m-1} y_m] \langle x, y_1 \rangle \langle x, y_m \rangle,$$

$$\omega_3 = [y_1 y_3]^2 [y_2 y_4][y_4 y_5] \cdots [y_{m-1} y_m] \langle x, y_2 \rangle \langle x, y_m \rangle,$$

$$\omega_4 = [y_2 y_4]^2 [y_1 y_3][y_3 y_5] \cdots [y_{m-1} y_m] \langle x, y_1 \rangle \langle x, y_m \rangle,$$

$$\omega_5 = [y_1 y_4]^2 [y_2 y_3][y_3 y_5] \cdots [y_{m-1} y_m] \langle x, y_2 \rangle \langle x, y_m \rangle,$$

$$\omega_6 = [y_2 y_3]^2 [y_1 y_4][y_4 y_5] \cdots [y_{m-1} y_m] \langle x, y_1 \rangle \langle x, y_m \rangle$$

and

$$\omega_0 = [y_1 y_2][y_2 y_3][y_3 y_4][y_1 y_4][y_5 y_6] \cdots [y_{m-1} y_m] \langle x, y_5 \rangle \langle x, y_m \rangle.$$

Then

$$2\omega = \omega_0 - \omega_1 + \omega_3 - \omega_5 - \omega_2 + \omega_4 - \omega_6.$$

Proof of the Lemma. All the symbolic expressions in the lemma contain a common part, namely

$$[y_5 y_6] \cdots [y_{m-1} y_m] \langle x, y_m \rangle,$$

which we won't write out in our computations below. We have

$$\omega+\omega_1-\omega_3+\omega_5$$
$$=[y_1y_2]\dots[y_4y_5]\langle x, y_1\rangle+[y_1y_2]^2[y_3y_4][y_4y_5]\langle x, y_3\rangle-\omega_3+\omega_5$$
$$=[y_1y_2][y_3y_4][y_4y_5]([y_2y_3]\langle x, y_1\rangle+[y_1y_2]\langle x, y_3\rangle)-\omega_3+\omega_5$$
$$=[y_1y_2][y_1y_3][y_3y_4][y_4y_5]\langle x, y_2\rangle-[y_1y_3]^2[y_2y_4][y_4y_5]\langle x, y_2\rangle+\omega_5$$
$$=[y_1y_3][y_4y_5]\langle x, y_2\rangle([y_1y_2][y_3y_4]-[y_1y_3][y_2y_4])+\omega_5$$
$$=-[y_1y_3][y_1y_4][y_2y_3][y_4y_5]\langle x, y_2\rangle+[y_1y_4]^2[y_2y_3][y_3y_5]\langle x, y_2\rangle$$
$$=[y_1y_4][y_2y_3]\langle x, y_2\rangle(-[y_1y_3][y_4y_5]+[y_1y_4][y_3y_5])$$
$$=[y_1y_4][y_2y_3][y_1y_5][y_3y_4]\langle x, y_2\rangle.$$

Here we have repeatedly used the identities on p. 140. With the same technique we can prove that

$$\omega+\omega_2-\omega_4+\omega_6$$
$$=[y_1y_4][y_2y_3][y_2y_5][y_4y_3]\langle x, y_1\rangle.$$

Hence

$$2\omega+\omega_1-\omega_3+\omega_5+\omega_2-\omega_4+\omega_6$$
$$=[y_1y_4][y_2y_3][y_3y_4]([y_1y_5]\langle x, y_2\rangle-[y_2y_5]\langle x, y_1\rangle)$$
$$=[y_1y_2][y_2y_3][y_3y_4][y_1y_4]\langle x, y_5\rangle=\omega_0 \qquad \text{Q.E.D.}$$

If we replace $\langle x, y_1\rangle\langle x, y_m\rangle$ in ω by $[y_1y_m]$, $\langle x, y_3\rangle\langle x, y_m\rangle$ in ω_1 by $[y_3y_m]$, etc., we get some new symbolic expressions, and the same computations as above show that the same relation holds between these new expressions. Later when we have introduced transvectants, this will be clear without any computations.

Proof of Prop. 1.3. The symbolic expressions for Δ and s_3 are

$$\omega_\Delta=[y_1y_2]^2,$$

and

$$\omega_{s_3}=[y_1y_2][y_2y_3][y_1y_3].$$

Let A be the algebra generated by Δ and a_3 and operations with the symmetric groups. We will first prove that $\tilde{I}_2^m\subseteq A$ for $m=2, 3, 4$.

$m=2$. The only possibility is ω_Δ.

$m=3$. The only possibility is ω_{s_3}.

$m=4$. Modulo permutations there are the possibilities

$$\omega_1=[y_1y_2]^2[y_3y_4]^2, \quad \text{and} \quad \omega_2=[y_1y_2][y_2y_3][y_3y_4][y_1y_4].$$

Clearly $\omega_1\in A$. Since

$$[y_1y_2][y_3y_4]+[y_1y_4][y_2y_3]=[y_1y_3][y_2y_4],$$

we have

$$2\cdot[y_1y_2][y_2y_3][y_3y_4][y_1y_4]$$
$$=[y_1y_3]^2[y_2y_4]^2-[y_1y_2]^2[y_3y_4]^2-[y_1y_4]^2[y_2y_3]^2,$$

whence $\omega_2\in A$, and $\tilde{I}_2^4\subseteq A$. We are going to prove that $\tilde{I}_2^m\subseteq A$ with induction over m. Suppose then that $\tilde{I}_2^k\subseteq A$ for $k<m$. Let us call a symbolic expression of the type

$$[y_{i_1}y_{i_2}][y_{i_2}y_{i_3}]\cdots[y_{i_{l-1}}y_{i_l}][y_{i_l}y_{i_1}]$$

a *cycle*. If $\omega\in\tilde{I}_2^m$ can be written as a product of two or more non-trivial cycles, then we are finished. Otherwise ω equals

$$[y_1y_2][y_2y_3]\cdots[y_{m-1}y_m][y_1y_m]$$

or a permutation of this cycle (here we may suppose that $m\geqq 5$). But by the remark following the lemma we can write ω as a linear combination of $\omega_0,\ldots,\omega_6$. Now $\omega_1,\ldots,\omega_6$ contain squares, and ω_0 is a product of two cycles. By the induction hypothesis, $\omega\in A$, and $\tilde{I}_2^m\subseteq A$. Q.E.D.

Proof of Prop. 1.4. The symbolic expressions are

$$\omega_\Delta=[y_1y_2]^2,\quad \omega_{s_3}=[y_1y_2][y_2y_3][y_1y_3],$$
$$\omega_\gamma=\langle x,y_1\rangle^2,\quad\text{and}\quad \omega_\delta=[y_1y_2]\langle x,y_1\rangle\langle x,y_2\rangle.$$

Let A be the algebra generated by these elements and operations with the symmetric groups. We will prove with induction over m that $\tilde{C}_{2me}\subseteq A$ for all m and e.

If $m=1$, then the only possibility is γ, and if $m=2$, then the only possibilities are Δ, δ, and γ^2. Suppose that $\tilde{C}_{2ke}\subseteq A$ for $k<m$, and let $f\in\tilde{C}_{2me}$ for some e. We may suppose that ω_f contains at least one scalar product $\langle\,,\rangle$, for otherwise f is an element of $\tilde{I}_2$, and this is generated by Δ and s_3. We may also suppose that ω_f doesn't contain any squares $[\ ]^2$, $\langle\,,\rangle^2$, for then we are finished by the induction hypothesis. Thus ω_f must be a product of cycles

$$[y_1y_2][y_2y_3]\cdots[y_{k-1}y_k]\langle x,y_1\rangle\langle x,y_k\rangle$$

and permutations of such cycles. If ω_f is a non-trivial product, then $f\in A$ by induction. Otherwise k equals m and

$$\omega_f=[y_1y_2]\cdots[y_{m-1}y_m]\langle x,y_1\rangle\langle x,y_m\rangle$$

or a permutation of this cycle. By the lemma, then, ω_f is a linear combination of $\omega_0,\ldots,\omega_6$. Since $\omega_1,\ldots,\omega_6$ contain squares, and ω_0 is a product of one element of $\tilde{I}_2$ and one element of some $\tilde{C}_{2m'e}$ with $m'<m$, we must have $\tilde{C}_{2me}\subseteq A$. Q.E.D.

Thus it seems to be much more difficult to find explicit S-algebra generators of $\tilde{I}_d$ and $\tilde{C}_d$ than to find algebra generators of their commutative counterparts.

This is at least partially due to the fact that the Hilbert series $H(\tilde{I}_d, t)$ and $H(\tilde{C}_d, t)$ seem to give little or no direct information on the degrees of the generators (see [4] and the last chapter below for some information on these Hilbert series).

2. *More on the Structure of the Algebra* $\tilde{I}_1$

As has been noted earlier, the space of invariants $\tilde{I}_1^{2q}$ has an S_{2q}-module structure, where S_{2q} acts by permuting the factors. We denote the irreducible S_{2q}-module corresponding to the partition λ of $2q$ by M^{λ}.

Proposition 2.1. *As S_{2q}-modules, $\tilde{I}_1^{2q} \cong M^{(q,q)}$.*

Proof. Let T be the tableau

1	3	...	$2q-1$
2	4	...	$2q$

corresponding to the partition (q, q) of $2q$. Let further $C_T(R_T)$ be the subgroup of S_{2q} stabilizing the columns (the rows) of T. Then

$$(a_0 a_1 - a_1 a_0)^q = e_T((a_0 a_1)^q),$$

where

$$e_T = \sum_{\substack{\pi \in C_T \\ \varrho \in R_T}} (\operatorname{sgn} \pi) \pi \varrho \in k[S_{2q}].$$

Hence

$$\tilde{I}_1^{2q} = k[S_{2q}] e_T((a_0 a_1)^q).$$

We have an S_{2q}-morphism

$$k[S_{2q}] e_T \to \tilde{I}_1^{2q}$$
$$\sigma \mapsto \sigma(a_0 a_1)^q),$$

which obviously is non-zero. Now e_T is a primitive idempotent of $k[S_{2q}]$ corresponding to projection onto the irreducible module $M^{(q,q)}$ (see [11]). Thus the above morphism is injective, and since it obviously is surjective, it is an isomorphism of S_{2q}-modules. Hence $\tilde{I}_1^{2q} \cong k[S_{2q}] e_T \cong M^{(q,q)}$. Q.E.D.

Corollary 2.2. $\dim_k \tilde{I}_1^{2q} = \dfrac{1}{q+1} \dbinom{2q}{q}$.

Proof. By [17], Ch. I, § 7 and § 6, Ex. 4, we have

$$\dim_k M^{(q,q)} = K_{(q,q),(1^{2q})} = (2q)!/h((q, q)),$$

where $h(\lambda)$ is the product of the hook lengths of the partition λ. Since

$$h((q,q)) = (q+1)q(q-1)\dots 2\cdot q(q-1)\dots 1$$
$$= (q+1)(q!)^2,$$

the corollary is proved. Q.E.D.

Hence, by a combinatorial coincidence (?), the dimensions of the invariant spaces $\tilde{I}_1^{2q}$ equal the Catalan numbers. From [7], p. 53, it follows that

$$H(\tilde{I}_1, t) = \frac{1}{2t^2}(1-\sqrt{1-4t^2}).$$

This is also proved in [4] by other methods.

3. *Gordan's Theorem in the Noncommutative Case*

Gordan, The King of Invariants, proved that the commutative algebra C_d can be generated by the element γ (see above, p. 138) and a certain kind of mappings $C_d \times C_d \to C_d$ called transvectants (*Überschiebungen* in German). We are here going to extend this theorem to the noncommutative case. It is easy to see that Gordan's own proof in [10] immediately carries over to our situation, wherefore the exposition here will be rather sketchy.

Before we begin proving the theorem, let us note that the symbolic expressions have "a life of their own", we can manipulate such expressions whether or not they can be interpreted as invariants or covariants.

First we will introduce the notion of *polars* ([10], § 2). Let

$$\omega = \langle x, y_1\rangle^{m_1}\dots\langle x, y_r\rangle^{m_r}$$

be a symbolic expression without brackets [ ]. Introduce a set of new symbols

$$y_{11}, \dots, y_{1m_1}, y_{21}, \dots, y_{2m_2}, \dots, y_{rm_r}$$

and define

$$\tilde{\omega} = \langle x, y_{11}\rangle\dots\langle x, y_{1m_1}\rangle\dots\langle x, y_{rm_r}\rangle.$$

Let further n be a non-negative integer $\leqq m_1+\dots+m_r$, and let x' be a new variable. Replace x in $\tilde{\omega}$ by x' in all possible ways and add the resulting symbolic expressions (which now contain the symbol x'). Divide by the number of terms

$$\binom{m_1+\dots+m_r}{n}$$

and finally replace $y_{11}, \dots, y_{1m_1}$ by $y_1, \dots, y_{r1}, \dots, y_{rm_r}$ by y_r. The resulting symbolic expression is denoted by $\omega_{x'^n}$, and is called *the n'th x'-polar of* ω. If $n > m_1+\dots+m_r$, we define $\omega_{x'^n}=0$. Note that if we replace x' by x in the nth x'-polar we recover ω (if $n\leqq m_1+\ldots+m_r$, of course). The definition seems complicated, but a few examples will make everything clear.

Example 1. If $\omega=\langle x, y\rangle^r$, and $n\leqq r$, then

$$\omega_{x'^n}=\langle x', y\rangle^n\langle x, y\rangle^{r-n}.$$

Example 2. If $\omega=\langle x, y_1\rangle^3\langle x, y_2\rangle^3$, then

$$\omega_{x'^4}=\tfrac{1}{15}(3\langle x', y_1\rangle^3\langle x', y_2\rangle\langle x, y_2\rangle^2+9\langle x', y_1\rangle^2\langle x, y_1\rangle\langle x', y_2\rangle^2\langle x, y_2\rangle$$
$$+3\langle x', y_1\rangle\langle x, y_1\rangle^2\langle x', y_2\rangle^3).$$

When ω contains brackets [ ], we consider these as constants when we compute polars, e.g., if $\omega=[y_1y_2]\langle x, y_1\rangle^3$, then

$$\omega_{x'^2}=[y_1y_2]\langle x', y_1\rangle^2\langle x, y_1\rangle.$$

Let us call n the *order* of the polar $\omega_{x'^n}$.

Example 3. If ω is as in Example 2, then

$$\omega_{x'^4}-\langle x', y_1\rangle^2\langle x, y_1\rangle\langle x', y_2\rangle^2\langle x, y_2\rangle$$
$$=\tfrac{1}{5}(\langle x', y_1\rangle^3\langle x', y_2\rangle\langle x, y_2\rangle^2-\langle x', y_1\rangle^2\langle x, y_1\rangle\langle x', y_2\rangle^2\langle x, y_2\rangle)$$
$$+\tfrac{1}{5}(\langle x', y_1\rangle\langle x, y_1\rangle^2\langle x', y_2\rangle^3-\langle x', y_1\rangle^2\langle x, y_1\rangle\langle x', y_2\rangle^2\langle x, y_2\rangle)$$
$$=\tfrac{1}{5}\langle x', y_1\rangle^2\langle x', y_2\rangle\langle x, y_2\rangle(\langle x', y_1\rangle\langle x, y_2\rangle-\langle x, y_1\rangle\langle x', y_2\rangle)$$
$$+\tfrac{1}{5}\langle x', y_1\rangle\langle x, y_1\rangle\langle x', y_2\rangle^2(\langle x, y_1\rangle\langle x', y_2\rangle-\langle x', y_1\rangle\langle x, y_2\rangle)$$
$$=-\tfrac{1}{5}[y_1y_2]\langle x', y_1\rangle^2\langle x', y_2\rangle\langle x, y_2\rangle[xx']$$
$$+\tfrac{1}{5}[y_1y_2]\langle x', y_1\rangle\langle x, y_1\rangle\langle x', y_2\rangle^2[xx']$$
$$=\tfrac{1}{5}[y_1y_2]\langle x', y_1\rangle\langle x', y_2\rangle(-\langle x', y_1\rangle\langle x, y_2\rangle+\langle x, y_1\rangle\langle x', y_2\rangle)[xx']$$
$$=\tfrac{1}{5}[y_1y_2]\langle x', y_1\rangle\langle x', y_2\rangle[xx']^2,$$

where we have used the identity

$$\langle x, y\rangle\langle x', y'\rangle-\langle x', y\rangle\langle x, y'\rangle=[yy'][xx'].$$

Hence

$$\langle x', y_1\rangle^2\langle x, y_1\rangle\langle x', y_2\rangle^2\langle x, y_2\rangle$$
$$= \omega_{x'^4} - \tfrac{1}{5}[y_1y_2]^2\langle x', y_1\rangle\langle x', y_2\rangle[xx']^2$$
$$= \omega_{x'^4} - \tfrac{1}{5}(\omega'_{x'}2)[xx']^2,$$

where $\omega' = [y_1y_2]^2\langle x, y_1\rangle\langle x, y_2\rangle$.

This can be generalized:

Lemma (see [10], p. 27). *Let t be a term in the n'th x'-polar of the symbolic expression ω. Then we can write*

$$t = \sum_{k=0}^{n} C_k(\omega_k)_{x'}k[xx']^{n-k}$$

where ω_k are new symbolic expressions and $\omega_n = \omega$.

Sketch of proof. If $\omega = \langle x, y_1\rangle^{m_1}\dots\langle x, y_r\rangle^{m_r}$, then a typical term in $\omega_{x'}n$ is

$$t = \langle x, y_1\rangle^{m_1-k_1}\langle x', y_1\rangle^{k_1}\dots\langle x, y_r\rangle^{m_r-k_r}\langle x', y_r\rangle^{k_r},$$

where $k_1+\dots+k_r = n$. From the identity

$$\langle x, y\rangle\langle x', y'\rangle - \langle x', y\rangle\langle x, y'\rangle = [yy'][xx'],$$

it follows, if we add and subtract sufficiently many new terms, that the difference between t and another term in the polar $\omega_{x'^n}$ contains a factor $[xx']$. The other factor is a term in a polar of order less than n of some symbolic expression. Now induction on n completes the proof. Q.E.D.

Remark. In Gordan's book on invariant theory [10], the "symbols" $y_1, y_2, \dots$ are denoted by $a, b, \dots$, and instead of $\langle x, y\rangle$, etc., Gordan writes a_x, etc. The brackets $[y_1y_2]$ are written (ab).

Next suppose that we have two symbolic expressions

$$\omega_1 = \langle x, y_1\rangle^{m_1}\dots\langle x, y_r\rangle^{m_r},$$
$$\omega_2 = \langle x, z_1\rangle^{n_1}\dots\langle x, z_p\rangle^{n_p}.$$

As before, introduce new symbols

$$y_{11}, \dots, y_{1m_1}, \dots, y_{rm_r}, z_{11}, \dots, z_{1n_1}, \dots, z_{pn_p},$$

and put

$$\tilde{\omega}_1 = \langle x, y_{11}\rangle\dots\langle x, y_{rm_r}\rangle,$$
$$\tilde{\omega}_2 = \langle x, z_{11}\rangle\dots\langle x, z_{pn_p}\rangle.$$

Let h be a non-negative integer less than or equal to $m = m_1+\dots+m_r$ and $n = n_1+\dots+n_p$. Take $y'_1, \dots, y'_h \in \{y_{11}, \dots, y_{rm_r}\}$, $z'_1, \dots, z'_h \in \{z_{11}, \dots, z_{pn_p}\}$ and form the new symbolic expression

$$[y_1' z_1'] \dots [y_h' z_h'] \Pi_{y_{ij} \neq y_k'} \langle x, y_{ij} \rangle \Pi_{z_{ij} \neq z_k'} \langle x, z_{ij} \rangle.$$

Add all such expressions for all possible choices of y_k', z_k'. Finally replace y_{ij} by y_i for all j and z_{ij} by z_i for all j and divide by the number of terms $\binom{m}{h}\binom{n}{h}$. The resulting symbolic expression is called the *h'th transvectant* of ω_1 and ω_2 and is denoted by $\tau_h(\omega_1, \omega_2)$.

If $h > \min(m, n)$, we let $\tau_h(\omega_1, \omega_2) = 0$.

As was the case with the polars, this definition seems complicated, and we give a few examples to make things clear.

Example 4.

$$\tau_h(\langle x, y_1 \rangle^{k_1}, \langle x, y_2 \rangle^{k_2}) = [y_1 y_2]^h \langle x, y_1 \rangle^{k_1 - h} \langle x, y_2 \rangle^{k_2 - h}, \quad \text{if} \quad h \leqq \min(k_1, k_2).$$

In particular, $\tau_h(\langle x, y_1 \rangle^h, \langle x, y_2 \rangle^h) = [y_1 y_2]^h$.

Example 5.

$$\tau_2(\langle x, y_1 \rangle^3 \langle x, y_2 \rangle, \langle x, y_3 \rangle^2) = \tfrac{1}{6}(3[y_1 y_3]^2 \langle x, y_1 \rangle \langle x, y_2 \rangle + 3[y_1 y_3][y_2 y_3] \langle x, y_1 \rangle^2).$$

Example 6.

$$\tau_1(\langle x, y_1 \rangle \langle x, y_2 \rangle, \langle x, y_3 \rangle \langle x, y_4 \rangle)$$

$$= \tfrac{1}{4}([y_1 y_3] \langle x, y_2 \rangle \langle x, y_4 \rangle + [y_1 y_4] \langle x, y_2 \rangle \langle x, y_3 \rangle + [y_2 y_3] \langle x, y_1 \rangle \langle x, y_4 \rangle$$

$$+ [y_2 y_4] \langle x, y_1 \rangle \langle x, y_3 \rangle).$$

We consider the brackets [ ] as constants when we compute transvectants.

So far the transvectants are just formal functions on symbolic expressions. Let f and g be elements of the algebra $\tilde{C}_d$. Then $\tau_h(\omega_f, \omega_g)$ is a symbolic expression, which restitutes to a new element of $\tilde{C}_d$. We denote this new covariant by $\tau_h(f, g)$, by a slight abuse of notation. Hence we have a method to generate new covariants.

Example 7. We proved above that the algebra $\tilde{C}_2$ is generated by the elements γ, δ, Δ, and s_3 as S-algebra, where

$$\omega_\gamma = \langle x, y_1 \rangle^2,$$

$$\omega_\delta = [y_1 y_2] \langle x, y_1 \rangle \langle x, y_2 \rangle,$$

$$\omega_\Delta = [y_1 y_2]^2,$$

and

$$\omega_{s_3} = [y_1 y_2][y_1 y_3][y_2 y_3].$$

We have

$$\tau_1(\omega_\gamma, \omega_\gamma) = [y_1 y_2]\langle x, y_1\rangle\langle x, y_2],$$

$$\tau_2(\omega_\gamma, \omega_\gamma) = [y_1 y_2]^2,$$

and

$$\tau_2(\omega_\gamma, \omega_\delta) = [y_1 y_2][y_1 y_3][y_2 y_3],$$

wherefore $\delta = \tau_1(\gamma, \gamma)$, $\Delta = \tau_2(\gamma, \gamma)$, and $s_3 = \tau_2(\gamma, \delta)$.

It is clear from the definitions of polars and transvectants that there is a relationship between them. To see this more clearly, let ω be a symbolic expression and form $\omega_{x'}n$. Replace x_1' by z_2 and x_2' by $-z_1'$, where z is a new symbol. Then $\langle x', y\rangle$ becomes $[yz]$ and $[xx']$ becomes $-\langle x, z\rangle$, whence $\omega_{x'^n}$ becomes $\tau_n(\omega, \langle x, z\rangle^n)$ and $(\omega_{x'^n})\cdot[xx']^k$ becomes $\pm\tau_n(\omega, \langle x, z\rangle^{k+n})$.

Example 7. If $\omega = \langle x, y_1\rangle^3\langle x, y_2\rangle^3$, then

$$\omega_{x'^2} = \tfrac{1}{15}(3\langle x, y_1\rangle\langle x', y_1\rangle^2\langle x, y_2\rangle^3 + 9\langle x, y_1\rangle^2\langle x', y_1\rangle\langle x, y_2\rangle^2\langle x, y_2\rangle$$
$$+3\langle x, y_1\rangle^3\langle x, y_2\rangle\langle x', y_2\rangle^2).$$

The substitution $x_1' \mapsto z_2$, $x_2' \mapsto -z_1$ gives the expression

$$\tfrac{1}{15}(3\langle x, y_1\rangle[y_1 z]^2\langle x, y_2\rangle^3 + 9\langle x, y_1\rangle^2[y_1 z]\langle x, y_2\rangle^2[y_2 z]$$
$$+3\langle x, y_1\rangle^3\langle x, y_2\rangle[y_2 z]^2),$$

which equals $\tau_2(\langle x, y_1\rangle^3\langle x, y_2\rangle^3, \langle x, z\rangle^2)$.

Suppose y is a symbol in a symbolic expression ω. Substitute $y_1 \mapsto x_2'$, $y_2 \mapsto -x_1'$. Then ω is transformed into $t\cdot[xx']^k$, where t is a symbolic expression not containing the factor $[xx']$ (but which might very well contain the symbol x'). This t is a term in an x'-polar of some order of some symbolic expression ω'. By the lemma on terms in a polar, we can write

$$t = \omega'_{x'^n} + (\omega_1')_{x'^{n-1}}[xx'] + \ldots + (\omega_n')[xx']^n,$$

where ω_n' does not contain the symbol x'.

Substituting back, we get

$$\omega = \pm\tau_n(\omega', \langle x, y\rangle^{n+k}) \pm \tau_{n-1}(\omega_1', \langle x, y\rangle^{n+k}) \pm \ldots \pm \tau_0(\omega_n', \langle x, y\rangle^{n+k}),$$

where $\omega', \omega_1', \ldots, \omega_n'$ do not contain the symbol y. With induction over the number of symbols, we get the analogue of Gordan's theorem in the noncommutative case:

Theorem 3.1. *The algebra of noncommutative covariants* $\tilde{C}_d$ *is generated by* $\gamma \in \tilde{C}_{d1d}$ *and the transvectants* τ_h.

Remark. One can define the transvectants by using the Clebsch—Gordan isomorphism

$$R_d \otimes_k R_e \cong R_{d+e} \oplus R_{d+e-2} \oplus \ldots \oplus R_{|d-e|}$$

also. See [21], p. 57.

4. *An Algebra Structure on* $\oplus_{d\geq 0} T^m(R_d^*)$ *and* $\oplus_{d\geq 0} \bar{I}_d^m$

The fundamental idea of the symbolic method is to treat the elements of the tensor power $T^m(R_d^*)$ as multilinear forms on R_d. Here we will exploit this idea in a slightly different direction.

Let f be an element of $T^m(R_d^*)$, and let $y_1, \dots, y_m$ be elements of V^*, i.e.,

$$y_i = \eta_{i1} X + \eta_{i2} Y.$$

Then $y_i^d \in R_d$, and f is determined by its values on elements of this type. This follows from the following

Lemma. *Let* l: $R_d \to k$ *be a linear form, and suppose that* $l(y^d)=0$ *for all* $y \in V^*$. *Then* $l=0$.

Proof. Let $a_0, \dots, a_d$ be different elements of k. Then

$$l((X+a_i Y)^d) = \sum_{j=0}^d \binom{d}{j} a_i^j \, l(X^{d-j} Y^j) = 0$$

for all i. But this is a system of linear equations in the unknown $l(X^{d-j} Y^j)$, whose determinant is $\det(a_i^j)$, hence is non-zero. Q.E.D.

Now let $f_i \in T^m(R_{d_i}^*)$, $i=1, 2$, and put

$$(f_1 * f_2)(y_1^{d_1+d_2}, \dots, y_m^{d_1+d_2}) = f_1(y_1^{d_1}, \dots, y_m^{d_1}) f_2(y_1^{d_2}, \dots, y_m^{d_2}).$$

Define $\chi_{k_1, \dots, k_m}^d \in T^m(R_d^*)$, $0 \leq k_i \leq d$, by

$$\chi_{k_1, \dots, k_m}^d \left(\sum_i a_{i1} \binom{d}{i} X^{d-i} Y^i, \dots, \sum_i a_{im} \binom{d}{i} X^{d-i} Y^i \right) = a_{k_1 1} \dots a_{k_m m}.$$

Then we get

$$(\chi_{k_1, \dots, k_m}^{d_1} * \chi_{l_1, \dots, l_m}^{d_2})(y_1^{d_1+d_2}, \dots, y_m^{d_1+d_2})$$

$$= \chi_{k_1, \dots, k_m}^{d_1}(y_1^{d_1}, \dots, y_m^{d_1}) \chi_{l_1, \dots, l_m}^{d_2}(y_1^{d_2}, \dots, y_m^{d_2})$$

$$= \chi_{k_1, \dots, k_m}^{d_1}\left(\sum_{i=0}^{d_1} \eta_{11}^{d_1-i} \eta_{12}^i \binom{d_1}{i} X^{d_1-i} Y^i, \dots, \sum_{i=0}^{d_1} \eta_{m1}^{d_1-i} \eta_{m2}^i \binom{d_1}{i} X^{d_1-i} Y^i\right) \cdot$$

$$\cdot \chi_{l_1, \dots, l_m}^{d_2} \left(\sum_{i=0}^{d_2} \eta_{11}^{d_2-i} \eta_{12}^i \binom{d_2}{i} X^{d_2-i} Y^i, \dots, \sum_{i=0}^{d_2} \eta_{m1}^{d_2-i} \eta_{m2}^i \binom{d_2}{i} X^{d_2-i} Y^i \right)$$

$$= \eta_{11}^{d_1-k_1} \eta_{12}^{k_1} \dots \eta_{m1}^{d_1-k_m} \eta_{m2}^{k_m} \eta_{11}^{d_2-l_1} \eta_{12}^{l_1} \dots \eta_{m_1}^{d_2-l_m} \eta_{m2}^{l_m}$$

$$= \eta_{11}^{d_1+d_2-(k_1+l_1)} \eta_{12}^{k_1+l_1} \dots \eta_{m1}^{d_1+d_2-(k_m+l_m)} \eta_{m2}^{k_m+l_m}$$

$$= \chi_{k_1+l_1, \dots, k_m+l_m}^{d_1+d_2} (y_1^{d_1+d_2}, \dots, y_m^{d_1+d_2}).$$

Here $0 \leqq k_i \leqq d_1$ and $0 \leqq l_i \leqq d_2$. Hence we have proved that

$$\chi^{d_1}_{k_1, \dots, k_m} * \chi^{d_2}_{l_1, \dots, l_m} = \chi^{d_1+d_2}_{k_1+l_1, \dots, k_m+l_m}.$$

(Of course, this relation can also be taken as the definition of $*$.)

This shows that $f_1 * f_2 \in T^m(R^*_{d_1+d_2})$. Obviously $*$ is commutative and associative, so we have found a commutative algebra structure on $\bigoplus_{d \geqq 0} T^m(R^*_d)$. We will denote this algebra by A_m.

It is graded, and by the relation above, it is generated by elements of degree 1, i.e., by the elements in $T^m(R^*_1)$. It is clear that $G = SL(2, k)$ acts as a group of homogeneous algebra automorphisms on A_m. Hence $*$ defines a commutative, graded algebra structure on $A^G_m = \bigoplus_{d \geqq 0} \tilde{I}^m_d$, too.

In fact, the multiplication $*$ looks very attractive on A^G_m: let $f_i \in \tilde{I}^m_{d_i}$, $i = 1, 2$. The symbolic expression ω_{f_i} for f_i consists of $\frac{1}{2} m d_i$ brackets [ ], filled with $d_i\ y_1$:s, ..., $d_i\ y_m$:s, and we have $\omega_{f_i} = f_i(y_1^{d_i}, \dots, y_m^{d_i})$. Hence $\omega_{f_1 * f_2}$ is obtained just by writing ω_{f_1} and ω_{f_2} beside each other. For instance, if

$$\omega_{f_i} = [y_1 y_2]^{d_i} \in \tilde{I}^2_{d_i}$$

then

$$\omega_{f_1 * f_2} = [y_1 y_2]^{d_1+d_2} \in \tilde{I}^2_{d_1+d_2}.$$

Since A_m is finitely generated, and G is reductive, the algebra of invariants A^G_m is also finitely generated. Furthermore, the Hilbert series $H(A^G_m, t)$ is rational. In fact, it can be computed explicitly:

$$H(A^G_2, t) = \frac{1}{1-t},$$

and if $m \geqq 3$, then

$$H(A^G_m, t) = \frac{1}{2t} \sum_{0 \leqq j < (1/2)m} \binom{m}{j} (-1)^{j+1} (\varphi_{m-2j} h_m)(t),$$

where φ is the Reynolds operator (see the introduction), and $h_m(t) = \big(t/(1-t^2)\big)^{m-2}$. For a proof, see p. 167 below.

Example 1. The algebra A_2 is generated by $\chi^1_{0,0}$, $\chi^1_{0,1}$, $\chi^1_{1,0}$, and $\chi^1_{1,1}$, whence

$$A_2 \cong k[x, y, z, u]/(xu - yz).$$

In symbolic notation the elements of the $T^2(R^*_d)$:s can be written as $[y_1 y_2]^d$, and so A^G_2 is generated by

$$[y_1 y_2] = \chi^1_{0,1} - \chi^1_{1,0}.$$

As is proved in [24], the Hilbert series of A^G_m has the form $g_m(t)/(1-t^2)^{m-2}$, where $g_m(t)$ is a polynomial. One might ask if this means that A^G_m is generated by elements of degree at most two. It seems to be rather difficult to prove or disprove this. The problem amounts to showing that a tableau of shape $((\frac{1}{2}md)^2)$ and weight (d^m) can be written as the union (with the obvious definition of this concept) of tableaux of shape $((\frac{1}{2}m))^2$, weight (1^m) (if m is even), and tableaux of shape (m^2), weight (2^m). To see that this is equivalent to our problem, just identify $[y_{i_1}y_{j_1}][y_{i_2}y_{j_2}], \ldots,$ written so that $i_k < j_k$ for all k, and $i_1 \leqq i_2 \leqq \ldots$, $j_1 \leqq j_2 \leqq \ldots$, with the tableau

i_1	i_2	$\cdots$
j_1	j_2	$\cdots$

(see also [25]; there the symbolic expressions are written as tableaux).

Proposition 4.1. *A_m is an integral domain.*

Proof. Suppose that $f_1 \neq 0$, but that $f_1 * f_2 = 0$. This means that

$$f_1(y_1^{d_1}, \ldots, y_m^{d_1}) f_2(y_1^{d_2}, \ldots, y_m^{d_2}) = 0$$

for all $y_i \in V^*$. But

$$(y_1, \ldots, y_m) \mapsto f_1(y_1^{d_1}, \ldots, y_m^{d_1})$$

is a polynomial function on $V^* \oplus \ldots \oplus V^*$ (m terms), whence the set of points $(y_1, \ldots, y_m)$ such that $f_1(y_1^{d_1}, \ldots, y_m^{d_1}) \neq 0$ is a Zariski-open subset, hence it is a dense subset. This implies that $f_2(y_1^{d_2}, \ldots, y_m^{d_2})$ is zero on a dense subset, and so $f_2 = 0$. Q.E.D.

Proposition 4.2. *The quotient field of A_m^G has transcendence degree $m-2$ over k (if $m \geqq 3$).*

Proof. The transcendence degree equals the order of the pole $t=1$ of $H(A_m^G, t)$. Expand $h_m(t)$ in a Laurent series about $t=1$:

$$h_m(t) = \left(\frac{t}{1-t^2}\right)^{m-2} = \frac{a_{-(m-2)}}{(1-t)^{m-2}} + \frac{a_{-(m-3)}}{(1-t)^{m-3}} + \ldots,$$

where

$$a_{-(m-2)} = \lim_{t \to 1} (1-t)^{m-2} h_m(t) = 2^{-(m-2)}, \quad \text{etc.}$$

Hence

$$(\varphi_{m-2j} h_m)(t) = \frac{(m-2j)^{m-3} \cdot 2^{-(m-2)}}{(1-t)^{m-2}} + \ldots.$$

If m is odd this immediately gives

$$\lim_{t \to 1} (1-t)^{m-2} H(A_m^G, t) = 2^{-(m-2)} \sum_{0 \leqq j < (1/2)m} \binom{m}{j} (-1)^{j+1} (m-2j)^{m-3}.$$

If m is even the pole $t=-1$ of $h_m(t)$ leads to the pole $t=1$ of $(\varphi_{m-2j}h_m)(t)$. Hence

$$\lim_{t\to 1}(1-t)^{m-2}H(A_m^G, t) = 2\cdot 2^{-(m-1)}\sum_{0\leq j<(1/2)m}\binom{m}{j}(-1)^{j+1}(m-2j)^{m-3}.$$

By [21], p. 63, this is non-zero. Q.E.D.

Remark. This looks very much like the situation when one considers $H(I_d, t)$. See [22].

5. *The Cayley—Sylvester Theorem*

In the commutative case, the Cayley—Sylvester theorem states that

$$\dim_k C_{dme} = A\left(\tfrac{1}{2}(md-e), m, d\right) - A\left(\tfrac{1}{2}(md-e)-1, m, d\right),$$

where $A(a, b, c)$ is the number of partitions of a into b non-negative parts of size $\leq c$. For a proof, see [21], Exercise 3.3.6 (1).

If we let $\tilde{A}(a, b, c)$ denote the number of *ordered* partitions of a into b parts of size $\leq c$, then Brion ([5]) has proved that

$$\dim_k \tilde{I}_d^m = \tilde{A}\left(\tfrac{1}{2}md, m, d\right) - \tilde{A}\left(\tfrac{1}{2}md-1, m, d\right).$$

Furthermore, Teranishi has proved ([24], p. 6) that $\dim_k \tilde{I}_d^m$ also equals the number of tableaux of shape $((\frac{1}{2}md)^2)$ and weight (d^m), i.e.,

$$\dim_k \tilde{I}_d^m = K_{(((1/2)md)^2),(d^m)},$$

where K is the Kostka matrix.

Here we will generalize these results to $\tilde{C}_{dme}$. Let us first note that $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ has the trace $\xi^{-l}+\xi^{-l+2}+\ldots+\xi^{l}=h_l(\xi, \xi^{-1})$ on R_l, where h denote the complete symmetric functions. Hence the trace of $\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ on $T^m(R_d^*)\otimes_k R_e$ is

$$h_d(\xi, \xi^{-1})^m h_e(\xi, \xi^{-1}) = h_{(d^m,e)}(\xi, \xi^{-1}),$$

where (d^m, e) should be read (e, d^m) if $e>d$. We also note that if $\lambda=(\lambda_1, \lambda_2)$ is a partition, then

$$s_\lambda(\xi, \xi^{-1}) = (\xi^{\lambda_1-\lambda_2+1}-\xi^{-(\lambda_1-\lambda_2+1)})/(\xi-\xi^{-1}),$$

and $s_\lambda(\xi, \xi^{-1})=0$ if $l(\lambda)>2$.

Proposition 5.1. *As G-modules,*

$$T^m(R_d^*)\otimes_k R_e \cong \bigoplus_{l\geq 0} R_l^{a_l},$$

where

$$\alpha_l = K_{(1/2(md+e+l)),1/2(md+e-l)),(d^m,e)},$$

which should be interpreted as zero if $\frac{1}{2}(md+e+l)\notin Z$.

Proof. By the general representation theory of G, we can write

$$h_{(d^m,e)}(\xi,\xi^{-1}) = \sum_{l\geqq 0}\alpha_l \frac{\xi^{l+1}-\xi^{-(l+1)}}{\xi-\xi^{-1}}$$

for some non-negative integers α_l. By the theory of symmetric functions, we have

$$h_{(d^m,e)}(\xi,\xi^{-1}) = \sum_{\substack{|\lambda|=md+e\\ l(\lambda)\leqq 2}} K_{\lambda,(d^m,e)} s_\lambda(\xi,\xi^{-1}).$$

Comparing these two expressions, the proposition is proved. Q.E.D.

Proposition 5.2.

$$\dim_k \tilde{C}_{dme} = K_{((1/2(md+e))^2),(d^m,e)} = \tilde{A}\left(\tfrac{1}{2}(md+e), m, d\right) - \tilde{A}\left(\tfrac{1}{2}(md-e)-1, m, d\right).$$

Proof. The first equality follows by taking $l=0$ in the foregoing proposition. To prove the second we note that $\alpha_0 = \dim_k \tilde{C}_{dme}$ is the difference between the coefficients of 1 and ξ^2 in

$$\sum_{l\geqq 0}\alpha_l(\xi^{-l}+\xi^{-l+2}+\ldots+\xi^l),$$

hence the difference between the coefficients of 1 and ξ^2 in

$$(\xi^{-d}+\ldots+\xi^d)^m(\xi^{-e}+\ldots+\xi^e) = \left(\sum_{i_1=0}^d \cdots \sum_{i_m=0}^d \xi^{(d-2i_1)+\cdots+(d-2i_m)}\right)(\xi^{-e}+\ldots+\xi^e).$$

In the first factor the coefficient of ξ^j equals the number of m-tuples $(i_1, \ldots, i_m)$ such that $0\leqq i_1, \ldots, i_m\leqq d$ and $i_1+\ldots+i_m=\frac{1}{2}(md-j)$, whence

$$(\xi^{-d}+\ldots+\xi^d)^m(\xi^{-e}+\ldots+\xi^e) = \sum_{i,j}\tilde{A}\left(\tfrac{1}{2}(md-j), m, d\right)\xi^{j+e-2i}.$$

This shows that

$$\begin{aligned}\alpha_0 &= \sum_{i=0}^e \tilde{A}\left(\tfrac{1}{2}(md+e)-i, m, d\right) - \sum_{i=0}^e \tilde{A}\left(\tfrac{1}{2}(md+e)-i-1, m, d\right)\\ &= \tilde{A}\left(\tfrac{1}{2}(md+e), m, d\right) - \tilde{A}\left(\tfrac{1}{2}(md-e)-1, m, d\right).\end{aligned}$$

Q.E.D.

The method of proof is taken from [4], p. 206.

Proposition 5.3. *A k-basis for* $\tilde{C}_{dme}$ *is symbolically given by*

$$[y_{i_1}y_{j_1}]\ldots[y_{i_{1/2(md-e)}}y_{j_{1/2(md-e)}}]\langle x, y_{i_{1/2(md-e)+1}}\rangle\ldots\langle x, y_{i_{1/2(md+e)}}\rangle,$$

where $i_1\leqq i_2\leqq\ldots$, $j_1\leqq j_2\leqq\ldots$, *and* $i_k<j_k$ *for* $k=1, \ldots, \frac{1}{2}(md-e)$.

Proof. Let us order the set of monomials $a_{\nu_1}a_{\nu_2}\dots a_{\nu_m}$ of degree m lexicog-graphically, i.e.,

$$a_{\nu_1}\dots a_{\nu_m} < a_{\nu'_1}\dots a_{\nu'_m}$$

if and only if the first index that separates the two monomials is less in $a_{\nu_1}\dots$ than in $a_{\nu'_1}\dots$. Let $\beta(k)$ be the number of $j's$ in the element in the proposition equal to k. Then it is easy to see that the least monomial appearing in the expansion of this element is

$$a_{\beta(1)}a_{\beta(2)}\dots a_{\beta(m)}$$

(it appears multiplied by x_1^e). Hence different such elements have different least terms, wherefore they must be linearly independent. If we identify these elements with the tableaux

i_1	i_2	$\dots$	$i_{1/2(md-e)}$	$i_{1/2(md-e)+1}$	$\dots$	$i_{1/2(md+e)}$
j_1	j_2	$\dots$	$j_{1/2(md-e)}$	$m+1$	$\dots$	$m+1$

we see that their number is precisely

$$K_{((1/2(md+e))^2),(d^m,e)} = \dim_k \tilde{C}_{dme}. \qquad \text{Q.E.D}$$

Remark. For $e=0$ this is a theorem by Teranishi ([24], p. 8).

The structure of $\tilde{I}_d^m$ as an S_m-module

As was noted in the introduction, the tensor space $T^m(R_d^*)$ carries the structure of a module over the symmetric group S_m, which acts by permutations of the factors. This action commutes with the action of $G=SL(2,k)$, whence the subspace of G-invariants $\tilde{I}_d^m$ is also an S_m-module. Here again it is more natural to consider the object $\oplus_d \tilde{I}_d^m$ than to study the algebra $\oplus_d \tilde{I}_d^m$, since the spaces $\tilde{I}_d^m$ for different m are modules over different symmetric groups. Hence we are led to study a formal power series

$$\sum_{d\geq 0} \Gamma_d^m t^d \in R[[t]]$$

where Γ_d^m is the S_m-character of $\tilde{I}_d^m$ (the ring R was introduced in the introduction).

1. The Decomposition of $T^m(R_d^*)$ into Irreducible S_m-Modules

We let $A=k[S_m]$ be the group algebra.

Definition. For $\mu=(\mu_0, \mu_1, \ldots, \mu_d)\in N^{d+1}$, we let

$$|\mu| = \mu_0+\mu_1+\ldots+\mu_d,$$

analogously to the definition for partitions.

If $|\mu|=m$, we let

$$a_\mu = a_0^{\mu_0} a_1^{\mu_1} \ldots a_d^{\mu_d} d\in T^m(R_d^*).$$

If P is a finite-dimensional S_m-module, let

$$P = \bigoplus_{|\lambda|=m} M_P(\lambda)$$

be its isotypic decomposition, i.e., $M_P(\lambda)$ is the sum of the submodules of P isomorphic to the irreducible module M^λ (see the introduction).

If $\mu\in N^{d+1}$, then we can rearrange the components of μ to get a partition. We denote this partition by $f(\mu)$.

Since a_μ is an element of $T^m(R_d^*)$, it generates a sub-S_m-module of $T^m(R_d^*)$, namely Aa_μ. We let its isotypic decomposition be

$$Aa_\mu = \bigoplus_\lambda M_\mu(\lambda)$$

for the sake of simplicity.

Lemma 1.1. *$Aa_\mu \cong \operatorname{ind}_{S_\mu}^{S_m}(1_{S_\mu})$, where $S_\mu=S_{\mu_0}\times\ldots\times S_{\mu_d}$, and 1_{S_μ} is the trivial character on S_μ.*

Proof. Obviously a_μ generates the trivial S_μ-module. Let $\sigma_1, \ldots, \sigma_r$ be a set of representatives for S_{m_i}/S_μ. Then

$$\operatorname{ind}_{S_\mu}^{S_m}(1_{S_\mu}) = 1_{S_\mu}\otimes_{k[S_\mu]} k[S_m]$$

is spanned by $\{a_\mu\otimes\sigma_i\}$. We have a surjection

$$\operatorname{ind}_{S_\mu}^{S_m}(1_{S_\mu}) \to Aa_\mu$$

$$a_\mu\otimes\sigma_i \mapsto \sigma_i a_\mu.$$

Since $\dim_k(\operatorname{ind}_{S_\mu}^{S_m}(1_{S_\mu}))=m!/\mu_0!\,\mu_1!\ldots\mu_d!=\dim_k Aa_\mu$, the lemma is proved. Q.E.D.

It now follows from [17], § 7, that if $\eta(\mu)$ is the S_m-character of Aa_μ, then $\operatorname{ch}(\eta(\mu))=h_{f(\mu)}$. But

$$h_{f(\mu)} = \sum_\lambda (K')_{f(\mu)\lambda} s_\lambda = \sum_\lambda K_{\lambda f(\mu)} s_\lambda$$

(here K' denotes the transpose of the Kostka matrix K), whence

$$Aa_\mu \cong \bigoplus_\lambda (M^\lambda)^{K_{\lambda f(\mu)}}$$

as S_m-modules. Now let ν be a partition of m with length $\leqq d+1$. Then we let c_ν be the number of $\mu \in N^{d+1}$ such that $f(\mu)=\nu$. It is easily seen that $c_\nu=(d+1)!/\prod_{i\geqq 0} m_i(\nu)$, where $m_i(\nu)$ is the number of i's in ν (and $m_0(\nu)=d+1-l(\nu)$). We can now describe the decomposition of $T^m(R_d^*)$:

Proposition 1.2. *$T^m(R_d^*)\cong\bigoplus_{|\lambda|=m}(M^\lambda)^{\sum_\nu c_\nu K_{\lambda\nu}}$, where ν runs through all partitions of m of length $\leqq d+1$.*

Proof. Let the character of $T^m(R_d^*)$ be η. Then

$$\eta = \textstyle\sum_\mu \eta(\mu)$$

where $\mu \in N^{d+1}$ and $|\mu|=m$. Thus

$$ch(\eta) = \textstyle\sum_\mu ch(\eta(\mu)) = \sum_\mu h_{f(\mu)} = \sum_\nu c_\nu h_\nu$$

where ν runs through all partitions of m of length $\leqq d+1$. Hence

$$ch(\eta) = \textstyle\sum_{\lambda,\nu} c_\nu K_{\lambda\nu} s_\lambda \qquad \text{Q.E.D.}$$

Lemma 1.3. *In the decomposition*

$$T^m(R_d^*) = \bigoplus_{|\lambda|=m} M_{T^m(R_d^*)}(\lambda)$$

the isotypic components $M_{T^m(R_d^*)}(\lambda)$ are sub-G-modules of $T^m(R_d^*)$.

The proof is obvious.

Consider the binary form $\sum a_i\binom{d}{i}X^{d-i}Y^i$. Take $g=\begin{pmatrix}\xi & 0\\ 0 & \xi^{-1}\end{pmatrix}$ in G. We have

$$g\cdot\left(\sum a_i\binom{d}{i}X^{d-i}Y^i\right) = \sum a_i\binom{d}{i}(g\cdot X)^{d-i}(g\cdot Y)^i$$

$$= \sum a_i\binom{d}{i}\xi^{-(d-i)}X^{d-i}\xi^i Y^i = \sum a_i\binom{d}{i}\xi^{-d+2i}X^{d-i}Y^i.$$

Hence $a_i \mapsto \xi^{-d+2i}a_i$. If now $\sigma a_\mu \in Aa_\mu \subseteq T^m(R_d^*)$, then

$$g\cdot(\sigma a_\mu) = \sigma(g\cdot a_\mu) = \sigma(\xi^{(-d\mu_0+(-d+2)\mu_1+\cdots+d\mu_d)}a_\mu) = \xi^{\Sigma(-d+2i)\mu_i}(\sigma a_\mu).$$

The character of $M_\mu(\lambda)$ as a T-module (where T is the subgroup of G consisting of all diagonal matrices; note that $M_\mu(\lambda)$ is not a sub-G-module) is therefore

$$(\dim_k M_\mu(\lambda))\,\xi^{\Sigma(-d+2i)\mu_i} = (\dim_k M_\lambda)\textstyle\sum_\nu K_{\lambda f(\mu)}\xi^{\Sigma(d+2i)\mu_i}.$$

Summing over μ, we get the character of $M_{T^m(R_d^*)}(\lambda)$ as a G-module:

$$(\dim_k M^\lambda) \sum_\mu K_{\lambda f(\mu)} \xi^{\Sigma(-d+2i)\mu_i} = (\dim_k M^\lambda) K_{\lambda\nu} \left(\sum_{\mu=f(\nu)} \xi^{\Sigma(-d+2i)\mu_i}\right)$$
$$= (\dim_k M^\lambda) \sum_\nu K_{\lambda\nu} m_\nu(\xi^d, \xi^{d-2}, \ldots, \xi^{-d}),$$

where in the first sum $\mu \in N^{d+1}$, $|\mu|=m$, and in the second ν is a partition of m of length $\leqq d+1$. In the third sum, ν runs over all partitions of m (if $l(\nu)>d+1$, we let $m_\nu(\xi^d, \ldots, \xi^{-d})=m_\nu(\xi^d, \ldots, \xi^{-d}, 0, \ldots, 0)$ with $l(\nu)-d-1$ zeros (and this is zero)).

But now K is the transition matrix $M(s, m)$, by definition, whence

$$\sum_\nu K_{\lambda\nu} m_\nu = s_\lambda.$$

Denote the G-character of $M_{T^m(R_d^*)}(\lambda)$ by $\chi_{d,m}(\lambda)$. Summing up, we have proved:

Theorem 1.4. *In the isotypic decomposition of $T^m(R_d^*)$ as S_m-module,*

$$T^m(R_d^*) = \bigoplus_{|\lambda|=m} M_{T^m(R_d^*)}(\lambda),$$

the isotypic components $M_{T^m(R_d^*)}(\lambda)$ are sub-G-modules with characters

$$\chi_{d,m}(\lambda)(\xi) = (\dim_k M^\lambda) s_\lambda(\xi^d, \xi^{d-2}, \ldots, \xi^{-d}).$$

Remark. By [17], p. 62, $\dim_k M^\lambda = K_{\lambda,(1^m)}$. Thus

$$\sum_{|\lambda|=m} \chi_{d,m}(\lambda)(\xi) = \sum_\lambda K_{\lambda,(1^m)} s_\lambda(\xi^d, \ldots, \xi^{-d}) = h_{(1^m)}(\xi^d, \ldots, \xi^{-d})$$
$$= \left(h_1(\xi^d, \ldots, \xi^{-d})\right)^m = (\xi^d + \xi^{d-2} + \ldots + \xi^{-d})^m.$$

This is a complicated way to see that $T^m(R_d^*)$ has the character $(\chi_d(\xi))^m$ as a G-module.

Corollary 1.5. *The character of $S^m(R_d^*)$ as a G-module is $h_m(\xi^d, \ldots, \xi^{-d})$.*

Proof. The space $S^m(R_d^*)$ consists of the symmetric tensors in $T^m(R_d^*)$, i.e., $S^m(R_d^*)=T^m(R_d^*)^{S_m}$. Now the trivial S_m-module corresponds to the partition (m), so the G-character of $S^m(R_d^*)$ is $s_{(m)}(\xi^d, \ldots, \xi^{-d})=h_m(\xi^d, \ldots, \xi^{-d})$. Q.E.D.

Corollary 1.6. *The character of the antisymmetric part $\Lambda^m(R_d^*)$ of $T^m(R_d^*)$ as a G-module is $e_m(\xi^d, \ldots, \xi^{-d})$.*

Proof. The antisymmetric par $\Lambda^m(R_d^*)=M_{T^m(R_d^*)}((1^m))$ corresponds to the sign character of S_m, hence its G-character is $s_{(1^m)}(\xi^d, \ldots, \xi^{-d})=e_m(\xi^d, \ldots, \xi^{-d})$. Q.E.D.

2. *The Decomposition of $\tilde{I}_d^m$*

We are now ready to describe the decomposition of the invariant space $\tilde{I}_d^m$ into irreducible S_m-modules.

Let Γ_d^m be the S_m-character of $\tilde{I}_d^m$. Then

$$\Gamma_d^m = \sum_{|\lambda|=m} a_\lambda(d, m)\chi^\lambda,$$

where χ^λ are the irreducible S_m-characters, and the coefficients can be written

$$a_\lambda(d, m) = \int (1-\xi^{-2}) s_\lambda(\xi^d, \ldots, \xi^{-d}),$$

since $(\dim_k M^\lambda) s_\lambda(\xi^d, \ldots, \xi^{-d})$ is the G-character of $M_{T^m(R_d^*)}(\lambda)$. Now this integral is not easy to evaluate directly. Instead we are going to study a formal power series

$$\sum_{d\geqq 0} \Gamma_d^m t^d \in R[[t]].$$

First a

Definition. If $\mu=(\mu_1, \mu_2, \ldots, \mu_n)\in N^n$, we put

$$|\mu| = \sum \mu_i, \quad \text{and} \quad n(\mu) = \sum (i-1)\mu_i,$$

as for partitions.

Let us also say that $\mu^{(1)}\subset\mu^{(2)}$ if $\mu_i^{(1)}\leqq\mu_i^{(2)}$ for all i. In this case we define a "generalized binomial coefficient":

$$\binom{\mu^{(2)}}{\mu^{(1)}} = \prod_{i\geqq 1} \binom{\mu_i^{(2)}}{\mu_i^{(1)}}.$$

When λ is a partition, and $\mu\subset\lambda'$, but not necessarily a partition, let

$$f_{\lambda,\mu}^{\pm}(t) = \frac{1-t^{\pm 2}}{\prod_{j\geqq 1}(1-t^{2j})^{\lambda'_j}} t^{2(|\mu|+n(\mu))}.$$

Theorem 2.1. *We have*

$$\sum_{d\geqq 0} \Gamma_d^m t^d = \sum_{|\nu|=m} \left[\sum_{|\lambda|=m} (K^{-1})_{\lambda\nu} \sum_{\substack{\mu\subset\lambda' \\ |\mu|<(1/2)m}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f_{\lambda,\mu}^{\pm})(t) \right] \chi^\nu$$

where $\{\chi^\nu;\ |\nu|=m\}$ are the irreducible characters on S_m. Hence the coefficient of χ^ν is a rational function.

Proof. We will compute in the ring of symmetric functions Λ, i.e., we will apply the characteristic map. By [17], Ch. 1, § 4, we have

$$\begin{aligned}\sum_{d\geqq 0} ch(\Gamma_d^m) t^d &= \sum_{d\geqq 0} \sum_{|\lambda|=m} \int (1-\xi^2) s_\lambda(\xi^d, \ldots, \xi^{-d}) s_\lambda(y) t^d \\ &= \sum_{d\geqq 0} \sum_{|\lambda|=m} \int (1-\xi^2) h_\lambda(\xi^d, \ldots, \xi^{-d}) m_\lambda(y) t^d,\end{aligned}$$

where y is a new set of polynomial variables.

Furthermore,

$$h_\lambda(\xi^d, \ldots, \xi^{-d})$$

$$= \xi^{-|\lambda|d} \prod_{i\geqq 1} h_{\lambda_i}(1, \xi^2, \ldots, \xi^{2d}) = \xi^{-|\lambda|d} \prod_{i\geqq 1} \begin{bmatrix} d+\lambda_i \\ \lambda_i \end{bmatrix} (\xi^2)$$

$$= \xi^{-|\lambda|d} \prod_{i\geqq 1} \frac{(1-\xi^{2(d+\lambda_i)})(1-\xi^{2(d+\lambda_i-1)})\ldots(1-\xi^{2(d+1)})}{(1-\xi^2)(1-\xi^4)\ldots(1-\xi^{2\lambda_i})}.$$

In this product, the factor

$$\frac{(1-\xi^{2(d+1)})}{(1-\xi^2)}$$

appears as many times as there are $\lambda_i's$ greater than or equal to 1, i.e., λ_1' times, and the factor

$$\frac{(1-\xi^{2(d+2)})}{(1-\xi^{2\cdot 2})}$$

appears as many times as there are $\lambda_i's \geqq 2$, i.e., λ_2' times, etc., wherefore

$$h_\lambda(\xi^d, \ldots, \xi^{-d})$$

$$= \xi^{-|\lambda|d} \prod_{j\geqq 1} \left(\frac{1-\xi^{2(d+j)}}{1-\xi^{2j}}\right)^{\lambda_j'} = \prod_{j\geqq 1} \left(\frac{\xi^{d+j}-\xi^{-(d+j)}}{\xi^j-\xi^{-j}}\right)^{\lambda_j'}$$

$$= \prod_{j\geqq 1} (\xi^j-\xi^{-j})^{-\lambda_j'} \prod_{j\geqq 1} \sum_{\mu_j=0}^{\lambda_j'} \binom{\lambda_j'}{\mu_j} (-1)^{\mu_j} \xi^{(d+j)(\lambda_j'-2\mu_j)}$$

$$= \prod_{j\geqq 1} (\xi^j-\xi^{-j})^{-\lambda_j'} \sum_{\substack{0\leqq \mu_j \leqq \lambda_j' \\ \text{for all } j}} \binom{\lambda_1'}{\mu_1}\binom{\lambda_1'}{\mu_1} \ldots (-1)^{\mu_1+\mu_2+\cdots} \xi^{\Sigma_j (d+j)(\lambda_j'-2\mu_j)}$$

$$= \prod_{j\geqq 1} (\xi^j-\xi^{-j})^{-\lambda_j'} \sum_{\mu\subset\lambda'} \binom{\lambda'}{\mu} (-1)^{|\mu|} \xi^{(d+1)(|\lambda|-2|\mu|)+n(\lambda')-2n(\mu)},$$

where in the last sum μ does not have to be a partition, just a sequence of integers.

Summing the geometric series, we get

$$\sum_{d\geqq 0} h_\lambda(\xi^d, \ldots, \xi^{-d})\, t^d$$

$$= \prod_{j\geqq 1} (\xi^j-\xi^{-j})^{-\lambda_j'} \sum_{\mu\subset\lambda'} \binom{\lambda'}{\mu} (-1)^{|\mu|} \frac{\xi^{|\lambda|-2|\mu|+n(\lambda')-2n(\mu)}}{1-t\xi^{|\lambda|-2|\mu|}}.$$

Instead of summing over μ, we sum over $\lambda'-\mu$ (the set-theoretic difference), and obtain

$$\sum_{d\geqq 0} h_\lambda(\xi^d, \ldots, \xi^{-d})t^d$$

$$= \xi^{|\lambda|+n(\lambda')}(-1)^{|\lambda|}\prod_{j\geqq 1}(1-\xi^{2j})^{-\lambda'_j}\sum_{\mu\subset\lambda'}\binom{\lambda'}{\mu}(-1)^{|\lambda|-|\mu|}\frac{\xi^{-|\lambda|+2|\mu|-n(\lambda')+2n(\mu)}}{1-t\xi^{-|\lambda|+2|\mu|}}$$

$$= \prod_{j\geqq 1}(1-\xi^{2j})^{-\lambda'_j}\sum_{\mu\subset\lambda'}\binom{\lambda'}{\mu}(-1)^{|\mu|}\frac{\xi^{|\lambda|+2n(\mu)}}{\xi^{|\lambda|-2|\mu|}-t}.$$

Hence

$$\sum_{d\geqq 0} ch(\Gamma_d^m)\,t^d$$

$$= \sum_{|\lambda|=m}\frac{m_\lambda(y)}{2\pi}\int_0^{2\pi}\frac{1-e^{2ix}}{\prod_{j\geqq 1}(1-e^{2ijx})^{\lambda'_j}}\sum_{\mu\subset\lambda'}\binom{\lambda'}{\mu}(-1)^{|\mu|}\frac{e^{i(|\lambda|+2n(\mu))x}}{e^{i(|\lambda|-2|\mu|)x}-t}\,dx.$$

Now let t be a real variable with $0<t<1$, put $z=e^{ix}$, and integrate around the unit circle C:

$$\sum_{d\geqq 0} ch(\Gamma_d^m)\,t^d$$

$$= \sum_{|\lambda|=m}\frac{m_\lambda(y)}{2\pi i}\int_C\frac{1-z^2}{\prod_{j\geqq 1}(1-z^{2j})^{\lambda'_j}}\sum_{\mu\subset\lambda'}\binom{\lambda'}{\mu}(-1)^{|\mu|}\frac{z^{|\lambda|+2n(\mu)-1}}{z^{|\lambda|-2|\mu|}-t}\,dz.$$

Write, for the sake of simplicity, $\varepsilon_n=\exp(2\pi i/n)$. The integrand above has the following poles in the unit disc:

$$\varepsilon^k_{|\lambda|-2|\mu|}\,t^{1/(|\lambda|-2|\mu|)},\quad 1\leqq k\leqq |\lambda|-2|\mu|,\quad |\mu|<\tfrac{1}{2}|\lambda|,$$

and the residue theorem gives

$$\sum_{d\geqq 0} ch(\Gamma_d^m)\,t^d$$

$$= \sum_{|\lambda|=m} m_\lambda(y)\sum_{\substack{\mu\subset\lambda'\\ |\mu|<1/2|\lambda|}}\Bigg(\binom{\lambda'}{\mu}(-1)^{|\mu|}\times$$

$$\times\sum_{k=1}^{|\lambda|-2|\mu|}\frac{(1-\varepsilon^{2k}_{|\lambda|-2|\mu|}\,t^{2/(|\lambda|-2|\mu|)})\,\varepsilon^{2(|\mu|+2n(\mu))k}_{|\lambda|-2|\mu|}\,t^{(2(|\mu|+2n(\mu)))/(|\lambda|-2|\mu|)}}{\prod_{j\geqq 1}(1-\varepsilon^{2jk}_{|\lambda|-2|\mu|}\,t^{2j/(|\lambda|-2|\mu|)})^{\lambda'_j}(|\lambda|-2|\mu|)}\Bigg).$$

By the definition of the Reynolds operator, this equals

$$\sum_{|\lambda|=m} m_\lambda(y)\sum_{\substack{\mu\subset\lambda'\\ |\mu|<(1/2)m}}\binom{\lambda'}{\mu}(-1)^{|\mu|}(\varphi_{m-2|\mu|}f^{\pm}_{\lambda,\mu})(t).$$

If we take $1-e^{-2ix}$ in the numerator of the integrand instead, we get $f^{-}_{\lambda,\mu}$ in the result. The theorem follows on noting that the transition matrix $M(m,s)$ equals K^{-1}.

Q.E.D.

3. Some Examples

Here we will explicitly compute $\sum_{d\geqq 0}\Gamma_d^m t^d$ for $m=2, 3,$ and 4. At the same time, we will once more see how powerful the symbolic method is.

Example 3.1. Let $m=2$. Then

$$\sum_{d\geqq 0} ch(\Gamma_d^m)\, t^d = \sum_{|\lambda|=2} m_\lambda(y) \sum_{\substack{\mu\subset\lambda' \\ |\mu|<1}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{2-2|\mu|} f_{\lambda,\mu}^{+})(t)$$

$$= m_{(2)}(y)(\varphi_2 f_{(2),0}^{+})(t) + m_{(1^2)}(y)(\varphi_2 f_{(1^2),0}^{+})(t)$$

$$= m_{(2)}(y)\,\varphi_2\left(\frac{1-t^2}{(1-t^2)(1-t^4)}\right) + m_{(1^2)}(y)\,\varphi_2\left(\frac{1-t^2}{(1-t^2)^2}\right)$$

$$= \frac{m_{(2)}(y)}{1-t^2} + \frac{m_{(1^2)}(y)}{1-t} = \frac{s_{(2)}(y)-s_{(1^2)}(y)}{1-t^2} + \frac{s_{(1^2)}(y)}{1-t} = \frac{s_{(2)}(y)+ts_{(1^2)}(y)}{1-t^2}.$$

Hence

$$\sum_{d\geqq 0} \Gamma_d^2 t^d = \frac{\chi^{(2)}+t\chi^{(1^2)}}{1-t^2}.$$

This can be seen in another way by use of the symbolic method. In fact, symbolically a basis element of $\tilde{\Gamma}_d^m$ has the form

$$[y_1 y_2]^d,$$

so if $1\neq\sigma\in S_2$, then

$$\sigma[y_1 y_2]^d = (-1)^d [y_1 y_2]^d.$$

If we interprete Γ_d^2 as a function on S_2, this means that

$$\Gamma_d^2(1) = 1, \quad \text{and} \quad \Gamma_d^2(\sigma) = (-1)^d.$$

This gives

$$\sum_{d\geqq 0} \Gamma_d^2(1)\, t^d = \frac{1}{1-t}, \quad \text{and} \quad \sum_{d\geqq 0} \Gamma_d^2(\sigma)\, t^d = \frac{1}{1+t}.$$

We now get (here $\langle\,,\rangle$ denotes the scalar product on the space of central functions on a group)

$$\sum_{d\geqq 0} \Gamma_d^2 t^d = \sum_{d\geqq 0} \sum_{|\lambda|=2} \langle \Gamma_d^2, \chi^\lambda\rangle \chi^\lambda t^d$$

$$= \sum_{d\geqq 0} \sum_{|\lambda|=2} \frac{1}{|S_2|} \sum_{\tau\in S_2} \Gamma_d^2(\tau)\chi^\lambda(\tau)\chi^\lambda t^d$$

$$= \tfrac{1}{2} \sum_{|\lambda|=2} \sum_{\tau\in S_2} \chi^\lambda(\tau) \left(\sum_{d\geqq 0} \Gamma_m^2(\tau)\, t^d\right) \chi^\lambda$$

$$= \frac{1}{2}\left(\left(\frac{1}{1-t}+\frac{1}{1+t}\right)\chi^{(2)} + \left(\frac{1}{1-t}-\frac{1}{1+t}\right)\chi^{(1^2)}\right) = \frac{\chi^{(2)}+t\chi^{(1^2)}}{1-t^2}.$$

Example 3.2. Let $m=3$. Then

$$\sum_{d\geq 0} ch(\Gamma_d^3)\, t^d = \sum_{|\lambda|=3} m_\lambda(y) \sum_{\substack{\mu\subset\lambda' \\ |\mu|\leq 1}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{3-2|\mu|} f_{\lambda,\mu})(t).$$

For $\lambda=(3)$, the possible $\mu\subset\lambda'$ are $\mu=(0,0,0)$, $\mu=(1,0,0)$, $\mu=(0,1,0)$, and $\mu=(0,0,1)$. For $\lambda=(2,1)$ we may take $\mu=(0,0)$, $\mu=(1,0)$, and $\mu=(0,1)$. Finally, for $\lambda=(1^3)$, we get $\mu=(0)$, and $\mu=(1)$.

Hence the coefficient of $m_{(3)}(y)$ is

$$(\varphi_3 f_{(3),(0,0,0)})(t) - \varphi_1(f_{(3),(1,0,0)} + f_{(3),(0,1,0)} + f_{(3),(0,0,1)})(t)$$

$$= \varphi_3\left(\frac{1-t^2}{(1-t^2)(1-t^4)(1-t^6)}\right) - \varphi_1\left(\frac{1-t^2}{(1-t^2)(1-t^4)(1-t^6)}(t^2+t^4+t^6)\right)$$

$$= \varphi_3\left(\frac{1+t^4+t^8}{(1-t^{12})(1-t^6)}\right) - \frac{t^2+t^4+t^6}{(1-t^4)(1-t^6)}$$

$$= \frac{1}{(1-t^2)(1-t^4)} - \frac{t^2}{(1-t^2)(1-t^4)} = \frac{1}{1-t^4}.$$

The coefficient of $m_{(2,1)}(y)$ is

$$(\varphi_3 f_{(2,1),(0,0)})(t) - \varphi_1(2f_{(2,1),(1,0)} + f_{(2,1),(0,1)})(t)$$

$$= \varphi_3\left(\frac{1-t^2}{(1-t^2)^2(1-t^4)}\right) - \varphi_1\left(\frac{1-t^2}{(1-t^2)^2(1-t^4)}(2t^2+t^4)\right) = \frac{1}{1-t^4},$$

and the coefficient of $m_{(1^3)}(y)$ is finally

$$(\varphi_3 f_{(1^3),(0)})(t) - 3(\varphi_1 f_{(1^3),(1)})(t)$$

$$= \varphi_3\left(\frac{1-t^2}{(1-t^2)^3}\right) - 3\varphi_1\left(\frac{1-t^2}{(1-t^2)^3}\right) = \frac{1}{1-t^2}.$$

The Kostka matrix is

$$K_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$K_3^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix},$$

whence

$$\sum_{d\geqq 0} ch(\Gamma_d^3)\, t^d = \frac{1}{1-t^4}\, m_{(3)}(y) + \frac{1}{1-t^4}\, m_{(2,1)}(y) + \frac{1}{1-t^2}\, m_{(1^3)}(y)$$

$$= \frac{1}{1-t^4}\big(s_{(3)}(y) - s_{(2,1)}(y) + s_{(1^3)}(y) + s_{(2,1)}(y) - 2s_{(1^3)}(y) + (1+t^2)s_{(1^3)}(y)\big)$$

$$= \frac{s_{(3)}(y) + t^2 s_{(1^3)}(y)}{1-t^4},$$

and so

$$\sum_{d\geqq 0} \Gamma_d^3\, t^d = \frac{\chi^{(3)} + t^2\chi^{(1^3)}}{1-t^4}.$$

We note that the coefficient of $\chi_{(2,1)}$ is zero. The formula can also be proved using the symbolic method. A basis element of $\tilde{I}_{2q}^3$ is

$$[y_1 y_2]^q [y_1 y_3]^q [y_2 y_3]^q$$

and

$$(1\ 2)[y_1 y_2]^q [y_1 y_3]^q [y_2 y_3]^q = (-1)^q [y_1 y_2]^q [y_1 y_2]^q [y_2 y_3]^q,$$

$$(1\ 2\ 3)[y_1 y_2]^q [y_1 y_3]^q [y_2 y_3]^q = [y_1 y_2]^q [y_1 y_3]^q [y_2 y_3]^q,$$

whence the value of Γ_{2q}^3 on an element of cycle type $(2, 1)$ is $(-1)^q$, and on (3) it is 1. Now the same method as in Example 3.1 can be applied.

Example 3.3. Let $m=4$. We leave out the computations, which are long, and only give the result:

$$\sum_{d\geqq 0} \Gamma_d^4\, t^d = \frac{\chi^{(4)} + t(1+t)\chi^{(2^2)} + t^3\chi^{(1^4)}}{(1-t^2)(1-t^3)}.$$

We note that the coefficients of $\chi^{(3,1)}$ and $\chi^{(2,1^2)}$ are zero. Of course, the symbolic method can be used to prove this formula also. Let us just record some results: base elements of $\tilde{I}_d^4$ are

$$F_s = [y_1 y_2]^s [y_1 y_3]^{d-s} [y_2 y_4]^{d-s} [y_3 y_4]^s, \quad 0 \leqq s \leqq d$$

(so $\mathrm{sim}_k\, \tilde{I}_d^4 = d+1$). This follows from Section 5 in the chapter with applications of the symbolic method. Some computations give

$$(1\ 4)F_s = F_{d-s}$$

$$(1\ 2\ 4)F_s = (-1)^d \sum_{i=0}^{d-s} \binom{d-s}{i} (-1)^i F_i$$

$$(1\ 4)(2\ 3)F_s = F_s$$

$$(1\ 2\ 3\ 4)F_s = \sum_{i=0}^{d-s} \binom{d-s}{i} (-1)^i \Big(\sum_{j=0}^{i} \binom{i}{j} (-1)^j F_j\Big).$$

Let the value of Γ_d^4 on an element of cycle type ν be $\Gamma_d^4(\nu)$. Then the above formulas give

$$\Gamma_d^4((1^4)) = d+1,$$

$$\Gamma_d^4((2, 1^2)) = \tfrac{1}{2}(1+(-1)^d)$$

$$\Gamma_d^4((3, 1)) = (-1)^d \sum_{s=0}^{[(1/2)d]}(-1)^s \binom{d-s}{s},$$

$$\Gamma_d^4((2^2)) = d+1,$$

and

$$\Gamma_d^4((4)) = \tfrac{1}{2}(1+(-1)^d).$$

The same method as in Example 3.1 gives the result.

4. *A functional equation*

Write

$$\sum_{d\geqq 0} \Gamma_d^m t^d = \sum_{|\lambda|=m} P_\lambda(t)\chi^\lambda$$

where $P_\lambda(t)$ are rational functions. As usual, we denote the conjugate of the partition λ by λ'. Remember that there is an involution ω on Λ defined by $\omega(e_r)=h_r$, and corresponding to multiplication by $\chi^{(1^m)}$ (the sign character) on R_m. Also note that $\chi^{\lambda'}=\chi^{(1^m)}\chi^\lambda$.

Theorem 4.1.

$$\sum_{|\lambda|=m} P_\lambda(1/t)\chi^\lambda = (-1)^m t^2 \sum_{|\lambda|=m} P_{\lambda'}(t)\chi^\lambda = (-1)^m t^2 \sum_{|\lambda|=m} P_\lambda(t)\chi^{\lambda'}.$$

Proof. First of all, we have

$$e_\lambda(\xi^d, \ldots, \xi^{-d}) = \xi^{-|\lambda|d} \prod_{i\geqq 1} e_{\lambda_i}(1, \xi^2, \ldots, \xi^{2d})$$

$$= \xi^{-|\lambda|d} \prod_{i\geqq 1} \xi^{\lambda_i(\lambda_i-1)} \begin{bmatrix} d+1 \\ \lambda_i \end{bmatrix}(\xi^2)$$

$$= \xi^{-|\lambda|d} \prod_{i\geqq 1} \xi^{\lambda_i(\lambda_i-1)} \frac{(1-\xi^{2(d+2-1)})(1-\xi^{2(d+2-2)})\ldots(1-\xi^{2(d+2-\lambda_i)})}{(1-\xi^2)(1-\xi^4)\ldots(1-\xi^{2\lambda_i})}$$

$$= \xi^{-|\lambda|d}\xi^{2n(\lambda')} \prod_{j\geqq 1} \left(\frac{1-\xi^{2(d+2-j)}}{1-\xi^{2j}}\right)^{\lambda'_j}$$

$$= \xi^{2n(\lambda')-|\lambda|d} \prod_{j\geqq 1} \xi^{(d+2-2j)\lambda'_j} \prod_{j\geqq 1} \left(\frac{\xi^{d+2-j}-\xi^{-(d+2-j)}}{\xi^j-\xi^{-j}}\right)^{\lambda'_j}$$

$$= \prod_{j\geqq 1} \left(\frac{\xi^{d+2-j}-\xi^{-(d+2-j)}}{\xi^j-\xi^{-j}}\right)^{\lambda'_j}.$$

Here we have used the identity

$$n(\lambda') = \sum_{i\geq 1} \binom{\lambda_i}{2}$$

(see [17], Ch. I, § 1). We now get

$$\sum_{d\geq 0} s_{\lambda'}(\xi^d, \ldots, \xi^{-d}) s_\lambda(y)\, t^d = \sum_{\substack{d\geq 0\\ |\lambda|=m}} e_\lambda(\xi^d, \ldots, \xi^{-d}) m_\lambda(y) t^d$$

$$= \sum_{|\lambda|=m} m_\lambda(y) \left(\sum_{d\geq 0} \prod_{j\geq 1} \left(\frac{\xi^{(d+2-j)} - \xi^{-(d+2-j)}}{\xi^j - \xi^{-j}} \right)^{\lambda'_j} t^d \right)$$

$$= \sum_{|\lambda|=m} m_\lambda(y) \prod_{j\geq 1} (\xi^j - \xi^{-j})^{-\lambda'_j} \sum_{d\geq 0} \left(\sum_{\mu_j=0}^{\lambda'_j} \binom{\lambda'_j}{\mu_j} (-1)^{\mu_j} \xi^{(d+2-j)(\lambda'_j - 2\mu_j)} \right) t^d$$

$$= \sum_{|\lambda|=m} m_\lambda(y) \prod_{j\geq 1} (\xi^j - \xi^{-j})^{-\lambda'_j} \sum_{\mu\subset\lambda'} \binom{\lambda'}{\mu} (-1)^{|\mu|} \frac{\xi^{|\lambda|-2|\mu|-n(\lambda')+2n(\mu)}}{1 - t\xi^{|\lambda|-2|\mu|}}$$

$$= \sum_{|\lambda|=m} \prod_{j\geq 1} (\xi^j - \xi^{-j})^{-\lambda'_j} \sum_{\mu\subset\mu'} \binom{\lambda'}{\mu} (-1)^{|\mu|} \frac{\xi^{n(\lambda')-2n(\mu)}}{\xi^{|\lambda|-2|\mu|} - t}\, m_\lambda(y).$$

Proceeding as in the proof of Theorem 2.1, we get

$$\sum_{d\geq 0} \omega\big(ch(\Gamma_d^m)\big) t^d = \sum_{d\geq 0} \Big(\int (1 - \xi^{\pm 2}) s_{\lambda'}(\xi^d, \ldots, \xi^{-d}) \Big) t^d s_\lambda(y)$$

$$= \sum_{|\lambda|=m} m_\lambda(y) \sum_{\mu\subset\lambda'} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|}\, g^{\pm}_{\lambda,\mu})(t),$$

where

$$g^{\pm}_{\lambda,\mu}(t) = \frac{1 - t^{\pm 2}}{\prod_{j\geq 1} (1 - t^{2j})^{\lambda'_j}}\, t^{2n(\lambda')+2|\mu|-2n(\mu)} = t^{2(n(\lambda')-2n(\mu))} f^{\pm}_{\lambda,\mu}(t).$$

Now

$$f^{-}_{\lambda,\mu}(1/t) = \frac{1 - t^2}{\prod_{j\geq 1} (1 - t^{-2j})^{\lambda'_j}}\, t^{-2(|\mu|+n(\mu))}$$

$$= \frac{1 - t^2}{\prod_{j\geq 1} (1 - t^{2j})^{\lambda'_j}}\, t^{2(|\lambda|+n(\lambda'))} t^{-2(|\mu|+n(\mu))} (-1)^{|\lambda|} = (-1)^{|\lambda|} t^{2(|\lambda|-2|\mu|)} g^{+}_{\lambda,\mu}(t),$$

wherefore

$$\sum_{|\lambda|=m} P_\lambda(1/t) s_\lambda(y) = \sum_{|\lambda|=m} m_\lambda(y) \sum_{\substack{\mu\subset\lambda'\\ |\mu|<(1/2)m}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f^{-}_{\lambda,\mu})(1/t)$$

$$= \sum_{|\lambda|=m} m_\lambda(y) \sum_{\substack{\mu\subset\lambda'\\ |\mu|<(1/2)m}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (-1)^{|\lambda|} t^2 (\varphi_{m-2|\mu|}\, g^{+}_{\lambda,\mu})(t)$$

$$= (-1)^m t^2 \sum_{d\geq 0} \omega\big(ch(\Gamma_d^m)\big) t^d = (-1)^m t^2 \sum_{|\lambda|=m} P_{\lambda'}(t) s_\lambda(y).$$

Q.E.D.

5. Some consequences of Theorems 1.4, 2.1, and 4.1

We denote by R_G the representation ring of $G=SL(2,k)$, i.e., R_G is the free abelian group on $R_0, R_1, R_2, \dots$, with multiplication induced by the tensor product over k (for the details on the structure of R_G, we refer to [1] and [2]).

Definition. Let λ be a partition. The *Schur module* (corresponding to λ) is defined by

$$S^\lambda(R_d^*) = \det\left(S^{\lambda_i - i + j}(R_d^*)\right)_{1 \leqq i,j \leqq m} \in R_G,$$

where $m \geqq l(\lambda)$. This definition should be compared with the relation

$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)_{1 \leqq i,j \leqq m}$$

between the s- and h-functions. We will prove below that the $S^\lambda(R_d^*)$ really are modules (this fact also follows from Schur's thesis, see [19], p. 43).

Proposition 5.1. a) *The Schur modules $S^\lambda(R_d^*)$ are modules.*
b)

$$M_{T^m(R_d^*)}(\lambda) \cong S^\lambda(R_d^*)^{K_{\lambda(1^m)}}$$

(i.e., $K_{\lambda(1^m)}$ copies of $S^\lambda(R_d^*)$) as G-modules.
c)

$$\sum_{d \geqq 0} \dim_k\left(S^\lambda(R_d^*)^G\right) t^d = \sum_{|\nu|=m} (K^{-1})_{\nu\lambda} \sum_{\substack{\mu \subset \nu' \\ |\mu| < (1/2)m}} \binom{\nu'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f^{\pm}_{\nu,\mu})(t).$$

Proof. The (possibly virtual) G-character of $S^\lambda(R_d^*)$ is

$$\det\left(h_{\lambda_i - i + j}(\xi^d, \dots, \xi^{-d})\right) = s_\lambda(\xi^d, \dots, \xi^{-d}),$$

since $S^{\lambda_i - i + j}(R_d^*)$ has the character $h_{\lambda_i - i + j}(\xi^d, \dots, \xi^{-d})$. Hence $K_{\lambda(1^m)}$ copies of $S^\lambda(R_d^*)$ has the same character as $M_{T^m(R_d^*)}(\lambda)$ by Theorem 1.4, which proves a) and b).

It follows that

$$\sum_{d \geqq 0} \dim_k\left(S^\lambda(R_d^*)^G\right) t^d$$

is the coefficient of χ^λ in $\sum \Gamma_d^m t^d$, and thus c) follows from Theorem 2.1. Q.E.D.

Proposition 5.2. *If $m \geqq 3$, then the Hilbert series of the algebra A_m^G is*

$$H(A_m^G, t) = \frac{1}{2t} \sum_{0 \leqq j < (1/2)m} \binom{m}{j} (-1)^{j+1} \varphi_{m-2j}\left(\left(\frac{t}{1-t^2}\right)^{m-2}\right).$$

Proof. We have $\dim_k \tilde{I}_d^m = \Gamma_d^m(1)$, where 1 is the identity element of S_m. Hence

$$H(A_m^G, t) = \sum_{|\nu|=m} \sum_{|\lambda|=m} (K^{-1})_{\lambda\nu} \sum_{\substack{\mu\subset\lambda' \\ |\mu|<(1/2)m}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f_{\lambda,\mu}^{\pm})(t)\chi^\nu(1)$$

$$= \sum_{|\nu|=m} \sum_{|\lambda|=m} (K^{-1})_{\lambda\nu} K_{\nu(1^m)} \sum_{\substack{\mu\subset\lambda' \\ |\mu|<(1/2)m}} \binom{\lambda'}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f_{\lambda,\mu}^{\pm})(t)$$

$$= \sum_{\substack{\mu\subset(m) \\ |\mu|<(1/2)m}} \binom{(m)}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f_{(1^m),\mu}^{\pm})(t),$$

since $\chi^\nu(1) = K_{\nu(1^m)}$ and $\sum_\nu (K^{-1})_{\lambda\nu} K_{\nu(1^m)} = 1$ if $\lambda = (1^m)$ and zero otherwise.

Now let $\mu^{(j)} = (j) \subset (m)$. Then

$$f_{(1^m),\mu^{(j)}}^{\pm}(t) = \frac{1-t^{\pm 2}}{(1-t^2)^m} t^{2j},$$

and

$$f_{(1^m),\mu^{(j)}}^{+}(t) + f_{(1^m),\mu^{(j)}}^{-}(t) = -\frac{1}{t^{m-2j}} \left(\frac{1}{1-t^2}\right)^{m-2},$$

whence

$$H(A_m^G, t) = \tfrac{1}{2} \sum_{\substack{\mu\subset(m) \\ |\mu|<(1/2)m}} \binom{(m)}{\mu} (-1)^{|\mu|} \left(\varphi_{m-2|\mu|} (f_{(1^m),\mu}^{+} + f_{(1^m),\mu}^{-})\right)(t)$$

$$= \frac{1}{2t} \sum_{0\leqq j<(1/2)m} \binom{m}{j} (-1)^{j+1} \varphi_{m-2j} \left(\left(\frac{1}{1-t^2}\right)^{m-2}\right). \qquad \text{Q.E.D.}$$

We note that by Example 1 in the chapter on the symbolic method, we have $\dim_k \tilde{I}_d^2 = 1$ for all d, and so $H(A_2^G, t) = 1/(1-t)$. The formula in the proposition is of course equivalent to

$$H(A_m^G, t) = \frac{1}{t} \sum_{0\leqq j<(1/2)m} \binom{m}{j} (-1)^j \varphi_{m-2j} \left(\frac{t^m}{(1-t^2)^{m-1}}\right),$$

which is also valid for $m=2$.

Finally we will give a new proof of Springer's formula for the Hilbert series of the commutative algebra I_m (see [1] and [22]).

Proposition 5.3. *We have*

$$H(I_m, t)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{t^{j(j+1)}}{(1-t^4)(1-t^6)\dots(1-t^{2(m-j)})(1-t^2)(1-t^4)\dots(1-t^{2j})}\right)$$

and

$$H(I_m, 1/t) = (-1)^m t^{m+1} H(I_m, t).$$

Proof. The coefficient of $\chi^{(m)}$ in $\sum \Gamma_d^m t^d$ is $\sum \dim_k S^m(R_d^*)^G t^d$ by Proposition 5.1c. But by Hermite's reciprocity law,

$$\dim_k S^m(R_d^*)^G = \dim_k S^d(R_m^*)^G,$$

wherefore the coefficient of $\chi^{(m)}$ equals $H(I_m, t)$. Noting that the coefficients of $m_{(m)}(y)$ and $s_{(m)}(y)$ are equal, we get

$$H(I_m, t) = \sum_{\substack{\mu\subset(1^m)\\ |\mu|<(1/2)m}} \binom{(1^m)}{\mu} (-1)^{|\mu|} (\varphi_{m-2|\mu|} f^+_{(m),\mu})(t)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \Big(\sum_{\substack{|\mu|=j\\ \mu\subset(1^m)}} f^+_{(m),\mu}\Big)(t)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{1}{(1-t^4)(1-t^6)\dots(1-t^{2m})} \sum_{\substack{|\mu|=j\\ \mu\subset(1^m)}} t^{2(j+n(\mu))}\right)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{1}{(1-t^4)\dots(1-t^{2m})} t^{2j} e_j(1, t^2, \dots, t^{2(m-1)})\right)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{1}{(1-t^4)\dots(1-t^{2m})} t^{2j} t^{j(j+1)} \begin{bmatrix} m \\ j \end{bmatrix} (t^2)\right)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{t^{j(j+1)}}{(1-t^4)\dots(1-t^{2m})} \cdot \frac{(1-t^{2m})(1-t^{2(m-1)})\dots(1-t^{2(j+1)})}{(1-t^2)(1-t^4)\dots(1-t^{2(m-j)})}\right)$$

$$= \sum_{0\leqq j<(1/2)m} (-1)^j \varphi_{m-2j} \left(\frac{t^{j(j+1)}}{(1-t^4)\dots(1-t^{2(m-j)})(1-t^2)\dots(1-t^{2j})}\right).$$

The G-character of $M_{T^m(R_d^*)}((1^m))$ (the antisymmetric part) equals

$$e_m(\xi^d, \dots, \xi^{-d}) = \xi^{-md} \xi^{m(m-1)} \begin{bmatrix} d+1 \\ m \end{bmatrix} (\xi^2) = \xi^{-m(d-(m-1))} \begin{bmatrix} (d-(m-1))+m \\ m \end{bmatrix} (\xi^2)$$

$$= \xi^{-m(d-(m-1))} h_m(1, \xi^2, \dots, \xi^{2(d-(m-1))}) = h_m(\xi^{d-(m-1)}, \xi^{d-(m-1)-2}, \dots, \xi^{-(d-(m-1))}),$$

whence the coefficient of $\chi^{(1^m)}$ is t^{m-1} times the coefficient of $\chi^{(m)}$, i.e.,

$$P_{(1^m)}(t) = t^{m-1} P_{(m)}(t).$$

By Theorem 4.1, we have

$$P_{(m)}(1/t) = t^{m-1} P_{(1^m)}(1/t) = t^{m-1}(-1)^m t^2 P_{(m)}(t) = (-1)^m t^{m+1} P_{(m)}(t).$$

Since $H(I_m, t) = P_{(m)}(t)$, this finishes the proof. Q.E.D.

Some weak analogues of classical theorems

1. The Cayley—Sylvester theorem again

We have earlier seen a noncommutative analogue of the Cayley—Sylvester theorem. We are now going to give an analogue in another direction. First a

Definition. When $\alpha=(\alpha_1, \ldots, \alpha_n)\in N^n$, we say that the length, $l(\alpha)$, of α is n, and we put

$$n(\alpha) = \sum_{i=1}^{n} (i-1)\alpha_i$$

(cf. [17] for the corresponding notion for partitions). When λ is a partition, we let $B(\lambda, d, j)$ be the number of distinct permutations α of λ of length $d+1$ such that $n(\alpha)=j$ (note that α may contain zeros). Hence $B(\lambda, d, j)=0$ if $l(\lambda)>d+1$.

Finally let $a_\lambda(d, m)$ be the number of times M^λ appears in $\tilde{I}_d^m$, considered as an S_m-module.

Proposition 1.2. $a_\lambda(d, m)=\sum_{|\mu|=m} K_{\lambda\mu}(B(\mu, d, \frac{1}{2}md)-B(\mu, d, \frac{1}{2}md-1))$.

Proof. We have

$$a_\lambda(d, m) = \int (1-\xi^{-2})s_\lambda(\xi^d, \ldots, \xi^{-d})$$

by Theorem 1.4 in the foregoing chapter. But $s_\lambda=\sum_\mu K_{\lambda\mu}m_\mu$, and

$$m_\mu(\xi^d, \ldots, \xi^{-d}) = \sum_\alpha (\xi^d)^{\alpha_1}(\xi^{d-2})^{\alpha_2}\ldots(\xi^{-d})^{\alpha_{d+1}},$$

where the sum is over all distinct permutations α of μ. Hence

$$m_\mu(\xi^d, \ldots, \xi^{-d}) = \sum_\alpha \xi^{md-2n(\alpha)},$$

and

$$\int (1-\xi^{-2})m_\mu(\xi^d, \ldots, \xi^{-d}) = B\left(\mu, d, \tfrac{1}{2}md\right)-B\left(\mu, d, \tfrac{1}{2}md-1\right). \quad \text{Q.E.D.}$$

Let, as usual, $A(j, m, d)$ be the number of partitions of j into m non-negative integers of size $\leq d$. If $\alpha\in N^{d+1}$, let $\theta(\alpha)$ denote the partition $(0^{\alpha_1}, 1^{\alpha_2}, \ldots, d^{\alpha_{d+1}})$. Then $|\theta(\alpha)|=n(\alpha)$ and $l(\theta(\alpha))\leq|\alpha|$. By mapping $\alpha\mapsto\theta(\alpha)$, we see that

$$\sum_{|\mu|=m} B(\mu, d, j) = A(j, m, d).$$

Since $\chi^{(m)}$ is the trivial S_m-character, we have

$$\begin{aligned}\dim_k I_d^m = a_{(m)}(d, m) &= \sum_{|\mu|=m}\left(B\left(\mu, d, \tfrac{1}{2}md\right)-B\left(\mu, d, \tfrac{1}{2}md-1\right)\right)\\ &= A\left(\tfrac{1}{2}md, m, d\right)-A\left(\tfrac{1}{2}md-1, m, d\right),\end{aligned}$$

since $K_{(m)\mu}=1$ for all μ. This is the ordinary Cayley—Sylvester theorem.

2. The Hermite reciprocity theorem

In the commutative case, the famous Hermite reciprocity theorem states that

$$\dim_k I_d^m = \dim_k I_m^d,$$

for all m and d.

In [1], Almkvist proves a generalized version of this:

$$S^m(R_d^*) \cong S^d(R_m^*)$$

as G-modules. Let us give a quick proof: the G-character of $S^m(R_d^*)$ is $h_m(\xi^d, \dots, \xi^{-d})$. Now

$$h_m(\xi^d, \dots, \xi^{-d}) = \xi^{-md} \begin{bmatrix} d+m \\ m \end{bmatrix} (\xi^2) = \xi^{-md} \begin{bmatrix} d+m \\ d \end{bmatrix} (\xi^2) = h_d(\xi^m, \dots, \xi^{-m}),$$

and we are done. We note that the crucial step is the symmetry relation

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n \\ n-r \end{bmatrix}$$

between Gaussian polynomials.

There seems to be no simple analogue of Hermite's theorem in the noncommutative case. For example,

$$\dim_k \tilde{I}_1^{2q} = \frac{1}{q+1}\binom{2q}{q}, \quad \text{but} \quad \dim_k \tilde{I}_{2q}^1 = 0.$$

However, it is quite possible that there are other symmetry relations between our G-modules. We will derive two such relations, one rather trivial and the other somewhat less obvious.

The S_m-decomposition of $\tilde{I}_d^m$ is

$$\textstyle\sum_{|\lambda|=m} \int (1-\xi^{-2}) s_\lambda(\xi^d, \dots, \xi^{-d}) \chi^\lambda.$$

In the ring Λ we have (see [17], Ch. I, § 4)

$$\textstyle\sum_\lambda s_\lambda(\xi^d, \dots, \xi^{-d}) s_\lambda(y) = \sum_\lambda m_\lambda(\xi^d, \dots, \xi^{-d}) h_\lambda(y).$$

Since h_λ corresponds to the character

$$\eta_\lambda = \operatorname{ind}_{S_\lambda}^{S_m}(1_{S_\lambda})$$

we have another decomposition of $\tilde{I}_d^m$, namely

$$\textstyle\sum_{|\mu|=m} \int (1-\xi^{-2}) m_\lambda(\xi^d, \dots, \xi^{-d}) \eta_\lambda.$$

Let the coefficient of η_λ be $b_\lambda(d, m)$ (which may be negative) and put

$$b(d, m) = \textstyle\sum_{|\lambda|=m} b_\lambda(d, m).$$

Then we have a very weak analogue of Hermite's theorem:

Proposition 2.1. $b(d,m)=b(m,d)$. *In fact,* $b(d,m)=\dim_k I_d^m$.

Proof.

$$b(d,m)=\sum_{|\lambda|=m}\int(1-\xi^{-2})m_\lambda(\xi^d,\dots,\xi^{-d})=\int(1-\xi^{-2})\sum_\lambda m_\lambda(\xi^d,\dots,\xi^{-d})$$
$$=\int(1-\xi^{-2})s_{(m)}(\xi^d,\dots,\xi^{-d})=\int(1-\xi^{-2})h_m(\xi^d,\dots,\xi^{-d})=\dim_k I_d^m. \quad \text{Q.E.D.}$$

To consider the sum of the coefficients $b_\lambda(d,m)$ is not as artificial as it may seem, because the ordinary dimension $\dim_k I_d^m$ equals the sum of the coefficients in the decomposition of I_d^m into irreducible S_m-modules (since I_d^m is a trivial S_m-module). Of course, from this point of view it is more natural to consider the sum

$$a(d,m)=\sum_{|\lambda|=m}a_\lambda(d,m),$$

where $a_\lambda(d,m)$ is the coefficient of χ^λ in the decomposition of $\tilde{I}_d^m$, but unfortunately, $a(d,m)$ does not follow the reciprocity law, e.g., $a(2,1)=0$, but $a(1,2)=1$ (see the section on the algebra $\tilde{I}_1$). We will consider the $a(d,m)'s$ more in the next section.

As was noted above, the Hermite reciprocity law hinges on a symmetry relation between Gaussian polynomials: $\begin{bmatrix} n \\ r \end{bmatrix}=\begin{bmatrix} n \\ n-r \end{bmatrix}$. Let us exploit this relation a little more:

Lemma 2.2. $e_m(\xi^d,\dots,\xi^{-d})=e_{d-m+1}(\xi^d,\dots,\xi^{-d})$ (both sides should be interpreted as zero if $m>d+1$).

Proof.

$$e_m(\xi^d,\dots,\xi^{-d})=\xi^{-md}e_m(1,\xi^2,\dots,\xi^{2d})=\xi^{-md}\xi^{m(m-1)}\begin{bmatrix} d+1 \\ m \end{bmatrix}(\xi^2)$$
$$=\xi^{-m(d-m+1)}\begin{bmatrix} d+1 \\ m \end{bmatrix}(\xi^2)=\xi^{-m(d-m+1)}\begin{bmatrix} d+1 \\ d-m+1 \end{bmatrix}(\xi^2)$$
$$=\xi^{d(d-m+1)}e_{d-m+1}(1,\xi^2,\dots,\xi^{2d})=e_{d-m+1}(\xi^d,\dots,\xi^{-d}). \qquad \text{Q.E.D.}$$

The antisymmetric part $M_{T^m(R_d^*)}((1^m))$ with G-character $e_m(\xi^d,\dots,\xi^{-d})$ can be identified with the m'th exterior power $\Lambda^m(R_d^*)$. Hence the lemma implies that

$$\Lambda^m(R_d^*)\cong\Lambda^{d-m+1}(R_d^*)$$

as G-modules, and

$$\dim_k(\Lambda^m(R_d^*))^G=\dim_k(\Lambda^{d-m+1}(R_d^*))^G,$$

a Λ-Hermite theorem.

Furthermore, since

$$\sum_{d\geq 0}\dim_k(\Lambda^m(R_d^*))^G t^d=t^{m-1}H(I_m,t)$$

by Section 3.5, we have by the commutative Cayley—Sylvester theorem,

$$\dim_k\left(\Lambda^m(R_d^*)\right)^G = \dim_k I_m^{d-m+1} = A\left(\tfrac{1}{2}m(d-m+1),\, d-m+1,\, m\right)$$
$$-A\left(\tfrac{1}{2}m(d-m+1)-1,\, d-m+1,\, m\right).$$

Finally, we get

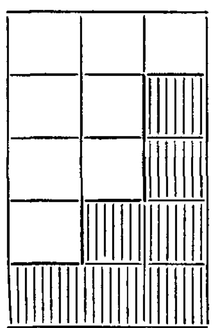
$$H\left((\Lambda(R_d^*))^G, t\right) = \sum_{m\geq 0}(1-\xi^{-2})e_m(\xi^d, \ldots, \xi^{-d})t^m = \int (1-\xi^{-2})\prod_{j=0}^{d}(1+\xi^{d-2j}t)$$
$$= \frac{1}{\pi}\int_0^{2\pi} \sin^2 x \prod_{j=0}^{d}(1+e^{(d-2j)ix}t)\,dx$$

(see also [1], p. 334).

Writing s_λ as a determinant in the e-functions, the lemma can be generalized. If λ is a partition of m of length $\leqq d+1$, let $\tilde{\lambda}$ be the partition defined by

$$\tilde{\lambda}' = (d+1-\lambda'_{l'},\, d+1-\lambda'_{l'-1},\, \ldots,\, d+1-\lambda'_1),$$

where $l'=l(\lambda')$ $(=\lambda_1)$. For instance, if $\lambda=(3, 2^2, 1)$, and $d=4$, then $\tilde{\lambda}$ is the shaded area in the diagram below, i.e., $\tilde{\lambda}=(3, 2, 1^2)$.



Proposition 2.3. $s_\lambda(\xi^d, \ldots, \xi^{-d}) = s_{\tilde{\lambda}}(\xi^d, \ldots, \xi^{-d})$, *i.e.,*

$$S^\lambda(R_d^*) \cong S^{\tilde{\lambda}}(R_d^*)$$

as G-modules. (Note that $|\tilde{\lambda}| = l'(d+1)-|\lambda'| = \lambda_1(d+1)-m$.*)*

Proof. By [17], Ch. 1, § 3, we have

$$s_\lambda(\xi^d, \ldots, \xi^{-d}) = \det\left(e_{\lambda'_i-i+j}(\xi^d, \ldots, \xi^{-d})\right)_{1\leq i,j\leq l'}.$$

As was noted above, this can be written

$$\det\left(e_{d+1-\lambda'_i+i-j}(\xi^d, \ldots, \xi^{-d})\right).$$

Now $d+1-\lambda_i'=\tilde{\lambda}_{l'}'-i+1$, whence

$$s_\lambda(\xi^d, \ldots, \xi^{-d}) = \det\left(e_{\tilde{\lambda}'_{l'-i+1}-(l'-i+1)+(l'-j+1)}(\xi^d, \ldots, \xi^{-d})\right)$$

$$= \det\left(e_{\tilde{\lambda}'_i-i+j}(\xi^d, \ldots, \xi^{-d})\right) = s_{\tilde{\lambda}}(\xi^d, \ldots, \xi^{-d}). \qquad \text{Q.E.D.}$$

Example 2.1. If $\lambda=(1^m)$, then $\tilde{\lambda}=(1^{d-m+1})$, and

$$s_{(1^m)}(\xi^d, \ldots, \xi^{-d}) = s_{(1^{d-m+1})}(\xi^d, \ldots, \xi^{-d}),$$

by the proposition.

If $\lambda=(m)$, then $\tilde{\lambda}=(m^d)$, and

$$s_{(m)}(\xi^d, \ldots, \xi^{-d}) = s_{(m^d)}(\xi^d, \ldots, \xi^{-d}) = \xi^{-md^2}s_{(m^d)}(1, \xi^2, \ldots, \xi^{2d})$$

$$= \xi^{-md^2}\xi^{md^2-md}\prod_{\substack{1\le i\le d\\1\le j\le m}}\frac{1-\xi^{2(d+1+j-i)}}{1-\xi^{2(d-i+m-j+1)}} = \xi^{-md}\prod\frac{1-\xi^{2(i+j)}}{1-\xi^{2(i+j-1)}}$$

$$= \xi^{-md}\frac{1-\xi^{2(m+1)}}{1-\xi^2}\cdot\frac{1-\xi^{2(m+2)}}{1-\xi^4}\cdot\ldots\cdot\frac{1-\xi^{2(m+d)}}{1-\xi^{2d}}$$

$$= \xi^{-md}\begin{bmatrix}d+m\\d\end{bmatrix}(\xi^2) = s_{(d)}(\xi^m, \ldots, \xi^{-m}).$$

We finish this section with a remark on the functions $s_\lambda(\xi^d, \ldots, \xi^{-d})$, and we freely use the notation of [17], p. 65. We have

$$s_\lambda(\xi^d, \ldots, \xi^{-d}) = (s_\lambda\circ s_{(d)})(\xi, \xi^{-1}),$$

where $\circ$ denotes plethysm. On the one hand, we can write

$$s_\lambda(\xi^d, \ldots, \xi^{-d}) = \sum_l \alpha_l\frac{\xi^{l+1}-\xi^{-(l+1)}}{\xi-\xi^{-1}},$$

and on the other

$$(s_\lambda\circ s_{(d)})(\xi, \xi^{-1}) = \sum_{|\varrho|=m} a^\varrho_{\lambda(d)}s_\varrho(\xi, \xi^{-1}).$$

But if $\varrho=(\varrho_1, \varrho_2)$, then

$$s_\varrho(\xi, \xi^{-1}) = \frac{\xi^{\varrho_1-\varrho_2+1}-\xi^{-(\varrho_1-\varrho_2+1)}}{\xi-\xi^{-1}}.$$

Since $\alpha_l\geqq 0$ for all l it follows that $a^\varrho_{\lambda(d)}\geqq 0$ for all ϱ with $l(\varrho)\leqq 2$ (this is a very special case of the discussion in the appendix to Ch. I in Macdonald's book [17]).

We also conclude that if md is even, then $\varrho_1-\varrho_2$ is also even, whence $a_l=0$ if l is odd. Conversely, if md is odd, then $\alpha_l=0$ if l is even (which once again shows that no invariants exist in this case).

We have an integral formula for the $a^{\varrho}_{\lambda(d)}$:s:

$$a^{\varrho}_{\lambda(d)} = \frac{1}{\pi}\int_0^{2\pi} \sin x \sin (l+1)x s_\lambda(e^{dix}, \ldots, e^{-dix})\,dx$$

where $l=\varrho_1-\varrho_2$.

For more information on the coefficients $a^{\varrho}_{\lambda(d)}$, see, e.g., [11], [16], [17], and [18].

Finally, we cannot resist giving yet another formulation of the classical Hermite theorem:

$$(s_{(m)}\circ s_{(d)})(\xi, \xi^{-1}) = (s_{(d)}\circ s_{(m)})(\xi, \xi^{-1}),$$

or

$$(h_m\circ h_d)(\xi, \xi^{-1}) = (h_d\circ h_m)(\xi, \xi^{-1}).$$

3. *An interesting power series*

Let as above $a(d, m)$ be the number of irreducible components in the S_m-decomposition of $\tilde{I}_d^m$. As was noted above, this is in a certain sense a generalization of the dimension $\dim_k I_d^m$ in the commutative case. In fact, this dimension is the number of elements that together with addition and multiplication by scalars generate I_d^m. In the noncommutative case we have another operation beside these two, namely permutation of the factors. The numbers $a(d, m)$ are at least upper limits for the number of elements that generate $\tilde{I}_d^m$ together with addition, multiplication by scalars, and operations with the symmetric group S_m. Inspired by this observation, let us consider the series

$$\tilde{H}(\tilde{I}_d, t) = \sum_{m\geq 0} a(d, m)\, t^m.$$

Theorem 1.4 in the foregoing chapter gives us

$$a(d, m) = \sum_{|\lambda|=m} \int (1-\xi^2)\, s_\lambda(\xi^d, \ldots, \xi^{-d}),$$

whence

$$\begin{aligned}\sum_{m\geq 0} a(d, m)\, t^m &= \int (1-\xi^2) \sum_{m\geq 0} \Big(\sum_{|\lambda|=m} s_\lambda(\xi^d, \ldots, \xi^{-d})\Big) t^m \\ &= \int (1-\xi^2) \sum_\lambda s_\lambda(\xi^d, \ldots, \xi^{-d})\, t^{|\lambda|} = \int (1-\xi^2) \sum_\lambda s_\lambda(t\xi^d, \ldots, t\xi^{-d}) \\ &= \int (1-\xi^2) \prod_{i=0}^d (1-\xi^{d-2i}t)^{-1} \prod_{0\leq i<j\leq d} (1-\xi^{2(d-i-j)}t^2)^{-1},\end{aligned}$$

by the beautiful formula in [17], Ch. I § 5, Ex. 4.

This proves

Proposition 3.1.

$$\tilde{H}(\tilde{I}_d, t) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-e^{2ix})\,dx}{\prod_{0\leq j\leq d}(1-e^{(d-2j)ix}t)\prod_{0\leq j<k\leq d}(1-e^{2(d-j-k)ix}t^2)}.$$

In particular, $\tilde{H}(\tilde{I}_d, t)$ is rational (see Proposition 3.3 below). This formula resembles Springer's integral formula for the Hilbert series of I_d (see [22]) — the difference is the very unpleasant second factor in the denominator.

Example 3.1. One can compute

$$\tilde{H}(\tilde{I}_1, t) = \frac{1}{1-t^2}, \quad \tilde{H}(\tilde{I}_2, t) = \frac{1}{(1-t^2)(1-t^3)(1-t^4)},$$

and, with some effort,

$$\tilde{H}(\tilde{I}_3, t) = \frac{1+2t^8+3t^{10}+5t^{12}+3t^{14}+2t^{16}+t^{24}}{(1-t^2)(1-t^4)^2(1-t^6)^2(1-t^8)(1-t^{10})}.$$

We have a reciprocity relation:

Proposition 3.2.

$$\tilde{H}(\tilde{I}_d, 1/t) = (-1)^{\binom{d}{2}} t^{(d+1)^2} \tilde{H}(\tilde{I}_d, t).$$

Proof. The series

$$\textstyle\sum_\lambda s_\lambda(t\xi^d, \ldots, t\xi^{-d})$$

obviously converges for $0<t<1$. Write

$$\tilde{H}(\tilde{I}_d, t) = \frac{1}{2\pi i} \int_C \frac{(1-z^2)\,dz}{z\prod(1-z^{d-2j}t)\prod(1-z^{2(d-j-k)}t^2)}.$$

We consider the poles of the integrand corresponding to the factors in the denominator with $d-2j<0$ and $d-j-k<0$. If $t>1$ so that $\tilde{H}(\tilde{I}_d, 1/t)$ converges, then, noting that the products in the denominator are symmetric in z, z^{-1},

$$\tilde{H}(\tilde{I}_d, 1/t) = \frac{1}{2\pi i} \int_C \frac{(1-z^2)\,dz}{z\prod(1-z^{d-2j}t^{-1})\prod(1-z^{2(d-j-k)}t^{-2})}$$

$$= \frac{1}{2\pi i} t^{d+1} t^{2d(d+1)/2} (-1)^{d+1+d(d+1)/2} \int_C \frac{(1-z^2)\,dz}{z\prod(1-z^{d-2j}t)\prod(1-z^{2(d-j-k)}t^2)}.$$

Here the poles corresponding to $d-2j<0$, $d-j-k<0$ lie outside C, and the result follows if we note that the sum of the residues of a rational function is 0. Q.E.D.

Denote by $c(d, m)$ the number of irreducible components in the S_m-decomposition of $T^m(R_d^*)$. Then $c(d, m)$ is the value of $\sum_\lambda s_\lambda(\xi^d, \ldots, \xi^{-d})$ for $\xi=1$, whence

$$\tilde{H}(T(R_d^*), t) = \textstyle\sum_{\text{all } \lambda} s_\lambda(t\xi^d, \ldots, t\xi^{-d})\big|_{\xi=1} = \frac{1}{(1-t)^{d+1}(1-t^2)^{(1/2)d(d+1)}}.$$

The $\tilde{H}$-series considered here are Hilbert series in the usual sense, since we have

Proposition 3.3.

$$\tilde{H}(T(R_d^*), t) = H(S(R_d^* \oplus \Lambda^2 R_d^*), t)$$
$$\tilde{H}(\tilde{I}_d, t) = H(S(R_d^* \oplus \Lambda^2 R_d^*)^G, t)$$

where we have given the elements of $\Lambda^2 R_d^*$ the degree 2.

Proof. This is essentially obvious. One way to see it is to identify $S(R_d^* \oplus \Lambda^2 R_d^*)$ with $S(R_d^*) \otimes_k S(\Lambda^2 R_d^*)$ and then note that

$$\sum_{m \geq 0} Tr(S^m(R_d^*), g)\, t^m) = \prod_{0 \leq j \leq d} (1 - \xi^{d-2j} t)^{-1}$$

and

$$\sum_{m \geq 0} Tr\left(S^m(\Lambda^2 R_d^*), g\right) t^{2m} = \prod_{0 \leq j < k \leq d} (1 - \xi^{2(d-j-k)} t^2)^{-1}$$

where g is the element $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ of G (since the eigenvalues of g as an endomorphism of $S^m(\Lambda^2 R_d^*)$ are $\xi^{2(d-j-k)}$, $j<k$). Q.E.D.

Example 3.2. As a k-algebra, $S(R_2^* \oplus \Lambda^2 R_2^*)^G$ is generated by

$$a_0 a_2 - a_1^2, \quad a_2(a_0 \wedge a_1) - a_1(a_0 \wedge a_2) + a_0(a_1 \wedge a_2),$$

and

$$4(a_0 \wedge a_1)(a_1 \wedge a_2) - (a_0 \wedge a_2)^2.$$

This case is especially simple since $R_2^* \cong \Lambda^2 R_2^*$ as G-modules; an isomorphism is given by

$$a_0 \wedge a_1 \mapsto a_0$$
$$a_0 \wedge a_2 \mapsto 2a_1$$
$$a_1 \wedge a_2 \mapsto a_2.$$

We will finish this section with a short discussion of finite groups. Let V be a finite-dimensional vector space, and let G be a finite subgroup of $GL(V)$. Denote by c_m the number of irreducible components in the S_m-decomposition of $T^m(V)^G$. Put

$$\tilde{H}(T(V)^G, t) = \sum_{m \geq 0} c_m t^m.$$

Then we have a nice analogue of Molien's theorem:

Proposition 3.4.

$$\tilde{H}(T(V)^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - tg)\det(1 - t^2 \Lambda^2 g)}.$$

Proof. Let the eigenvalues of $g \in GL(V)$ be $\varrho_1, \ldots, \varrho_n$. As in the proof of Theorem 1.4 in the foregoing chapter, we see that $M_{T^m(V)}(\lambda)$ is stable under $GL(V)$, and that the trace of g on this space is $\dim_k M^\lambda \cdot s_\lambda(\varrho_1, \ldots, \varrho_n)$. Hence

$$c_m = \frac{1}{|G|} \sum_{|\lambda| = m} \sum_{g \in G} s_\lambda(\varrho_1(g), \ldots, \varrho_n(g)).$$

Multiplying by t^m and summing over m gives

$$\tilde{H}(T(V)^G, t) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\prod_i (1-\varrho_i(g)t) \prod_{i<j}(1-\varrho_i(g)\varrho_j(g)t^2)}. \quad \text{Q.E.D.}$$

Remark. Let V have dimension 2, and let G be a finite subgroup of $SL(2, k)$ (i.e., a finite cyclic group, a dihedral group, or a binary polyhedral group). Then the proposition gives

$$\tilde{H}(T(V)^G, t) = \frac{1}{1-t^2} H(S(V)^G, t).$$

Some results on covariants

We have earlier defined

$$\tilde{C}_{dme} = (T^m(R_d^*) \otimes_k R_e)^G$$
$$\tilde{C}_{dm} = (T^m(R_d^*) \otimes_k R)^G$$
$$\tilde{C}_d = (T(R_d^*) \otimes_k R)^G.$$

The G-character of $T^m(R_d^*) \otimes_k R_e$ is $\chi_d(\xi)^m \chi_e(\xi)$ (see the introduction). For any invariants to exist in this space, $\frac{1}{2}(md-e)$ must be a non-negative integer, as we saw in the chapter on the symbolic method. We note the following, which will be used later:

$$\sum_{1/2(md-e)\in N} \chi_e(\xi) = \begin{cases} (\xi^{md+2}+\xi^{-(md+2)}-2)/(\xi-\xi^{-1})^2 & \text{if } md \text{ is even,} \\ (\xi^{md+2}+\xi^{-(md+2)}-(\xi+\xi^{-1}))/(\xi-\xi^{-1})^2 & \text{if } md \text{ is odd.} \end{cases}$$

1. The Hilbert series of $\tilde{C}_d$

When defining the Hilbert series $H(\tilde{C}_d, t)$ we use the grading in the m-index, i.e.,

$$H(\tilde{C}_d, t) = \sum_{m\geq 0} \dim_k \tilde{C}_{dm} t^m.$$

Since there are only finitely many e:s involved in

$$\bigoplus_e \tilde{C}_{dme} = \tilde{C}_{dm},$$

we see that $\dim_k \tilde{C}_{dm}$ is finite, and the series above is well-defined.

Theorem 1.1.

$$H(\tilde{C}_d, t) = -\frac{1}{t} \sum_{j=1}^{d} \frac{1-\eta_j^2}{(1-\eta_j^{\varepsilon(d)})(d\eta_j^d+(d-2)\eta_j^{d-2}+\ldots-d\eta_j^{-d})}$$

where $\varepsilon(d)=1$ *if d is odd and* 2 *if d is even, and* $\eta_1, \ldots, \eta_d$ *are the distinct roots of* $z^{2d}+z^{2d-2}+\ldots+1-t^{-1}z^d=0$ *which lie in the unit disc for small t (equivalently, which lie in* $\mathbf{C}[[t^{1/d}]]$*).*

(This is not surprising, the theorem bears the same relationship to the formula for $H(\tilde{I}_d, t)$ obtained by Almkvist, Dicks and Formanek in [4] as the Hilbert series $H(C_d, t)$ does to $H(I_d, t)$, see, e.g., [1].)

Proof. We will consider t as a real variable with $0<t<(d+1)^{-1}$.

(i) d even. We compute:

$$\sum_{m\geq 0}\chi_d(\xi)^m\left(\sum_e \chi_e(\xi)\right)t^m = \sum_{m\geq 0}\left(\frac{\xi^{d+1}-\xi^{-(d+1)}}{\xi-\xi^{-1}}\right)^m \frac{\xi^{md+2}+\xi^{-(md+2)}-2}{(\xi-\xi^{-1})^2}\,t^m.$$

Now

$$\frac{1}{\pi}\int_0^{2\pi}\sin^2 x(e^{dix}+e^{(d-2)ix}+\ldots+e^{-dix})^m\,\frac{e^{(md+2)ix}+e^{-(md+2)ix}}{(e^{ix}-e^{-ix})^2}\,dx$$

$$=-\frac{1}{4\pi}\int_0^{2\pi}(e^{dix}+\ldots+e^{-dix})^m(e^{(md+2)ix}+e^{-(md+2)ix})\,dx = 0,$$

and so

$$H(\tilde{C}_d, t)=\frac{1}{\pi}\int_0^{2\pi}\sin^2 x\left[\sum_{m\geq 0}\left(\frac{e^{(d+1)ix}-e^{-(d+1)ix}}{e^{ix}-e^{-ix}}\right)^m\frac{(-2)}{(e^{ix}-e^{-ix})^2}\,t^m\right]dx$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{dx}{1-(e^{dix}+\ldots+e^{-dix})t}=\frac{1}{2\pi i t}\int_C\frac{dz}{t^{-1}z-(z^{d+1}+z^{d-1}+\ldots+z^{-(d-1)})}.$$

Applying the residue theorem proves the theorem in this case.

(ii) d odd. We compute:

$$\sum_{m\geq 0}\chi_d(\xi)^m\left(\sum_{1/2(md-e)\in N}\chi_e(\xi)\right)t^m$$

$$=\sum_{m\geq 0}\chi_d(\xi)^m\frac{\xi^{md+2}+\xi^{-(md+2)}-2}{(\xi-\xi^{-1})^2}\,t^m+\sum_{m \text{ odd}}\chi_d(\xi)^m\frac{2-\xi-\xi^{-1}}{(\xi-\xi^{-1})^2}\,t^m.$$

The first sum was considered above; the second equals

$$(\xi-\xi^{-1})^{-2}(2-\xi-\xi^{-1})\frac{\xi^{d+1}-\xi^{-(d+1)}}{\xi-\xi^{-1}}\,t\cdot\frac{1}{1-\left(\dfrac{\xi^{d+1}-\xi^{-(d+1)}}{\xi-\xi^{-1}}\,t\right)^2},$$

whence its contribution to $H(\tilde{C}_d, t)$ is (we use the symmetry in ξ, ξ^{-1})

$$\frac{2}{2\pi(2i)^2}\int_0^{2\pi}(1-e^{ix})\left(\frac{1}{1-\dfrac{e^{(d+1)ix}-e^{-(d+1)ix}}{e^{ix}-e^{-ix}}\,t}-\frac{1}{1+\dfrac{e^{(d+1)ix}-e^{-(d+1)ix}}{e^{ix}-e^{-ix}}\,t}\right)dx$$

$$=\left(-\tfrac{1}{2}\right)\left(I(t)-I(-t)\right).$$

By residue calculus, $I(t)$ equals

$$-\frac{1}{t}\sum_{j=1}^{d}\frac{1-\eta_j}{d\eta_j^d+\ldots-d\eta_j^d}.$$

If η is a root of the equation $z^{2d}+\ldots+1-t^{-1}z^d=0$, then $(-\eta)^{2d}+\ldots+1=\eta^{2d}+\ldots+1=t^{-1}\eta^d=(-t^{-1})(-\eta)^d$, since d is odd. Hence

$$I(-t)=\left(-\frac{1}{t}\right)\sum_{j=1}^{d}\frac{1+\eta_j}{d\eta_j^d+\ldots-d\eta_j^{-d}},$$

and

$$\left(-\tfrac{1}{2}\right)\left(I(t)-I(-t)\right)=\left(-\frac{1}{t}\right)\sum_{j=1}^{d}\frac{\eta_j}{d\eta_j^d+\ldots-d\eta_j^{-d}}.$$

Adding this to the expression obtained in (i), we get the desired result. Q.E.D.

Example 1.1. We can compute

$$H(\tilde{C}_1, t)=\frac{2}{1-2t+\sqrt{1-4t^2}}$$

$$H(\tilde{C}_2, t)=\frac{1}{\sqrt{(1-3t)(1+t)}}.$$

2. $\tilde{C}_{dm}$ *as an* S_m*-module*

By permutation of the factors in $T^m(R_d^*)$ we define an S_m-module structure on $T^m(R_d^*)\otimes_k R$, i.e., also on $\tilde{C}_{dm}$. We let the S_m-character be $\Gamma_d^m(\tilde{C})$. There are analogues of Theorems 2.1 and 4.1 in the chapter on the S_m-structure of $\tilde{I}_d^m$:

Theorem 2.1.

$$\sum_{d\geq 0}\Gamma_d^m(\tilde{C})t^d$$

$$=\sum_{|\nu|=m}\left(\sum_{|\lambda|=m}(K^{-1})_{\lambda\nu}\sum_{\substack{\mu\subset\lambda'\\|\mu|<(1/2)m}}\binom{\lambda'}{\mu}(-1)^{|\mu|}\varphi_{m-2|\mu|}\left(\frac{f_{\lambda,\mu}^{\pm}(t)}{1-t^{\varepsilon(m)}}\right)\right)\chi^\nu,$$

where $f_{\lambda,\mu}^{\pm}(t)$ *and* $\varepsilon(m)$ *have the same meaning as before. Furthermore, if this expression is written*

$$\sum_{|\nu|=m}Q_\nu(t)\chi^\nu,$$

then

$$\sum_{|\nu|=m}Q_\nu(1/t)\chi^\nu=(-1)^m t^2\sum_{|\nu|=m}Q_\nu(t)\chi^{\nu'}.$$

The proof is a copy of the proofs of the results for $\tilde{I}_d^m$.

Example 2.1. With some effort, we can compute

$$\sum_{d\geq 0} \Gamma_d^2(\tilde{C})t^d = \frac{\chi^{(2)}+t\chi^{(1^2)}}{(1-t)(1-t^2)},$$

$$\sum_{d\geq 0} \Gamma_d^3(\tilde{C})t^d = \frac{(1+t^3)\chi^{(3)}+t(1+t+t^2+t^3)\chi^{(2,1)}+t^2(1+t^3)\chi^{(1^3)}}{(1-t)(1-t^2)(1-t^4)};$$

As before, we get two corollaries:

Corollary 2.2. *Let* $F_m(\tilde{C}, t)=\sum_{d\geq 0}(\dim_k \tilde{C}_{dm})t^d$. *Then*

$$F_m(\tilde{C}, t) = \frac{1}{t}\sum_{0\leq j<(1/2)m}\binom{m}{j}(-1)^j\varphi_{m-2j}\left(\frac{t^m}{(1-t^{\varepsilon(m)})(1-t^2)^{m-1}}\right),$$

and $F_m(\tilde{C}, 1/t)=(-1)^m t^2 F_m(\tilde{C}, t)$.

Example 2.2.

$$F_1(\tilde{C}, t) = \frac{1}{1-t}$$

$$F_2(\tilde{C}, t) = \frac{1}{(1-t)^2}$$

$$F_3(\tilde{C}, t) = \frac{1+t+t^2}{(1-t)^2(1-t^2)}.$$

Corollary 2.3 (Springer [22], Almkvist [1]). *The Hilbert series of the commutative algebra* C_m *is*

$$H(C_m, t)$$
$$= \sum_{0\leq j<(1/2)m}(-1)^j\varphi_{m-2j}\left(\frac{t^{j(j+1)}}{(1-t^{\varepsilon(m)})(1-t^4)\ldots(1-t^{2(m-j)})(1-t^2)\ldots(1-t^{2j})}\right).$$

Furthermore, $H(C_m, 1/t)=(-1)^m t^{m+1} H(C_m, t)$.

Proof. Just take the coefficient of $\chi^{(m)}$. Q.E.D.

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