# Traces on curves of Sobolev spaces of holomorphic functions

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## 1. Introduction and statement of results

Let $B^n$ denote the unit ball of $\mathbf{C}^n$. For an holomorphic function $f$ on $B^n$ with homogeneous expansion $f(z)=\sum_k f_k(z)$, the fractional derivative $D^\beta f$ of order $\beta\in\mathbf{C}$ is defined by

(1) $$D^\beta f(z) = \sum_k (k+1)^\beta f_k(z).$$

Thus $D=R+\mathrm{Id}$, where $R=\sum_j z_j \frac{\partial}{\partial z_j}$ is the radial derivative. (The reason for using $D$ instead of $R$ is to get a one to one map and to have simple expressions for the inverse map.) The *Hardy—Sobolev* space $H_\beta^p$, $1\leqq p<\infty$, $\beta>0$, is defined

$$H_\beta^p = \left\{f\colon \sup_r \int_S |D^\beta f(r\zeta)|^p d\sigma(\zeta) = \|f\|_{p,\beta}^p < +\infty\right\},$$

where $S$ denotes the boundary of $B^n$ and $d\sigma$ its Lebesgue measure. It is well-known (see [5] and [6]) that in case $\beta$ is an integer one can use $R^\beta$ instead of $D^\beta$ and if $f\in H_\beta^p$ then all derivatives up to order $\beta$ of $f$ belong to $H^p$ (i.e. they are in $L^p(S)$), and thus $H_\beta^p$ can be thought as the analogue of the Sobolev and potential spaces in real analysis. The space $H_\beta^p$ has been the subject of several recent papers and a theory of holomorphic Sobolev spaces is being systematically developed, in many aspects analogous to the real-variable theory of Sobolev and potential spaces (see [5], [7], [12], [14], [1], [2], [3] and the forthcoming book [6]).

The object of this paper is to describe the trace of the spaces $H_\beta^p$ along certain submanifolds of $S$, i.e. to find the analogue of the "trace theorem" (see [8] or [16]).

To simplify the exposition, here and in the main part of the paper, we will limit ourselves to (simple) closed smooth curves $\Gamma$ on $S$ and to the regular range

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of the spaces $H^p_\beta$, i.e., when all functions in $H^p_\beta$ are continuous up to $S$ (which is $\beta>n/p$ if $p>1$ and $\beta\geqq n$ if $p=1$, see [12], [14], [6]). More general cases will be considered in another section.

As it was to be expected from the real-variable theory, the description of the trace involves *Besov spaces*, whose definition we proceed now to recall. Without loss of generality we may assume our curve $\Gamma$ parametrized by $[-\pi, \pi]$ and identify functions on $\Gamma$ with functions on the unit circle $T$. For $1\leqq p<\infty$ and $\alpha>0$, the Besov space $B^\alpha_p$ is the subspace of $L^p(T)$ defined by the condition, independent of the integer $k>\alpha$,

$$\int_{-\pi}^{+\pi} \frac{\|\Delta^k_t f\|^p_p}{|t|^{1+\alpha p}}\, dt < +\infty \tag{2}$$

where $\Delta^k_t$ denotes the $k$-th difference operator. The index $\alpha$ must be thought as an index of smoothness. We refer to [8] and [16] as a general reference for Besov spaces.

A complete description of the trace (in real analysis terms) can just be expected on those curves in which any nice function can be interpolated by a nice holomorphic function. These are the *complex-tangential curves*, or curves having a tangent at any point in the complex-tangent space at $S$. In terms of a parametrization $\gamma(s)$, this is the condition

$$\gamma'(s)\cdot\overline{\gamma(s)} = 0.$$

Our first main result is a restriction theorem:

**Theorem 1.** *The trace of $H^p_\beta$ on any curve is contained in $B^\alpha_p$, with $\alpha=\beta-\frac{n}{p}+\frac{1}{p}$. If the curve is complex-tangential, the trace is contained in $B^\alpha_p$, with $\alpha=2(\beta-\frac{n}{p})+\frac{1}{p}$.*

The better index of smoothness along complex-tangential curves is in correspondence with the better regularity properties of holomorphic functions along complex-tangential directions (this is a general fact and has been made precise for the spaces $H^p_\beta$ in [3]). From the inclusion $B^\alpha_p\subset \mathrm{Lip}_{\alpha-1/p}(T)$, the usual Lipschitz space of order $\alpha-\frac{1}{p}$, we recover the result that the functions in $H^p_\beta$ are in $\mathrm{Lip}_{\beta-n/p}(B^n)$, and in $\mathrm{Lip}_{2(\beta-n/p)}$ along any complex-tangential curve, again in accordance with the above general fact.

For complex-tangential curves, A. Nagel introduced in [15] a family $(I_q)_{q>1/2}$ of interpolation operators

$$I_q\colon C(\Gamma)\to A(B),\quad I_q\varphi=\varphi \quad \text{on} \quad \Gamma$$

from the space of continuous functions on the curve to the ball algebra.

*Definition.* The curve $\Gamma$ is said to satisfy the condition (I) if given $m\in N$, there exists $q_0=q_0(m)$ such that for $q\geqq q_0(m)$, $I_q$ maps $C^\infty$ functions on $\Gamma$ to functions with bounded radial derivatives up to order $m$.

We point out that, as said before, it is known that $C^\infty$ functions on $\Gamma$ can be interpolated by holomorphic functions in $A^\infty=C^\infty(\bar{B}^n)\cap H(B^n)$, but this uses others methods of interpolation. Our second main result is then:

**Theorem 2.** *If $\Gamma$ is complex-tangential and satisfies condition* (I), *the trace of $H^p_\beta$ is exactly $B^\alpha_p$ with* $\alpha=2(\beta-\frac{n}{p})+\frac{1}{p}$.

We prove that the "model" complex-tangential curve $t\mapsto(\cos t, \sin t)$ in $\mathbf{C}^2$ satisfies the condition (I). We also give a method to check that $q_0(m)=2m$ works for any complex-tangential curve and any concrete value of $m$. For $m=1$ this is easily done, but for higher values of $m$ the computations become more and more involved and long. We have implemented our method in a computer and have checked this property for several values of $m$. All this leads us to conjecture that condition (I) always holds.

The paper is structured as follows. In Section 2 some preliminaries and auxiliary results are collected. In Sections 3 and 4 we prove Theorems 1 and 2, respectively. In Section 5 we discuss some generalizations of Theorems 1 and 2. There we consider the situation for the so-called Bergman—Sobolev spaces, for the non-regular range of the spaces $H^p_\beta$ and also for higher dimensional submanifolds of $S$. In Section 6 we prove our results concerning the condition (I).

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## 2. Preliminaries

**2.1.** The spaces $H^p_\beta$ have a good behaviour under the complex interpolation method, which will allow us to reduce the proofs of the restriction and interpolation theorems to some particular cases. One property we will need is

(3) $$[H^p_{\beta_1}, H^p_{\beta_2}]_\theta = H^p_\beta \quad \beta=(1-\theta)\beta_1+\theta\beta_2.$$

The space $H^p_\beta$ being by definition the domain (in $H^p$) of $D^\beta$, a general result in interpolation (see [17, pg. 103]) shows that (3) is true if $D$ is a positive operator and for any $t\in\mathbf{R}$

(4) $$\|D^{it}f\|_p \leqq C(1+|t|)^N\|f\|_p$$

for some $C$, $N$. In fact something more general than (4) holds. Namely, for any $t, \sigma\in\mathbf{R}$

$$\|D^{\sigma+it}\|_p \cong \|D^\sigma\|_p$$

as operators in $H^p$ (see [6] and [10]), so that (4) holds with $N=0$. Alternatively, (4) can be reduced to the one-variable case by slice integration, and proved using the analogue in the periodic case of Mihlin's multiplier theorem.

On the other hand, if $l\geqq 0$ it is immediately seen that

$$\int_0^1 f(tz)t^l\,dt = (D+l)^{-1}f(z)$$

by checking both members with $f(z)=z^I$ for a multi-index $I$. By the continuous Minkowski inequality and the subharmonicity of $f$ it is then clear that

$$\|(D+l)^{-1}f\|_p \leqq \frac{1}{l+1}\|f\|_p$$

so that $D$ is a positive operator in the sense of [17, Def. 1.14.1].

An analogous result to (3) is also true when $\beta$ is kept fixed:

$$(5)\qquad [H_\beta^{p_1}, H_\beta^{p_2}]_\theta = H_\beta^p,\ \frac{1}{p} = \frac{1-\theta}{p_1}+\frac{\theta}{p_2},\quad 1\leqq p_1,\ p_2<\infty.$$

It is enough to prove this for $\beta=0$, and then the result follows, as in [8, Ex. 14, p. 18] as a consequence of two facts: for $1<p<\infty$, the Szegö projector is bounded from $L^p(S)$ to $H^p$, and the factorization theorem for functions in $H^1(B^n)$ of Coifman, Rochberg and Weiss ([11, Thm. III]).

**2.2.** The Besov spaces $B_p^\alpha$ can be alternatively defined in terms of harmonic functions. Under the Poisson transform, $B_p^\alpha$ corresponds to the space of harmonic functions $u$ in the unit disc such that

$$(6)\qquad \int_{|\lambda|<1}(1-|\lambda|)^{kp-\alpha p-1}|\nabla^k u(\lambda)|^p\,dm(\lambda) < +\infty.$$

Here $\nabla^k u$ denotes the $k$-th gradient of $u$ and the condition is again independent of the integer $k>\alpha$.

It is well-known that if $j$ is an integer less than $\alpha$, then $f\in B_p^\alpha$ if and only if $f$ has derivatives in the sense of distributions up to order $j$ in $B_p^{\alpha-j}$. In particular, for $\alpha$ non-integer, $k-1<\alpha<k$, $B_p^\alpha\subset W_p^{k-1}$, the classical Sobolev space. We will use the following property of functions in $B_p^\alpha$:

**Lemma.** *Suppose $\alpha$ is non-integer, $k-1<\alpha<k$, let $f\in B_p^\alpha$ and let $E_t^k f$ denote the Taylor remainder, defined for a.e. $s$ as*

$$E_t^k f(s) = f(s+t)-\sum_{i=0}^{k-1}\frac{f^{(i)}(s)}{i!}\,t^i.$$

*Then*

(7) $$\int_{-\pi}^{+\pi} \frac{\|E_t^k f\|_p^p}{|t|^{1+\alpha p}}\, dt < +\infty.$$

*Proof.* We assume $k \geqq 2$ and use the integral form of the remainder

$$E_t^k f(s) = \frac{1}{(k-2)!} \int_s^{s+t} \left(f^{(k-1)}(x) - f^{(k-1)}(s)\right)(x-s)^{k-2}\, dx$$
$$= \frac{1}{(k-2)!} \int_0^t \Delta_x f^{(k-1)}(s) x^{k-2}\, dx.$$

By Hardy's inequality this is bounded by

$$\int_{-\pi}^{+\pi} \frac{\|\Delta_x f^{(k-1)}\|_p^p}{|x|^{1+(\alpha-k+1)p}}\, dx,$$

which is finite because $f^{(k-1)} \in B_p^{\alpha-k+1}$. ∎

We point out that the converse of the lemma also holds. Namely, if $k-1<\alpha<k$, $f \in L^p$ and there are functions $g_0, g_1, \dots, g_{k-1} \in L^p$ such that

$$E_t^k f(s) = f(s+t) - \sum_{i=0}^{k-1} \frac{g_i(s)}{i!} t^i$$

satisfies (7), then $f \in B_p^\alpha$ (and $g_i = f^{(i)}$ a.e.). See the remark at the end of Section 4. This characterization of $B_p^\alpha$ for $\alpha$ non-integer is probably known to specialists but we have found no reference for it.

We will also need the fact that Besov spaces behave nicely under the complex-interpolation method ([8, pg. 153]):

(8) $$[B_{p_1}^{\alpha_1}, B_{p_2}^{\alpha_2}]_\theta = B_p^\alpha, \quad \alpha = (1-\theta)\alpha_1 + \theta\alpha_2, \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

**2.3.** We will need the notion of *Carleson measure*. A positive measure $\mu$ on $B^n$ is called a Carleson measure if there exists a constant $C$ such that for each $\zeta \in S$ and $\delta > 0$

$$\mu(\{z \in B^n \colon |1 - z \cdot \bar{\zeta}| < \delta\}) = O(\delta^n).$$

The importance of Carleson measures is given by the following inequality, due to Fefferman and Stein: whenever $\mu$ is a Carleson measure one has for any $h\colon B^n \to \mathbf{C}$

(9) $$\int_{B^n} |h(z)|^p\, d\mu(z) \leqq C_r \int_S M_r h(\zeta)^p\, d\sigma(\zeta)$$

where $M_r h$ is the *admissible maximal function*

$$M_r h(\zeta) = \sup\{|h(z)| \colon z \in D_r(\zeta)\}.$$

Here $D_r(\zeta)$ is the *admissible approach region*

$$\{z\in B^n\colon |1-z\cdot\bar\zeta| < \tau(1-|z|^2)\}.$$

Combined with the maximal characterization of $H^p$,

$$f\in H^p \Leftrightarrow M_\alpha f\in L^p(S)$$

(9) gives the well-known result that Carleson measures operate in $H^p$.

**2.4.** The following lemma will be required

**Lemma.** *If $g(r)$ is differentiable in* $[0, 1)$ *and* $g(0)=0$

$$\int_0^1 |g(r)|^p(1-r)^{\gamma-1}\,dr \leqq C_{\gamma,p}\int_0^1 |g'(r)|^p(1-r)^{p+\gamma-1}\,dr, \quad \gamma>0.$$

*Proof.* Let $h(r)=(|g(r)|^2+\varepsilon)^{p/2}$ and call $I$ the left-hand term. Integrating

$$\int_0^1 h(r)(1-r)^{\gamma-1}\,dr$$

by parts and making $\varepsilon\to 0$ leads to

$$I\leqq \frac{p}{\gamma}\int_0^1 |g'(r)|\,|g(r)|^{p-1}(1-r)^\gamma\,dr.$$

Hölder's inequality then gives

$$I\leqq \frac{p}{\gamma}\Big\{\int_0^1 |g'(r)|^p(1-r)^{p+\gamma-1}\,dr\Big\}^{1/p} I^{1/p'}$$

and hence the result follows with $C_{\gamma,p}=(\frac{p}{\gamma})^p$ (strictly speaking, one should prove the inequality for the integral between 0 and $R$ and then make $R\to 1$). ∎

## 3. The restriction theorem

**3.1.** In this section we prove Theorem 1, i.e.

$$H^p_\beta|_\Gamma \subset B^{\beta-(n/p)+(1/p)}, \quad \text{any}\quad \Gamma. \tag{10}$$

$$H^p_\beta|_\Gamma \subset B_p^{2(\beta-(n/p))+(1/p)}, \quad \Gamma \text{ complex-tangential.} \tag{11}$$

In view of (3), (5) and (8), in proving (10) we can assume that $\beta$ is a positive integer and $\beta-\frac{n}{p}+\frac{1}{p}$ is not. Similarly, for (11) we can assume $\beta$ is a positive integer and $2(\beta-\frac{n}{p})+\frac{1}{p}$ not integer.

We can also assume without loss of generality that the parametrization $\gamma(s)$ of $\Gamma$ satisfies that $|\gamma'(s)|$ is a constant $l$. We consider the *index of transversality*

$T(s)$ defined by

$$T(s) = |\gamma'(s)\cdot\overline{\gamma(s)}|,$$

the complex-tangential curves corresponding to $T(s)\equiv 0$. We denote by $\Pi$ the surface

$$\Pi = \{r\gamma(s),\ 0 \leqq r \leqq 1,\ 0 \leqq s \leqq 2\pi\}$$

and for a given function $f\in H^p_\beta$ we write $u(r, s)=f(r\gamma(s))$.

The following geometrical lemma is proved in [2]:

**Lemma.** *Given $\tau$, $\varrho$, $1<\tau<\varrho$ and a vector $v$, there exists $\varepsilon=\varepsilon(\tau, \varrho, \|v\|)$ such that whenever $z=r\eta\in D_\tau(\zeta)$ one has*

$$z+\lambda v\in D_\varrho(\zeta)$$

*for all $\lambda\in\mathbf{C}$ satisfying*

$$|\lambda| \leqq \varepsilon\frac{1-r}{(1-r)^{1/2}+|v\cdot\bar{\eta}|}. \tag{12}$$

*Moreover, $1-|z+\lambda v|$ is comparable to $1-r$ for these $\lambda$.*

Let $f\in H^p_\beta$ and assume $z=r\eta\in D_\tau(\zeta)$. Applying the lemma with $v=\eta$ and Cauchy inequality in the disc given by (12) to the function $R^\beta f(z+\lambda v)$ we obtain for $m\geqq\beta$

$$(1-|z|)^{m-\beta}|R^m f(z)| \leqq \mathrm{const}\, M_\varrho(R^\beta f)(\zeta),\quad z\in D_\tau(\zeta).$$

Therefore $F(z)\overset{\mathrm{def}}{=}(1-|z|)^{m-\beta}R^m f(z)$ has admissible maximal function in $L^p(S)$.

Assume now that $z\in\Pi$, $z=r\gamma(s)\in D_\tau(\zeta)$. Applying the lemma with $v=\gamma'(s)$ and again Cauchy inequality to $R^m f(z+\lambda v)$ we obtain

$$\frac{(1-r)^k}{((1-r)^{1/2}+T(s))^k}\left|\frac{\partial^{k+m}u}{\partial s^k\partial r^m}(r, s)\right| \leqq \mathrm{const}\sup_\lambda |R^m f(z+\lambda v)|$$

the supremum being taken over the $\lambda$'s satisfying (12). Multiplication of this inequality with $(1-r)^{m-\beta}$, together with the fact that $1-|z+\lambda v|$ is comparable to $1-r$ leads to

$$\frac{(1-r)^{k+m-\beta}}{((1-r)^{1/2}+T(s))^k}\left|\frac{\partial^{k+m}u}{\partial s^k\partial r^m}(r, s)\right| \leqq \mathrm{const}\, M_\varrho(F)(\zeta),$$

$$z = r\gamma(s)\in D_\tau(\zeta).$$

We conclude that the function $h(z)$, defined by the above expression for $z=r\gamma(s)\in\Pi$ and 0 outside, has admissible maximal function $M_\tau h$ pointwise bounded by $M_\varrho(F)$ and hence in $L^p(S)$.

Next we consider the measure $\mu$ supported on $\Pi$ defined by

$$\int h\, d\mu = \int_0^1\int_{-\pi}^{+\pi} h(r\gamma(s))(1-r)^{n-2}((1-r)^{1/2}+T(s))\,dr\,ds.$$

It is proved in [2] that $\mu$ is a Carleson measure. By (9) we reach the conclusion that for $m \geqq \beta$ and $k=0, 1, \ldots$

$$\int_0^1 \int_{+\pi}^{+\pi} \left| \frac{\partial^{k+m} u}{\partial s^k \partial r^m}(r, s) \right|^p \frac{(1-r)^{p(k+m-\beta)+n-2}}{((1-r)^{1/2}+T(s))^{pk-1}} \, dr ds < +\infty, \tag{13}$$

with $u(r, s)=f(r\gamma(s))$, whenever $f \in H^p_\beta$. Note that in the special case that $\Gamma$ is a slice, say $\Pi$ is $z_2=z_3=\ldots=z_n=0$, then (13) for $k=0$ and $m=\beta$ reduces to

$$\int_{|z_1|<1} |R^\beta f(z_1)|^p (1-|z_1|)^{n-2} dm(z_1) < +\infty$$

which is indeed equivalent to $f(e^{is})$ being in $B_p^{\beta-n/p+1/p}$ (subsection 2.2). In the general case $\Pi$ is not of course an analytic disk, so $u(r, s)$ is no longer analytic or harmonic. Our proof that $f(\gamma(s))$ is in $B_p^{\beta-n/p+1/p}$ will nevertheless imitate the one giving the equivalence between the two definitions (2) and (6) of the Besov spaces, but compensating the lack of harmonicity by the fact that we have condition (13) for bigger values of $k$, too.

**3.2.** Let us now prove (10) and (11). With $\alpha$ either $\beta-\frac{n}{p}+\frac{1}{p}$ or $2(\beta-\frac{n}{p})+\frac{1}{p}$, respectively and if $k-1<\alpha<k$, we have to prove

$$\int_{-\pi}^{+\pi} \frac{\|\Delta_t^k f\|_p^p}{|t|^{1+\alpha p}} \, dt < \infty. \tag{14}$$

The method is as usual to pass a bit inside in order to evaluate this $k$-th difference. We write

$$r(s, t) = 1-c(t^2+T(s)|t|),$$

where $c$ is some constant chosen so that $0 \leqq r \leqq 1$ when $|t| \leqq \pi$. Now

$$\Delta_t^k f(s) = \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} f(s+\nu t) = \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} u(r(s+\nu t, t), s+\nu t)$$
$$+\sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} \big(f(s+\nu t)-u(r(s+\nu t, t), s+\nu t)\big).$$

It will be convenient to introduce the notation

$$U_t(s) = u(r(s, t), s).$$

Then the above can be written

$$\Delta_t^k f(s) = \Delta_t^k U_t(s)+\Delta_t^k (f-U_t)(s)$$

and we treat each term separately. For the first we use that $U_t$ is a $C^\infty$ function and hence

$$\|\Delta_t^k U_t\|_p \leqq |t|^k \left\| \frac{d^k}{ds^k} U_t \right\|_p.$$

It is easily seen that $\left|\frac{d^k}{ds^k}U_t(s)\right|$ is bounded by a sum of terms

$$|t|^{k_1}\left|\frac{\partial^{k_1+k_2}u}{\partial r^{k_1}\partial s^{k_2}}(r(s,t),s)\right|$$

with $k_1+k_2\leqq k$. Hence

$$\int_{-\pi}^{+\pi}\frac{\|\Delta_t^k U_t\|_p^p}{|t|^{1+\alpha p}}dt \leqq c\sum\int_0^{\pi}|t|^{p(k+k_1-\alpha)-1}dt\int_{-\pi}^{+\pi}\left|\frac{\partial^{k_1+k_2}u}{\partial r^{k_1}\partial s^{k_2}}(r(s,t),s)\right|^p ds. \tag{15}$$

In the $t$-integral we change now $t$ by $r=r(s,t)$. Since

$$t\cong\frac{1-r}{(T(s)^2+1-r)^{1/2}},\quad dt\cong\frac{dr}{(T(s)^2+1-r)^{1/2}},$$

we get the bound

$$\iint(1-r)^{p(k+k_1-\alpha)-1}\left(T(s)^2+1-r\right)^{-(p/2)(k+k_1-\alpha)}\left|\frac{\partial^{k_1+k_2}u}{\partial r^{k_1}\partial s^{k_2}}(r,s)\right|^p drds.$$

By the lemma in 2.4 this is finite if

$$\iint(1-r)^{p(k+m-\alpha)-1}\left(T(s)^2+1-r\right)^{-(p/2)(k+k_1-\alpha)}\left|\frac{\partial^{m+k_2}u}{\partial r^{m}\partial s^{k_2}}(r,s)\right|^p drds \tag{16}$$

is finite, where $m$ is any integer bigger than $k_1$ and $\beta$.

Next, we bound $\|\Delta_t^k(f-U_t)\|_p$ by

$$\|\Delta_t^{k-1}(f-U_t)\|_p\leqq|t|^{k-1}\left\|\frac{d^{k-1}}{ds^{k-1}}(f-U_t)\right\|_p.$$

Now

$$(f-U_t)(s)=\int_{r(s,t)}^1\frac{\partial u}{\partial y}(y,s)\,dy.$$

In a similar way as before, $\frac{d^{k-1}}{ds^{k-1}}(f-U_t)$ is bounded by the sum of

$$\int_{r(s,t)}^1\frac{\partial^k u}{\partial y\,\partial s^{k-1}}(y,s)\,dy \tag{17}$$

with terms $|t|^{k_1}\frac{\partial^{k_1+k_2}u}{\partial r^{k_1}\partial s^{k_2}}(r(s,t),s)$, with $k_1+k_2\leqq k-1$, terms that have already been treated. For (14) it remains to bound

$$\int_0^{\pi}t^{p(k-1-\alpha)-1}dt\int_{-\pi}^{+\pi}\left(\int_{r(s,t)}^1\frac{\partial^k u}{\partial y\,\partial s^{k-1}}(y,s)\,dy\right)^p ds.$$

With $s$ fixed, we apply the same change of variable as before, from $t$ to $r=r(s,t)$, to obtain

$$\int_{-\pi}^{+\pi} ds \int_0^1 (1-r)^{p(k-1-\alpha)-1} \left(T(s)^2+1-r\right)^{-(p/2)(k-1-\alpha)}$$

$$\left(\int_r^1 \frac{\partial^k u}{\partial y\, \partial s^{k-1}}(y, s)\, dy\right)^p dr.$$

Since $k-1-\alpha<0$, we can apply Hardy's inequality in the $dr\, dy$ integral to get the bound

$$\int_{-\pi}^{+\pi} ds \int_0^1 (1-y)^{p(k-\alpha)-1} \left(T(s)^2+1-y\right)^{-(p/2)(k-1-\alpha)} \left|\frac{\partial^k u}{\partial y\, \partial s^{k-1}}(y, s)\right|^p dy ds.$$

Again the application of the lemma in 2.4 gives as bound $(m \geqq \beta)$

$$\int_{-\pi}^{+\pi} \int_0^1 (1-r)^{p(k-\alpha+m-1)-1} \left(T(s)^2+1-r\right)^{-(p/2)(k-1-\alpha)} \left|\frac{\partial^{k+m-1} u}{\partial r^m s^{k-1}}(r, s)\right|^p dr\, ds,$$

which is of the same type as (16).

In conclusion, we have seen that (14) holds if the integrals in (16) are finite. Recall that in (16) $m \geqq \beta$ and $k_1+k_2 \leqq k$.

An easy computation shows that the integrand in (16) is bounded by the one of (13) for $k=k_2$ if

$$(1-r)^{-p\alpha+1+\beta p-n} \left(T^2(s)+1-r\right)^{(\alpha p/2)-(1/2)} = 0(1). \tag{18}$$

Here we see that the choice $\alpha=\beta-\frac{n}{p}+\frac{1}{p}$ works for all curves (because then $\alpha p \geqq 1$ by the regularity assumption $\beta \geqq \frac{n}{p}$), thus proving (10) and the better choice $\alpha=2\left(\beta-\frac{n}{p}\right)+\frac{1}{p}$ works for complex-tangential curves, proving (11).

We remark that if the curve $\Gamma$ is transverse, i.e. $T(s)$ is bounded below, everything simplifies defining instead of $r(s, t)$

$$r(t) = 1-t$$

(in this case only the term with $k_1=0$, $k_2=k$ arises in the estimate of $\|\Delta_t^k U_t\|_p$ and (17) in that of $\Delta_t^k(f-U_t)$). Similarly, all the above is simpler when $\Gamma$ is complex-tangential, in which case $r(t)=1-t^2$.

## 4. The interpolation problem

**4.1.** To prove Theorem 2, we use linear operators of interpolation from complex-tangential manifolds constructed by A. Nagel in [15] that we proceed to recall.

We assume as before that our closed, simple, complex-tangential curve $\Gamma$ is parametrized by $[-\pi, \pi]$ and also that $\gamma'(s)$ has constant length. For each $q>1/2$, let $h_q(z)$ be the function

$$h_q(z)=\int_{-\pi}^{+\pi}\frac{ds}{\left(1-\overline{\gamma(s)}\cdot z\right)^q},\quad z\in B^n.$$

For $z\in B^n$, we denote by $d(z)$ the pseudodistance from $z$ to the curve

$$d(z)=\inf\{|1-\overline{\gamma(s)}\cdot z|,\ -\pi\leqq s\leqq\pi\}.$$

The following estimates are proved in [15]:

**Lemma.**

(a) $|h_q(z)|\simeq d(z)^{1/2-q}$

(b) $|R^i h_q(z)|=O\left(d(z)^{1/2-q-i}\right),\quad i=1,2,\ldots.$

We will fix in the following a tubular neighbourhood $U$ of the curve $\Gamma$. Each point $z\in U\cap\bar{B}^n$ has a unique euclidian projection $\gamma(s_z)$ so that $\operatorname{Re}\left(z-\gamma(s_z)\right)\cdot\overline{\gamma'(s_z)}=\operatorname{Re} z\cdot\overline{\gamma'(s_z)}=0$. It is easy to see using Taylor's development and again the condition $\gamma'(s)\cdot\overline{\gamma(s)}\equiv 0$ that

$$|1-\overline{\gamma(s)}\cdot z|\cong|1-\overline{\gamma(s_z)}\cdot z|+|s-s_z|^2. \tag{19}$$

In particular, $d(z)\cong|1-\overline{\gamma(s_z)}\cdot z|$. Using this, it is immediate to check the following estimates:

$$\int_{-\pi}^{+\pi}\frac{ds}{|1-\overline{\gamma(s)}\cdot z|^q}=O\left(|h_q(z)|\right).$$

$$\int_{|s-s_z|\geqq\delta}\frac{ds}{|1-\overline{\gamma(s)}\cdot z|^q}=o\left(|h_q(z)|\right)\quad\text{as}\quad d(z)\to 0,\quad\text{for fixed}\quad\delta>0.$$

Assuming without loss of generality that $h_q(z)\neq 0$ for $z\in U$, it follows that the kernel

$$K_q(s,z)\stackrel{\text{def}}{=}\frac{1}{h_q(z)}\frac{1}{\left(1-\overline{\gamma(s)}\cdot z\right)^q},\quad z\in U,\ |s|\leqq\pi$$

is an approximation of the identity in the sense that

$$\int_{-\pi}^{+\pi} K_q(s, z)\, ds = 1,$$

$$\int_{-\pi}^{+\pi} |K_q(s, z)|\, ds = O(1)$$

and

$$\int_{\pi \geqq |s-s_z| \geqq \delta} |K_q(s, z)|\, ds \to 0 \quad \text{as} \quad d(z) \to 0, \quad z \in U.$$

As a consequence, if $f(s)$ is a continuous function on the curve, the holomorphic function $T_q f(z)$ defined in $U \cap B^n$ by

$$T_q f(z) = \int_{-\pi}^{+\pi} K_q(s, z) f(s)\, ds$$

interpolates $f$, i.e. $T_q f(z) \to f(s)$ as $z \to \gamma(s)$ (and it is clearly $C^\infty$ in $U \setminus \Gamma$). To obtain a globally defined interpolating function we solve a $\bar\partial$ equation as usual: if $V \subset\subset U$ is another tubular neighbourhood of $\Gamma$, we let $\chi$ be a test function supported in $U$ and equal to 1 in $V$, and we consider the (0, 1) form $w = T_q f \bar\partial \chi / h$. Here $h$ is a function in $A^\infty(B)$, vanishing exactly on $\Gamma$ (one can even choose $h$ as flat on $\Gamma$, see [13]). The form $w$ has coefficients in $C^\infty(\bar B)$ and hence there is a function $u \in C^\infty(\bar B)$ such that $\bar\partial u = w$. Moreover, $u$ can be chosen $u = Tw$ where $T$ is a linear integral operator (see [9, Lemma 5.2]). Then the function

$$I_q f \stackrel{\text{def}}{=} \chi T_q f - hu$$

is a function in the ball algebra that interpolates $f$ on the curve, and the operator $I_q$ is linear. The function $hu$ is $C^\infty$ up to the boundary, and hence the behavior of $T_q f$ determines whether $I_q f$ belongs or not to a certain function space.

**4.2.** As before we write $\alpha = 2(\beta - \frac{n}{p}) + \frac{1}{p}$. We will complete the proof of Theorem 2 in the introduction by proving:

**Theorem.** *If $\Gamma$ satisfies condition* (I) *and* $q \geqq 2\beta$, $I_q$ *maps* $B_p^\alpha$ *to* $H_\beta^p$.

As in the previous section, and given the linearity of $I_q$, we may assume that $\beta$ is a positive integer and $\alpha$ is not.

Assume $k-1 < \alpha < k$ and let $f \in B_p^\alpha$. By the lemma in 2.2, the Taylor remainder $E_t^k f$ satisfies (7). We begin by estimating $R^i T_q f$, $i = 0, \dots, \beta$. With $\gamma(s_z)$ being the projection of $z \in U$,

$$R^i T_q f(z) = \int_{-\pi}^{+\pi} f(s) R_z^i K_q(s, z)\,ds = \int_{-\pi}^{+\pi} f(t+s_z) R_z^i K_q(t+s_z, z)\,dt$$

$$= \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(s_z) \int_{-\pi}^{+\pi} t^j R_z^i K_q(t+s_z, z)\,dt$$

$$+ \int_{-\pi}^{+\pi} E_t^k f(s_z) R_z^i K_q(t+s_z, z)\,dt.$$

According to property (I), the first integrals of the last term are all bounded in $z$. It is easy to check, using (19) and part (b) of the lemma in 4.1 that

$$|R_z^i K_q(t+s_z, z)| \leqq \text{const}\,\frac{d(z)^{q-1/2-i}}{(d(z)+t^2)^q}. \tag{20}$$

In conclusion, we obtain, for $i=0, \dots, \beta$

$$|R^i T_q f(z)| \lesssim g(s_z) + d(z)^{q-1/2-i} \int_{-\pi}^{+\pi} \frac{|E_t^k f(s_z)|}{(d(z)+t^2)^q}\,dt, \quad z \in U,$$

for some $g \in B_p^{\alpha-k+1} \subset L^p(T)$.

Next we introduce coordinates in $U$ suitable for these estimates. In what follows we will assume that the dimension $n$ is 2, the computations being just a bit more involved if $n>2$. That $U$ is a tubular neighbourhood of $\Gamma$ means that the map

$$(t_1, t_2, t_3, t_4) \xrightarrow{\Lambda} \gamma(t_1) + (-t_2 + it_3)\gamma(t_1) + it_4\gamma'(t_1)$$

is a coordinate map if $t_1 \in [-\pi, \pi]$ and $t' = (t_2, t_3, t_4)$ has $|t'|$ small enough, say, $|t'| \leqq \varepsilon$ (this is because $\gamma(t_1)$, $i\gamma(t_1)$ and $\gamma'(t_1)$ span the orthogonal of $\gamma'(t_1)$). If $\Lambda(t_1, t_2, t_3, t_4) = z$, then $t_1 = s_z$ and $d(z) \cong |1 - \overline{\gamma(s_z)} \cdot z| \cong |t_2| + |t_3|$. Our estimate reads in these coordinates

$$|R^i T_q f(z)| \lesssim g(t_1) + (|t_2| + |t_3|)^{q-1/2-\beta} \int_{-\pi}^{+\pi} \frac{|E_t^k f(t_1)|}{(t^2 + |t_2| + |t_3|)^q}\,dt, \tag{21}$$

$i=0, \dots, \beta$, an estimate independent of $t_4$. From the formula for $I_q f$, and since $\chi$, $h$, $u$ are $C^\infty$, we conclude that (21) holds for $R^\beta I_q f$.

In this parametrization the sphere of radius $R<1$ is given by

$$(1-t_2)^2 + t_3^2 + t_4^2 = R^2.$$

Since the bound we have obtained is independent of $t_4$ we choose $t_1$, $t_2$, $t_3$ to coordinate $S_R \cap U$, where $S_R = \{|z| = R\}$, for $R$ close enough to 1. Then

$$d\sigma_R = O\left(\frac{1}{\sqrt{R^2 - (1-t_2)^2 - t_3^2}}\right) dt_1\,dt_2\,dt_3,$$

(strictly speaking, we should consider two regions, one corresponding to $t_4>0$ and the other to $t_4<0$). The region of integration $S_R \cap U$ is contained in

$$\{|t_1| \leqq \pi,\ 0 < t_2 < \varepsilon,\ |t_3| \leqq \varepsilon,\ (1-t_2)^2+t_3^2 \leqq R^2\}.$$

To finish the proof we have to show that

$$I = \int_{S_R\cap U} |g(t_1)|^p \frac{dt_1\,dt_2\,dt_3}{\sqrt{R^2-(1-t_2)^2-t_3^2}},$$

$$II = \int_{S_R\cap U} (|t_2|+|t_3|)^{p(q-1/2-\beta)}\left(R^2-(1-t_2)^2-t_3^2\right)^{1/2}\left\{\int_{-\pi}^{+\pi} \frac{|E_t^k f(t_1)|\,dt}{(t^2+|t_2|+|t_3|)^q}\right\}^p dt_1\,dt_2\,dt_3$$

are bounded independently of $R<1$.

Using continuous Minkowski's inequality in II and the fact that $g$ is in $L^p(T)$ in $I$, we see that

$$I \leqq \iint \frac{dt_2\,dt_3}{\sqrt{R^2-(1-t_2^2)-t_3^2}},$$

$$(22)\quad II \leqq \iint (|t_2|+|t_3|)^{pq-p/2-p\beta}\left(R^2-(1-t_2^2)-t_3^2\right)^{-1/2}\left\{\int_{-\pi}^{+\pi} \frac{\|E_t^k f\|_p\,dt}{(t^2+|t_2|+|t_3|)^q}\right\}^p dt_2\,dt_3.$$

It is convenient to introduce the new variables $\varrho$, $\psi$ defined by

$$1-t_2 = \varrho\cos\psi,$$

$$t_3 = \varrho\sin\psi,$$

with $1-\varepsilon \leqq \varrho \leqq R$ and $\psi$ small. Writing

$$|t_2|+|t_3| \simeq (t_2^2+t_3^2)^{1/2} = [(1-\varrho)^2+2\varrho(1-\cos\psi)]^{1/2} \overset{\text{def}}{=} A(\varrho,\psi)$$

we are led to the following integrals:

$$I \leqq \text{const}\int_{1-\varepsilon}^{R} \varrho(R^2-\varrho^2)^{-1/2}\,d\varrho,$$

$$II \leqq \text{const}\int_{1-\varepsilon}^{R}\int_{-\delta}^{+\delta} A(\varrho,\psi)^{pq-p/2-p\beta}\,\varrho(R^2-\varrho^2)^{-1/2}\left\{\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+A(\varrho,\psi))^q}\right\}^p d\varrho\,d\psi.$$

It is clear that the first is bounded independently of $R$. To evaluate the second, we break it into two parts, $II_1$ and $II_2$, corresponding respectively to the regions $(1-\varrho)^2 \geqq$

$2\varrho(1-\cos\psi)$ and $(1-\varrho)^2 \leqq 2\varrho(1-\cos\psi)$ which imply, respectively, $A(\varrho,\psi)\cong (1-\varrho)$ and $A(\varrho,\psi)\cong(1-\cos\psi)^{1/2}$. In $II_1$, $\psi^2\cong 1-\cos\psi \lesssim (1-\varrho)^2$ and hence

$$II_1 \lesssim \int_{1-\varepsilon}^{R} (1-\varrho)^{pq-p/2-p\beta+1}(R^2-\varrho^2)^{-1/2}\left\{\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+1-\varrho)^q}\right\}^p d\varrho.$$

In this one, for $R$ close to 1, we break the $d\varrho$ integral according to whether $\varrho\leqq 2R-1$ or $\varrho\geqq 2R-1$, obtaining two integrals that we call $II_{1,1}$ and $II_{1,2}$. In $II_{1,2}$, $1-\varrho\cong 1-R$ and so

$$II_{1,2} \lesssim \int_{2R-1}^{R} (1-R)^{pq-p/2-p\beta+1}(R-\varrho)^{-1/2}\left(\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+1-R)^q}\right)^p d\varrho$$

$$\lesssim (1-R)^{3/2+pq-p/2-p\beta}\left(\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+1-R)^q}\right)^p$$

$$\leqq (1-R)^{3/2+pq-p/2-p\beta}\left(\int_{-\pi}^{+\pi}\|E_t^k f\|_p^p\,|t|^{2-2\beta p}\,dt\right)\left(\int_{-\pi}^{+\pi}\frac{t^{(2\beta-2/p)p'}}{(1-R+t^2)^{qp'}}\right)^{p-1}$$

with $p'$ the conjugate exponent of $p$. Now it is easy to see that the last integral is for $q$ big enough (in fact $q>\beta+1/2-3/2p$) dominated by $(1-R)^{-3/2-pq+p/2+p\beta}$. Thus

$$II_{1,2} \lesssim \int_{-\pi}^{+\pi}\|E_t^k f\|_p^p\, t^{2-2\beta p}\,dt$$

which is finite, in view of the lemma in 2.2.

In $II_{1,1}$, $R-\varrho$ is comparable to $1-\varrho$ and we bound $II_{1,1}$ by making $R=1$, i.e.

$$II_{1,1} \lesssim \int_{1-\varepsilon}^{1}(1-\varrho)^{pq-p/2-p\beta+1/2}\left(\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+1-\varrho)^q}\right)^p d\varrho \tag{23}$$

$$\leqq \int_0^1 \varrho^{pq-p/2-p\beta+1/2}\left(\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p\,dt}{(t^2+\varrho)^q}\right)^p d\varrho.$$

Here we use Hardy's inequality to obtain that if $q$ is big enough (specifically $q>\beta+\frac{1}{2}-\frac{3}{2p}$) then $II_{1,1}$ is also bounded by

$$\int \|E_t^k f\|_p^p t^{2-2\beta p}\,dt < +\infty.$$

It remains now to estimate $II_2$. In this case $A(\varrho,\psi)\cong(1-\cos\psi)^{1/2}\cong|\psi|$ and also $(1-\varrho)\leqq c|\psi|$

$$II_2 \lesssim \int_{\min(R,1-c|\psi|)}^{R}\int_{-\delta}^{+\delta}|\psi|^{pq-p/2-p\beta}(R-\varrho)^{-1/2}\left\{\int_{-\pi}^{+\pi}\frac{\|E_t^k f\|_p}{(t^2+|\psi|)^q}\right\}^p d\varrho\,d\psi.$$

The integral in $d\varrho$ is bounded by $|\psi|^{1/2}$ and hence $II_2$ is bounded by the same integral appearing in (23).

This completes the proof of the theorem. ▮

*Remark 1.* The same method of proof but using the Poisson transform $P$ instead of the operator $I_q$ serves to show the converse of the lemma in 2.2. If $E_t^k f$ satisfies (7), then $u=P[f]$ is an harmonic function for which (6) holds, an hence $f\in B_p^\alpha$.

## 5. Generalizations

**5.1.** The first generalization concerns the so-called Bergman—Sobolev spaces $A_{\beta,\gamma}^p$ consisting of the holomorphic functions $f$ such that

$$\int_{B^n} |R^\beta f(z)|^p (1-|z|)^{\gamma-1} dV(z) < +\infty, \quad p \geqq 1,\ \gamma > 0. \tag{24}$$

In a certain sense, $H_\beta^p$ corresponds to the limiting case $\gamma=0$. It is well-known that $A_{\beta_1,\gamma_1}^p = A_{\beta_2,\gamma_2}^p$ if $\beta_1 p-\gamma_1=\beta_2 p-\gamma_2$ (in fact, the class of harmonic functions satisfying (24) corresponds to the Besov space $B_p^{\beta-\gamma/p}(S^{2n-1})$). Also it is known (see [6]) that the regular range is here $\beta > \frac{n+\gamma}{p}$ for $p>1$ and $\beta \geqq n+\gamma$ for $p=1$ (in which cases $A_{s,\gamma}^p \subset \mathrm{Lip}_{s-(n+\gamma)/p}$).

We state the two corresponding results:

**Theorem.** *If $\Gamma$ is as before, the trace of $A_{\beta,\gamma}^p$ is contained in $B_p^\alpha$, with $\alpha=\beta-\frac{n+\gamma}{p}+\frac{1}{p}$. If $\Gamma$ is complex tangential, the trace of $A_{\beta,\gamma}^p$ is contained in $B_p^\alpha$, with $\alpha=2(\beta-\frac{n+\gamma}{p})+\frac{1}{p}$.*

**Theorem.** *If $\Gamma$ is a complex-tangential curve, for which property* (I) *holds, the trace of $A_{\beta,\gamma}^\alpha$ is exactly $B_p^\alpha$, with $\alpha=2(\beta-\frac{n+\gamma}{p})+\frac{1}{p}$.*

From these results we draw the conclusion that the Hardy—Sobolev spaces and Bergman—Sobolev spaces (i.e. Besov spaces) of holomorphic functions both have Besov spaces as traces on complex-tangential curves, in a complete parallelism with the real-variable theory.

The ideas to prove the above theorems are analogous to the ones used before, but there are some technical differences that we next indicate.

The analogue of (13) for $f\in A_{\beta,\gamma}^p$ is

$$\int_0^1\!\!\int \left|\frac{\partial^{m+k}u}{\partial r^m \partial s^k}(r,s)\right|^p \frac{(1-r)^{p(k+m-\beta)+n-2+\gamma}}{[(1-r)^{1/2}+T(s)]^{pk-1}}\, dr\, ds < +\infty, \tag{25}$$

$$m \geqq \beta, \quad k = 0, 1, 2, \ldots.$$

Here $u(r,s)=f(r\gamma(s))$. To see this, for each $z$ let $P_z^\varepsilon$ be a polydisc centered at $z$

with size $\varepsilon(1-|z|)$ in the normal direction and size $\varepsilon(1-|z|)^{1/2}$ in the complex-tangential directions, so that its volume is comparable to $(1-|z|)^{n+1}$. We need the following one-variable result proved in [3]:

**Lemma.** *If g is an holomorphic function in a disc* $|\lambda-\lambda_0|\leqq R$, *then*

$$|g^{(k)}(\lambda_0)|^p \leqq \frac{c}{R^{2+pk}}\int_{|\lambda-\lambda_0|\leqq R}|g(\lambda)|^p\,dm(\lambda).$$

Fix $r$, $s$; we apply the lemma to $g(\lambda)=R^m f(r\gamma(s)+\lambda\gamma'(s))$, $\lambda_0=0$ and $R=c(1-r)/((1-r)^{1/2}+T(s))$, where $c$ is chosen so that $1-|r\gamma(s)+\lambda\gamma'(s)|\cong 1-r$ for all $\lambda$, $|\lambda|\leqq R$. We obtain

$$\left|\frac{\partial^{m+k}u}{\partial r^m\partial s^k}(r,s)\right|^p\left(\frac{1-r}{(1-r)^{1/2}+T(s)}\right)^{pk} \leqq \frac{c}{R^2}\int_{|\lambda|\leqq R}\left|R^m f(r\gamma(s)+\lambda\gamma'(s))\right|^p dm(\lambda).$$

On the other hand, by plurisubharmonicity, for each $z=r\gamma(s)+\lambda\gamma'(s)$

$$|R^m f(z)|^p \leqq \frac{\text{const}}{(1-r)^{n+1}}\int_{P_z^\varepsilon}|R^m f(w)|^p dm(w).$$

Here we also used the fact that $1-|z|\cong 1-r$. An easy computation shows that, for $z=r\gamma(s)+\lambda\gamma'(s)$, $|\lambda|\leqq R$, one has $P_z^\varepsilon\subset P_{r\gamma(s)}^{\varepsilon'}$ with some $\varepsilon'=\varepsilon'(\varepsilon)$. Therefore we conclude that

$$\left|\frac{\partial^{m+k}u}{\partial r^m\,\partial s^k}(r,s)\right|^p\left(\frac{1-r}{(1-r)^{1/2}+T(s)}\right)^{pk} \leqq \frac{c}{(1-r)^{n+1}}\int_{P_{r\gamma(s)}^{\varepsilon'}}|R^m f(w)|^p\,dm(w)$$

and hence the integral in (25) is bounded by

$$\int_0^1\int_{-\pi}^{+\pi}(1-r)^{p(m-\beta)+\gamma-3}[(1-r)^{1/2}+T(s)]$$
$$\left\{\int_{P_{r\gamma(s)}^{\varepsilon'}}|R^m f(w)|^p\,dm(w)\right\}dr\,ds.$$

Next we apply Fubini's theorem and use the fact that for fixed $w$

$$\int_{\{(r,s):\,w\in P_{r\gamma(s)}^{\varepsilon'}\}}\left((1-r)^{1/2}+T(s)\right)dr\,ds = O(1-|w|)^2$$

together with the fact that $1-|w|\cong 1-r$ for $w\in P_{r\gamma(s)}^{\varepsilon'}$ (choosing $\varepsilon$ small enough) to bound (25) by

$$\int_{B^n}(1-|w|)^{p(m-\beta)+\gamma-1}|R^m f(w)|^p\,dm(w).$$

This finishes the proof of (25), because $f\in A^p_{\beta,\gamma}$ is equivalent, by the remarks made before, to the finiteness of this integral.

We point out that (25) for $k=0$ and $m=\beta$ simply says that the measure $d\mu=(1-r)^{n-2+\gamma}[(1-r)^{1/2}+T(s)]\,dr\,ds$ is a Carleson measure for the weighted Bergman space $A^p_{0,\gamma}$ (those are characterized by the condition $\mu\{z\in B^n\colon |1-z\cdot\bar\zeta|<\delta\}=O(\delta^{n+\gamma})$, $\zeta\in S$, $\delta>0$). In this sense, all the above corresponds in the limiting case $\gamma=0$ to the argument used in Section 3.

Observe now that (24) is exactly (13) but with $\beta$ replaced by $\beta-\frac{\gamma}{p}$, and so it is clear that it will lead to the Besov space $B_p^{\beta-(n+\gamma)/p+1/p}$, for the general curve, and $B_p^{2(\beta-(n+\gamma)/p)+1/p}$ for the complex-tangential curves, thus completing the proof of the restriction part of the theorems.

In the interpolation part of the second theorem we use the same interpolation operator $I_q$. Given $f\in B^\alpha_p$, $\alpha=2(\beta-\frac{n+\gamma}{p})+\frac{1}{p}$, we check now that $I_qf\in A^p_{\beta,\gamma}$, using estimate (21) for $R^\beta I_q f$, again only in the case $n=2$. In the coordinates $t_1$, $t_2$, $t_3$, $t_4$ used in Section 4, we have $|z|^2=(1-t_2)^2+t_3^2+t_4^2$ and $1-|z|^2=2t_2-t_2^2-t_3^2-t_4^2$. In the following we replace $t_2$ by the (equivalent) coordinate $2t_2-t_2^2$, which we continue to denote by $t_2$. Using (21), we have to check that

$$\iiiint |g(t_1)|^p(t_2-t_3^2-t_4^2)^{\gamma-1}\,dt_1\,dt_2\,dt_3\,dt_4,$$

$$\iiiint (t_2+|t_3|)^{p(q-(1/2)-\beta)}(t_2-t_3^2-t_4^2)^{\gamma-1}\left\{\int_{-\pi}^{+\pi}\frac{|E_t^kf(t_1)|}{(t^2+t_2+|t_3|)^q}\,dt\right\}^p dt_1\,dt_2\,dt_3\,dt_4$$

are finite, both extended over the region of integration $t_2\geqq t_3^2+t_4^2$ and $-\pi\leqq t_1\leqq\pi$. This is clear for the first integral because $g\in L^p$. In the second integral we use the continuous Minkowski's inequality in $dt_1$ and perform the integration in $t_4$ to bound it by

$$\iint (t_2+|t_3|)^{p(q-(1/2)-\beta)}(t_2-t_3^2)^{\gamma-1/2}\left\{\int_{-\pi}^{+\pi}\frac{\|E_t^kf\|_p\,dt}{(t^2+|t_2|+|t_3|)^q}\right\}^p dt_2\,dt_3$$

the first integral being extended over $t_3^2\leqq t_2$. Notice now that this is bounded by the integral in (22), replacing $\beta$ by $\beta-\frac{\gamma}{p}$, with $R=1$. Since in Subsection 4.2 we showed that (22) is finite when $f\in B_p^{2(\beta-n/p)+1/p}$, the result is completely proved.

**5.2.** In our theorems we have considered up to now the regular case $\beta>\frac{n}{p}$ for $p>1$ and $\beta\geqq n$ for $p=1$ i.e., when the functions in $H^p_\beta$ are continuous up to the boundary, in order to have a well-defined (continuous) trace on $\Gamma$. Beyond the regular range one can still speak about traces on curves in some cases. A recent result of P. Ahern [1] states the following: if $p=1$ and $m=n-\beta p>0$, and if $\mu$ is a measure on $S$ such that $\mu(B_\delta)=O(\delta^m)$ for a Koranyi ball $B_\delta=\{z\colon |1-z\cdot\bar\zeta|<\delta\}$ of radious $\delta$, then each function in $H^p_\beta$ has admissible limits (i.e. within the admissible approach regions $D_\tau(\zeta)$) almost everywhere $d\mu$. If $p>1$ and $m=n-\beta p\geqq 0$, the result is still true if $\mu(B_\delta)=O(\delta^{m+\varepsilon})$ for some $\varepsilon>0$ (this follows using standard arguments from part (iii) in Theorem 1.9 of [4]).

If $d\mu$ is arc-length on a curve, then $\mu(B_\delta)=O(\delta^{1/2})$, and thus, as a consequence of Ahern's results, we see that if $0\leq n-\beta p<1/2$, then all functions in $H^p_\beta$ have an admissible limit almost everywhere along the curve. When the curve is transverse, then $\mu(B_\delta)=O(\delta)$ and there is a trace when $n-\beta p<1$. Next theorems are generalizations of our main results to the non-regular range.

**Theorem.** *Assume* $0\leq n-\beta p<1/2$. *Then the trace of* $H^p_\beta$ *in* $\Gamma$ *is included in* $B_p^{2(\beta-n/p)+1/p}$. *If* $\Gamma$ *is a transverse curve and* $\beta>\frac{n-1}{p}$, *the trace of* $H^p_\beta$ *in* $\Gamma$ *is contained in* $B_p^{\beta-(n-1)/p}$.

Note that contrary to the regular case, the best trace here is for transverse curves and the worst trace occurs in the complex-tangential case. For the general curve, one has to live with the worst case, i.e., index of differentiability $\alpha$ equal to $2(\beta-\frac{n}{p})+\frac{1}{p}$.

**Theorem.** *Assume* $\Gamma$ *complex-tangential satisfying condition* (I) *and* $n-\beta p<1/2$. *Then the trace of* $H^p_\beta$ *along* $\Gamma$ *is exactly* $B_p^{2(\beta-n/p)+1/p}$.

We comment briefly on the proofs of these generalizations. The restriction part is proved exactly as in Section 3. The only difference is that now in (18) the choice $\alpha=2(\beta-\frac{n}{p})+\frac{1}{p}$ works for all curves (because $\alpha p-1\leq 0$), whereas the better choice $\alpha=\beta-\frac{n}{p}+\frac{1}{p}$ works only for the transverse curves.

As for the proof of the interpolation part, everything works equally well. The only detail to be checked is that for a general function $f(s)$ which is just integrable on the curve, $T_qf(z)$ has admissible limit $f(s)$ at almost every point $\gamma(s)$ of $\Gamma$. Since we already know that the limit exists and that the result holds when $f$ is continuous, by standard arguments it is enough to see that

$$(T_qf)^*(s)\leq \text{const}\, Mf(s),$$

where $(T_qf)^*(s)$ denotes the radial maximal function and $Mf(s)$ the Hardy—Littlewood maximal function. To prove this, if $z=r\gamma(s)$ then $s_z=s$ and by estimate (20) we obtain

$$\left|T_qf(r\gamma(s))\right|\leq c\int_{-\pi}^{+\pi}|f(t)|\frac{(1-r)^{q-1/2}}{(1-r+(t-s)^2)^q}\,dt.$$

Let now $I_k$ be the interval centered at $s$ with length $2^{k/2}(1-r)^{1/2}$, $k=0,1,2,\dots$. Then, with $|I_k|=2^k(1-r)^{1/2}$

$$\left|T_qf(r\gamma(\theta))\right|\leq \text{const}\frac{1}{|I_0|}\int_{I_0}|f(t)|\,dt+c\sum_{k=1}^{\infty}(1-r)^{q-1/2}\frac{1}{2^{kq}(1-r)^q}\int_{I_k}|f(t)|\,dt$$

$$\leq Mf(s)\left\{c+c\sum_{k=1}^{\infty}2^{k/2-kq}\right\}=cMf(s).$$

**5.3.** The third generalization concerns the replacement of a curve by a higher dimensional submanifold $M$ of the sphere. If the dimension of $M$ is $m$, analogous interpolation operators $T_q$ were introduced by Nagel for $q>\frac{m}{2}$ and a property (I) can be stated. We state without proof the corresponding results.

**Theorem.** *Let $M$ be a complex-tangential submanifold of dimension $m$ (i.e. $T_\zeta(M)\subset T_\zeta^c(S)$ for each $\zeta\in M$, and $m\leqq n-1$) for which property* (I) *holds. Then, if $n-\beta p<\frac{m}{2}$, the space $H_\beta^p$ has a trace on $M$ exactly equal to $B_p^{2(\beta-n/p)+m/p}(M)$.*

For a general manifold the description of the trace would involve non-isotropic Besov spaces in the sense of Nikol'skiĭ, with more regularity in some directions than in others. We simply mention that the trace is always contained in $B_p^{\beta-n/p+(m+1)/2p}(M)$, when $n-\beta p<\frac{m+1}{2}$.

## 6. Appendix

In this section we consider the property (I) used in the proof of the interpolation results. We first prove it for the model curve $s\mapsto(\cos s,\, \sin s)$ in $\mathbf{C}^2$ in Subsection 6.1, with $q_0(m)=2m$. In Subsection 6.2 we give, for a general complex-tangential curve, a combinatorial type argument to check this property for a given value of $m$. We will give details only for $m=1$.

**6.1.** For the curve $\gamma(s)=(\cos s, \sin s)$, and with the notations used in Section 4 we prove first:

**Lemma.** *The function*

$$\int_{-\pi}^{+\pi} K_q(s, z)\sin^k(s-s_z)\,ds,$$

*is of class $C^\infty$ up to the boundary in a neighbourhood of $\Gamma$ if $q>k$, $k\in\mathbf{N}$.*

*Proof.* It will be enough to show the result for

$$h(z)=\int_{-\pi}^{+\pi} K_q(s, z)e^{iks}\,ds, \quad q>|k|,\ k \text{ integer}.$$

In fact we will show that $h$ extends holomorphically across $\Gamma$. We first consider a point $z=(r\cos\theta,\, r\sin\theta)$ with real coordinates. In this case

$$h(z)=e^{ik\theta}\left(\int_{-\pi}^{+\pi}\frac{ds}{(1-r\cos s)^q}\right)^{-1}\left(\int_{-\pi}^{+\pi}\frac{e^{iks}}{(1-r\cos s)^q}\,ds\right).$$

Write

$$I_{q,k}(r)=\int_{-\pi}^{+\pi}\frac{e^{iks}}{(1-r\cos s)^q}\,ds=\sum_{j\geqq 0}\binom{-q}{j}(-r)^j\int_{-\pi}^{+\pi}e^{iks}\cos^j s\,ds$$

$$=\sum_{l\geqq 0}\binom{-q}{k+2l}(-r)^{k+2l}\frac{1}{2^{k+2l}}\binom{k+2l}{l}2\pi$$

$$=\frac{2\pi r^k(q)_k}{2^k k!}F\left(\frac{q+k+1}{2},\ \frac{q+k}{2};\ k+1:r^2\right)$$

where $F(a,b;c:\lambda)$ denotes the hypergeometric function

$$F(a,\ b;\ c:\lambda)=\sum_{l=0}^{\infty}\frac{(a)_l(b)_l}{l!(c)_l}\lambda^l$$

and $(a)_0=1$, $(a)_l=a(a+1)\dots(a+l-1)$.

Using the identity

$$F(a,b;c:\lambda)=(1-\lambda)^{c-a-b}F(c-a,c-b;c:\lambda)$$

we obtain

$$I_{q,k}(r)=\frac{2\pi r^k(q)_k}{2^k\cdot k!}(1-r^2)^{(1/2)-q}F\left(\frac{1+k-q}{2},\ \frac{2+k-q}{2};\ k+1:\ r^2\right).$$

Note that the function $G_{q,k}(\lambda)=F(\frac{1+k-q}{2},\ \frac{2+k-q}{2};\ k+1:\lambda)$ is a polynomial of degree $[(q-k-1)/2]$, if $q$ is an integer bigger than $k$ and that $G_{q,0}(1)\neq 0$.

We can then write

$$I_{q,k}(r)=(1-r^2)^{(1/2)-q}r^{-2k}P_{q,k}(1-r^2)$$

where $P_{q,k}$ is a polynomial with $P_{q,0}(0)\neq 0$.

Therefore we conclude that for $z=(r\cos\theta,\ r\sin\theta)$

$$h(z)=e^{ik\theta}r^{-2k}\frac{P_{q,k}(1-r^2)}{P_{q,0}(1-r^2)}$$

with $P_{q,0}(0)\neq 0$. By analytic continuation it then follows that

$$h(z)=\left(\frac{z_1+iz_2}{z_1^2+z_2^2}\right)^k\frac{P_{q,k}(1-z_1^2-z_2^2)}{P_{q,0}(1-z_1^2-z_2^2)}$$

for $z$ close to $\Gamma$, which proves the lemma. ∎

Let us now prove the result. Let $f(s)$ be a $C^\infty$ function on the curve. It is enough to deal with $T_q f$ for $z$ close to $\Gamma$. Fixed $z$, we consider the change of variable $u=\sin(s-s_z)$ from a neighbourhood of $s_z$ to a neighbourhood of $u=0$. We apply Taylor expansion to $f(s(u))$ up to order $2m-1$ at $u=0$ to obtain

$$f(s)=a_0(s_z)+a_1(s_z)\sin(s-s_z)+\dots+a_{2m-1}(s_z)\sin^{2m-1}(s-s_z)+O(|s-s_z|^{2m})$$

with the $a_i(s_z)$ being $C^\infty$ functions of $z$ in the tubular neighbourhood $U$. From the lemma we conclude that in order to show that $T_q f(z)$ has radial derivatives up to order $m$ bounded it is enough to check it for

$$\int_{-\pi}^{+\pi} K_q(s, z) \mathrm{O}(|s-s_z|^{2m})\, ds.$$

From the estimate (20) it follows that such derivatives are bounded by

$$d(z)^{q-(1/2)-m} \int_{-\pi}^{+\pi} \frac{\mathrm{O}(|s-s_z|^{2m})\, ds}{\left(d(z)+(s-s_z)^2\right)^q}$$

which proves the result because the integral is $\mathrm{O}\left(d(z)^{m-q+1/2}\right)$ if $q>m+1/2$.

**6.2.** We indicate now the procedure to prove the result for a general curve. Exactly as before we must prove that for $q \geqq 2m$

$$\int_{-\pi}^{+\pi} K_q(s, z)(s-s_z)^k\, ds, \quad k \leqq 2m-1$$

have bounded radial derivatives up to order $m$ (in fact, this was the result needed in Subsection 4.2). We introduce the notation

$$h_{q,k}(z) = \int_{-\pi}^{+\pi} \frac{(s-s_z)^k}{\left(1-\overline{\gamma(s)} \cdot z\right)^q}\, ds$$

so that $h_{q,0}=h_q$. We must show that $R^i\left(\frac{h_{q,k}}{h_q}\right)$ are bounded for $k \leqq 2m-1$, $i \leqq m$. Since $Rh_q = qh_{q+1} - qh_q$ and $Rh_{q,k} = qh_{q+1,k} - qh_{q,k}$ it is easily seen that

$$R^i\left(\frac{h_{q,k}}{h_q}\right) = \sum_\alpha c_\alpha h_{q+\alpha_1} \dots h_{q+\alpha_i} h_{q+\alpha_{i+1},k} / h_q^{i+1}.$$

Here $\alpha=(\alpha_1, \alpha_2, \dots, \alpha_{i+1})$ is a multi-index of length $|\alpha| \leqq i$ and $c_\alpha$ absolute constants. Recall that $|h_q(z)| \cong d(z)^{1/2-q}$ and hence we must show that

(26) $$\sum_\alpha c_\alpha h_{q+\alpha_1}(z) \dots h_{q+\alpha_i}(z) h_{q+\alpha_{i+1},k}(z) = \mathrm{O}\left(d(z)^{(i+1)((1/2)-q)}\right).$$

Note that each term of the sum is $\mathrm{O}\left(d(z)^{(1/2-q)(i+1)-i+k/2}\right)$, so that we have to show that appropriate cancellations occur in the sum. For this we will consider asymptotic expansions of the functions $h_{q,k}$ as a sum of terms, each having a growth exactly as a power of $d(z)$, as done in [15]. We write

$$1-z \cdot \overline{\gamma(s)} = G-E$$

where

$$G(s, z) = 1-z \cdot \overline{\gamma(s_z)} - z \cdot \overline{\gamma'(s_z)}(s-s_z) + \frac{l}{2}(s-s_z)^2,$$

$$E(s, z) = \frac{1}{2}\left(l+z \cdot \overline{\gamma''(s_z)}\right)(s-s_z)^2 + \frac{1}{3!} z \cdot \overline{\gamma'''(s_z)}(s-s_z)^3 + \dots.$$

Here $l$ is the constant such that $l=\gamma'(s)\cdot\overline{\gamma'(s)}$. Differentiating $\gamma(s)\cdot\overline{\gamma'(s)}=0$ we obtain $\gamma(s)\cdot\overline{\gamma''(s)}=-l$ and hence $|l+z\cdot\overline{\gamma''(s_z)}|=O(|z-\gamma(s_z)|)$. Since $z\cdot\overline{\gamma'(s_z)}$ is pure imaginary, $|\mathrm{Re}\, G|\geqq\frac{1}{2}(s-s_z)^2$ and therefore

$$\left|\frac{E}{G}(z)\right| = O(|z-\gamma(s_z)|+|s-s_z|).$$

We can then write

$$(1-z\overline{\gamma(s)})^{-q} = \sum_{j=0}^{2m-2}\binom{-q}{j}\frac{(-R)^j}{G^{q+j}}+\frac{1}{G^q}[O(|z-\gamma(s_z)|+|s-s_z|)]^{2m-1}.$$

Since each factor $O(|z-\gamma(s_z)|+|s-s_z|)$ improves the corresponding integral in $d(z)^{1/2}$ we will have

$$h_q(z) = \sum_{j=0}^{2m-2}\binom{-q}{j}\int_{-\pi}^{+\pi}\frac{(-E)^j}{G^{q+j}}\,ds+O\big(d(z)^{(1/2)-q+m-(1/2)}\big).$$

Next, we consider Taylor's development of $(-E)^j$ at $s_z$ up to order $2m+2j-2$

$$(-E)^j = T^{2m+2j-2}((-E)^j)+O(|s-s_z|^{2m+2j-1}).$$

Using

$$\int_{-\pi}^{+\pi}\frac{O(|s-s_z|^{2m+2j-1})}{G^{q+j}}\,ds = O(d(z)^{m-q})$$

we obtain

$$h_q(z) = \sum_{j=0}^{2m-2}\binom{-q}{j}\int_{-\infty}^{+\infty}\frac{T^{2m+2j-2}((-E)^j)}{G^{q+j}}\,ds+O(d(z)^{m-q}) \tag{27}$$

and analogously

$$h_{q,k}(z) = \sum_{j=0}^{2m-2}\binom{-q}{j}\int_{-\infty}^{+\infty}\frac{T^{2m+2j-2-m}((-E)^j)(s-s_z)^m}{G^{q+j}}\,ds+O(d(z)^{m-q}). \tag{28}$$

Finally, one explicitly computes integrals of the type

$$\int_{-\infty}^{+\infty}\frac{(s-s_z)^r}{G^{q+j}} = \int_{-\infty}^{+\infty}\frac{t^r}{(1-z\overline{\gamma(s_z)}-z\overline{\gamma'(s_z)}t+\frac{l}{2}t^2)^{q+j}}\,dt$$

by contour integration, obtaining a polynomial

$$\sum_{i=0}^{r}\binom{r}{i}B^iA^{-(q+j)+(r-i+1)/2}\int_{-\infty}^{+\infty}\frac{t^{r-i}}{(1+\frac{l}{2}t^2)^{q+j}}\,dt$$

where $A=1-z\overline{\gamma(s_z)}-\frac{1}{2l}(z\cdot\overline{\gamma'(s_z')^2})$, $B=\frac{1}{l}z\cdot\overline{\gamma'(s_z)}$. It is easy to see that $|A|\cong d(z)$ and $|B|=O(|z-\gamma(s_z)|)=O(d(z)^{1/2})$. Inserting this in (27) and (28) gives the desired expansions of $h_q$ and $h_{q,k}$. To illustrate we write the first three terms of the expansion of $h_q$. We assume $l=1$ and write for short

$$E = \sum_{i\geqq 2} e_i(s-s_z)^i$$

$$c_{i,j} = \int_{-\infty}^{+\infty} \frac{t^j}{\left(1+\frac{1}{2}t^2\right)^i}\,dt.$$

Then

$$h_q(z) = c_{q,0}A^{(1/2)-p}+\{e_2(qc_{q+1,2}A^{-p+(1/2)}+B^2c_{q+1,0}A^{-p-(1/2)})$$

$$+e_3q(3Bc_{q+1,2}A^{-q+(1/2)}+B^3c_{q+1,0}A^{-q-(1/2)})\}$$

$$+\left\{e_4q\left(B^4c_{q+1,0}A^{-q-(1/2)}+\binom{4}{2}B^2c_{q+1,2}A^{-q+(1/2)}+c_{q+1,4}A^{-q+(3/2)}\right)\right.$$

$$+e_2^2\frac{q(q+1)}{2}\left(B^4c_{q+2,0}A^{-q-(3/2)}+\binom{4}{2}B^2c_{q+2,2}A^{-q-(1/2)}+c_{q+2,4}A^{-q+(1/2)}\right)$$

$$+e_3^2\frac{q(q+1)}{2}\left(B^6c_{q+2,0}A^{-q-(3/2)}+\binom{6}{2}B^4c_{q+2,2}A^{-q-(1/2)}\right.$$

$$\left.+\binom{6}{4}B^2c_{q+2,4}A^{-q+(1/2)}+c_{q+2,6}A^{-q+(3/2)}\right)+$$

$$\left.+e_2e_3q(q+1)\left(B^5c_{q+2,0}A^{-q-(3/2)}\binom{5}{3}B^3c_{q+2,2}A^{-q-(1/2)}+\binom{5}{1}Bc_{q+2,4}A^{-q+(1/2)}\right)\right\}$$

are the terms in $d(z)^{(1/2)-q}$, $d(z)^{1-q}$ and $d(z)^{(3/2)-q}$ of $h_q$ (recall that $e_2=O(d(z)^{1/2})$).

The use of these expansions of the $h_{q,k}|s$ in (26), together with the explicit formula for the $C_\alpha$, shows that the desired cancellation takes place.

The case $i=1$ is short enough to be written here and will serve as illustration. To show that

$$R\frac{h_{q,1}}{h_q} = q\,\frac{h_qh_{q+1,1}-h_{q,1}h_{q+1}}{h_q^2}$$

is bounded it is enough to consider the leading terms of $h_q$ and $h_{q,1}$

$$h_q(z) = c_{q,0}A^{(1/2)-q}+O\left(d(z)^{1-q}\right)$$

$$h_{q,1}(z) = c_{q,0}BA^{(1/2)-q}+O\left(d(z)^{(3/2)-q}\right).$$

Then

$$h_qh_{q+1,1}-h_{q,1}h_{q+1} = (c_{q,0}c_{q+1,0}-c_{q,0}c_{q+1,0})BA^{-2q}+O\left(d(z)^{1-2q}\right)$$

$$=O\left(d(z)^{1-2q}\right) = O(h_q^2).$$

To deal with $R^2\frac{h_{q,1}}{h_q}$ it already requires a considerably longer computation. For higher values of $i$ we have checked all cancellations with the help of a computer.

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