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On surfaces in $\mathbf{P}^6$ with no trisecant lines

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Dedicated to the memory of F. Serrano

Abstract. Examples of surfaces in $\mathbf{P}^6$ with no trisecant lines are constructed. A partial classification recovering them is given and conjectured to be the complete one.

0. Introduction

Surfaces in $\mathbf{P}^6$ are expected to have a 1-parameter family of trisecant lines. More precisely, for a smooth surface, there is a 1-dimensional cycle class in the Grassmannian of lines in $\mathbf{P}^6$, defined in terms of the invariants of the surface, representing the set of lines with a subscheme of length at least 3 in common with the surface (cf. [12]). Of course, a particular surface may actually have a 2-dimensional family of trisecants or contain infinitely many lines, in which case the 1-dimensional class has no obvious meaning. On the other hand, if a surface contains no lines and has no trisecants at all, then the class of the trisecants must be the 0-class. The 1-dimensional cycle of trisecants has two numerical invariants. These are its degree with respect to the Plücker embedding and the number of tangential trisecants. These invariants are effective also for surfaces which contain a finite number of lines. In this case there is a contribution of each line to the number of tangential trisecants computed by Le Barz.

It is not hard to think of examples of smooth surfaces in $\mathbf{P}^6$ with no trisecant lines. Any surface cut out by quadrics cannot, by Bezout, have any trisecant lines. It is harder to find examples which are not cut out by quadrics. On the other hand, the invariants computed by Le Barz give, when set equal to zero, relations among the invariants of surfaces cut out by quadrics. Thus they provide a tool to classify these surfaces. The aim of the paper is twofold. First we use standard techniques to construct surfaces cut out by quadrics. Although the variety of constructions may seem ad hoc, they are all very classical. Surfaces without trisecants which are not cut out by quadrics, are subsequently constructed. Their invariants are computed using the formulas of Le Barz, but we have chosen to present the construction of these surfaces first. This emphasizes our preference for constructing examples.

Secondly we use the numerical relations to prove that our list of examples cover all possible sets of invariants for surfaces cut out by quadrics, and surfaces with no lines and no proper trisecants. We do not, however, claim that our examples are the generic members of the irreducible components of the corresponding Hilbert scheme of surfaces, although we believe they are. Our main result is the following theorem.

Theorem 0.1. *Let S be a smooth surface embedded in $\mathbf{P}^6$ with no trisecant lines. If S is cut out by quadrics or contains no lines, it belongs to the list given in Table* 1.

In the last section we analyze more closely the case when the surface contains lines and prove the following result.

Proposition 0.2. *Let S be a smooth surface embedded in $\mathbf{P}^6$ with no trisecant lines. Unless S has a line L with $L^2 \leq -4$, or a finite number of disjoint (-1)-lines, each one meeting some other line L' on the surface with $(L')^2 \leq -2$, and S is not an inner projection from $\mathbf{P}^7$, the surface belongs to the list in Table* 1.

The study of varieties embedded in $\mathbf{P}^N$ with no trisecant lines is a very classical problem in algebraic geometry. The simplest case, i.e. the case of space curves goes back to Castelnuovo.

For surfaces the problem has been studied in codimension 2 and 3. In [2] Aure classifies smooth surfaces in $\mathbf{P}^4$ with no trisecant lines through the general point in the space. In [3] Bauer classifies smooth surfaces in $\mathbf{P}^5$ with no trisecant lines through the general point on the surface.

In this vein, the next step would be a classification of surfaces in $\mathbf{P}^7$ without trisecants. The expected cycle of trisecants is in this case 0-dimensional and has only one numerical invariant; its degree. Therefore this classification seems much harder than the others.

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0.1. Notation

The ground field is the field of complex numbers $\mathbf{C}$. We use standard notation in algebraic geometry, as in [10]. The surface S is always assumed to be a nonsingular projective surface.

Table 1.

Surface	Degree	Linear system	Example
$\mathbf{P}^2$	1, 4	$\mathcal{O}_{\mathbf{P}^2}(1)$, $\mathcal{O}_{\mathbf{P}^2}(2)$	1.1
Rational scroll	2, 3, 4, 5	E_0+af, $2\le E_0^2+2a\le 4$	1.7
Elliptic scroll	7	E_0+3f, $E_0^2=1$	1.15
$\mathrm{Bl}_t(\mathbf{P}^2)$, $t=5,\ 4,\ 3$	4, 5, 6	$3l-\sum_{i=1}^{t} E_i$	1.8
$\mathrm{Bl}_6(\mathbf{P}^1\times\mathbf{P}^1)$	6	$(2,3)-\sum_{i=1}^{6} E_i$	1.9
$\mathrm{Bl}_5(\mathbf{P}^1\times\mathbf{P}^1)$	7	$(2,3)-\sum_{i=1}^{5} E_i$	1.10
$\mathrm{Bl}_7(\mathbf{P}^2)$	8	$6l-\sum_{i=1}^{7} 2E_i$	1.2
$\mathrm{Bl}_8(\mathbf{P}^1\times\mathbf{P}^1)$	8	$(2,4)-\sum_{i=1}^{8} E_i$	1.11
$\mathrm{Bl}_8(\mathbf{P}^2)$	8	$4l-\sum_{i=1}^{8} E_i$	1.12
$K3$	8		1.3
$\mathrm{Bl}_9(\mathbf{P}^1\times\mathbf{P}^1)$	9	$(3,3)-\sum_{i=1}^{9} E_i$	1.13
$\mathrm{Bl}_{11}(\mathbf{P}^2)$	10	$6l-\sum_{i=1}^{5} 2E_i-\sum_{j=6}^{11} E_j$	1.14
$K3$	10		1.4
$\mathrm{Bl}_1(K3)$	11	$p^*(\overline{\mathcal{H}})-E$	1.16
$\mathrm{Bl}_{11}(\mathbf{P}^2)$	12	$9l-\sum_{i=1}^{5} 3E_i-\sum_{j=6}^{11} 2E_j$	2.2
$\mathrm{Bl}_1(K3)$	12	$p^*(\overline{\mathcal{H}})-2E$	2.1
Regular elliptic, $p_g=2$	12		1.5
Abelian	14	$(1,7)$-polarization	2.3
General type	16		1.6

By abuse of notation $\mathcal{H}_S$ will denote the hyperplane section and the line bundle giving the embedding, with no distinction.

When S is the blow up of S_0 in n points, S will be denoted by $\mathrm{Bl}_n(S_0)$.

1. Construction of surfaces defined by quadrics

In this and the following section we present a list of examples, to show that surfaces without trisecant lines do exist. The reader may skip to Section 3 to find the argument that these examples cover precisely the possibilities of Theorem 0.1 and then come back to have a proof of the existence. We choose to present the examples first since we find the construction of these surfaces, especially the Examples 2.1

and 2.2, particularly interesting on their own.

We start off with the more familiar examples. If S is a surface whose homogeneous ideal I_S is generated by quadrics then clearly it cannot have trisecant lines.

1.1. Surfaces defined by quadrics with no lines

Consider surfaces with no lines. These surfaces all move in a family in the Hilbert scheme, and in some cases they may degenerate to smooth surfaces containing lines but still without proper trisecants. This is in particular the case for the Examples 1.3–1.6.

The first examples in our list are the following ones.

Example 1.1. (*The Veronese surface in* $\mathbf{P}^5$.) This is $\mathbf{P}^2$ embedded in $\mathbf{P}^5$ by the line bundle $\mathcal{O}_{\mathbf{P}^2}(2)$.

Example 1.2. (*Del Pezzo surfaces of degree* 8.) Let S be the blow up of $\mathbf{P}^2$ in seven points embedded in $\mathbf{P}^6$ by the linear system $|-2K_S|$. The line bundle $\mathcal{H}_S=-2K_S$ is 2-very ample and thus embeds S without trisecant lines. See [7] for the definition and the proof of the 2-very ampleness. One can construct this surface in $\mathbf{P}^6$ as the intersection of the cone over a Veronese surface in $\mathbf{P}^5$ with a quadric hypersurface. Thus the surface is defined by quadrics, and it has no lines as soon as the quadric does not contain the vertex of the cone. These surfaces are defined by seven quadrics in $\mathbf{P}^6$.

Example 1.3. (*Minimal nontrigonal* $K3$*-surface of degree* 8.) Consider a $K3$-surface of degree 8 in $\mathbf{P}^5$ such that the general hyperplane section is not trigonal. It is the complete intersection of three quadrics, and the general one has Picard group generated by the hyperplane section so it has no lines.

Example 1.4. (*Minimal nontrigonal* $K3$*-surfaces of degree* 10.) A nontrigonal $K3$-surface of degree 10 in $\mathbf{P}^6$ is a linear section of the Plücker embedding of the Grassmannian $\mathrm{Gr}(2,5)$ intersected with a quadric hypersurface. The surface is defined by quadrics, in fact six quadrics, and the general one has Picard group generated by the hyperplane section, so it has no lines. These surfaces may also be constructed by linkage. Consider a rational surface $\mathrm{Bl}_7(\mathbf{P}^2)$ embedded in $\mathbf{P}^5$ by the line bundle $4l-2E-\sum_{i=1}^{6} E_i$. This surface is defined by four quadrics in $\mathbf{P}^5$ (see Example 1.9), and in a complete intersection $(2,2,2,2)$ in $\mathbf{P}^6$ it is linked to $K3$-surfaces of the above type.

Example 1.5. (*Two families of elliptic surfaces of degree* 12.) Let V be a rational normal 4-fold scroll of degree 3 in $\mathbf{P}^6$ and consider $S=V\cap\mathcal{Q}_1\cap\mathcal{Q}_2$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two general quadrics, which do not have any common point in the singular locus of V. Note that this is possible only if the cubic 4-fold has vertex a point or a line. This gives us two separate cases, of which the latter is a degeneration of the former. In both cases the intersection of V with the two quadrics is a smooth surface. Now $K_{\bar{V}}=-4\mathcal{H}+\mathcal{F}$, where $\bar{V}$ is the $\mathbf{P}^3$-bundle over $\mathbf{P}^1$ which is mapped to V by $\mathcal{H}$, and $\mathcal{F}$ is a member of the ruling. Then $K_S=\mathcal{F}|_S$ gives a fibration of elliptic quartic curves without multiple fibers onto $\mathbf{P}^1$.

The surface S does not have proper trisecant lines since it is defined by quadrics. Moreover, S cannot contain lines, for a general choice of quadrics $\mathcal{Q}_i$. In fact, if there were a line $L\subset S$, then $L\subset V$ and $L\subset\mathcal{Q}_i$. The biggest component of the Fano variety of lines in V is the 5-dimensional component of lines in the pencil of $\mathbf{P}^3$'s. Since a line imposes three conditions on quadrics, we need at least a 6-dimensional family of lines on V in order for two general quadrics to contain a line on V. Thus we have constructed two families of elliptic surfaces of degree 12, one on the cubic 4-fold cone with vertex a point, and one on the cubic 4-fold cone with vertex a line. Note that the latter is a specialization of the former. In the first case any two canonical curves span $\mathbf{P}^6$, while in the other case any two canonical curves span a $\mathbf{P}^5$. These surfaces are defined by five quadrics in $\mathbf{P}^6$.

One may also construct these surfaces by linkage. Consider a Del Pezzo surface of degree 4 embedded in $\mathbf{P}^4$. This surface is a complete intersection of two quadrics in $\mathbf{P}^4$, and in a complete intersection $(2,2,2,2)$ in $\mathbf{P}^6$ it is linked to an elliptic surface of the above type.

Example 1.6. (*Complete intersections of four quadric hypersurfaces in* $\mathbf{P}^6$.) This is a degree 16 surface of general type. Since the Fano variety of lines in a quadric has codimension 3 in the Grassmannian of lines in $\mathbf{P}^6$ and this Grassmannian has dimension 10, there are no lines in a general complete intersection of four quadrics. A similar argument is made more precise in Section 2.1 below.

1.2. Construction of surfaces with lines

We proceed to construct examples of surfaces containing lines but with no proper trisecant lines. In all the cases below the surfaces are defined by quadrics, so naturally there are no proper trisecant lines. In fact, we do not know of any surface with lines on it, but with no proper trisecant, which is not defined by quadrics.

1.2.1. Rational surfaces

Example 1.7. (*Rational scrolls.*) Natural examples of surfaces with at least a one dimensional family of lines are given by surfaces of minimal degree, i.e. the rational normal scrolls of degree $N-1$ in $\mathbf{P}^N$, $3 \le N \le 6$.

Example 1.8. (*Del Pezzo surfaces.*) The Del Pezzo surfaces of degree 4, 5 and 6 form natural families of surfaces defined by quadrics. They contain a finite number of lines, which are all (-1)-lines.

In the following we construct the other rational surfaces occurring on our list.

Example 1.9. (*Conic bundles of degree* 6.) Consider a cubic 3-fold scroll $V \subset \mathbf{P}^5$ and let $S = V \cap Q$ be the surface of degree 6 given by the intersection with a general quadric hypersurface. S is smooth as soon as Q avoids the vertex of V. Therefore we have two cases, when V is smooth and when V is the cone over a smooth cubic surface scroll in $\mathbf{P}^4$.

Let $\bar{V}$ be the $\mathbf{P}^2$-scroll which is mapped to V by $\mathcal{H}$, and let $\mathcal{F}$ be a member of the ruling. By abuse of notation we denote the pullback of S to $\bar{V}$ by S, it is isomorphic anyway. Then by adjunction

$$K_S = (-3\mathcal{H} + \mathcal{F} + 2\mathcal{H})|_S = -\mathcal{H}|_S + \mathcal{F}|_S$$

and thus $(K_S)^2 = 2$ and $K_S \cdot \mathcal{H}_S = -4$. Furthermore $p_g(S) = q(S) = 0$, so S is rational. The ruling of the scroll defines a conic bundle structure on S, and there are six singular fibers, i.e. twelve (-1)-lines in the fibers since $K_S^2 = 2$. Let $S = \mathrm{Bl}_6(\mathbf{F}_s)$ and $\mathcal{H}_S = 2E_0 + af - \sum_{i=1}^{6} E_i$, where $\mathbf{F}_s$ is a Hirzebruch surface and E_0 a section with $E_0^2 = -s$. Numerical computations yield $a = 3$ and $s = 0$. Note that these surfaces do not contain lines L with $L^2 \le -4$, since in that case, by adjunction, $2 \le L \cdot K = L \cdot (-\mathcal{H}_S + \mathcal{F}_S) = -1 + L \cdot \mathcal{F}_S \le 1$, a contradiction. These surfaces are defined by four quadrics in $\mathbf{P}^5$.

Example 1.10. (*Conic bundles of degree* 7.) Consider a rational normal 3-fold scroll $V \subset \mathbf{P}^6$ of degree 4, and let Q be a general quadric hypersurface containing a member of the ruling $\mathcal{F}$. Then the complete intersection $V \cap Q = \mathcal{F} \cup S$, where S is a surface of degree 7 in $\mathbf{P}^6$. As soon as V is smooth and Q general, S is smooth. If V is singular, then S is singular at the singular point of V. By adjunction

$$K_S = (-3\mathcal{H} + \mathcal{F} + 2\mathcal{H})|_S = -\mathcal{H}|_S + \mathcal{F}|_S,$$

where $\mathcal{H}$ is a hyperplane section. Thus $K_S^2 = 3$ and $K_S \cdot \mathcal{H}_S = -5$. Like in the previous example the ruling of V define a conic bundle structure on S with five singular fibers, i.e. ten (-1)-lines altogether in the fibers. If S had a trisecant line, L, then L would be a line in $\mathcal{F}$ intersecting the curve $C=S\cap\mathcal{F}$ in three points. In this case C is a conic section so there is no trisecant. In fact one can also show that S is defined by eight quadrics. As in Example 1.9 we see that $S=\mathrm{Bl}_5(\mathbf{F}_0)$ and $\mathcal{H}_S=(2,3)-\sum_{i=1}^{5}E_i$ and that S contains no lines with selfintersection ≤ -4.

Example 1.11. (*Conic bundles of degree* 8.) Let V be a rational normal 3-fold scroll of degree 4 in $\mathbf{P}^6$, and let $\mathcal{Q}$ be a general quadric hypersurface not containing any singular point on V. Then V is smooth or is a cone with vertex a point, and $S=V\cap\mathcal{Q}$ is a smooth surface of degree 8 in $\mathbf{P}^6$. We get two cases like in Example 1.9. Proceeding with notation like in that case we get $K_S=2\mathcal{F}_S-\mathcal{H}_S$ and $K_S^2=0$. Thus we get conic bundles with sectional genus 3 with eight singular fibers, i.e. sixteen (-1)-lines in fibers. These surfaces are defined by seven quadrics. Again $S=\mathrm{Bl}_8(\mathbf{F}_0)$ and $\mathcal{H}_S=(2,4)-\sum_{i=1}^{8}E_i$. Moreover, as above, S contains no lines with selfintersection ≤ -4.

Example 1.12. (*A family of surfaces of degree* 8.) Let V be a cone over the Segre embedding of $\mathbf{P}^1\times\mathbf{P}^2$ as in Example 1.5 and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two quadrics containing a member $\mathcal{F}$ of the ruling, in particular they pass through the vertex of the cone. Then $V\cap\mathcal{Q}_1\cap\mathcal{Q}_2=S\cup\mathcal{F}$, where S is a smooth surface of degree 8 in $\mathbf{P}^6$. Let $\bar{V}$ be the $\mathbf{P}^3$-bundle over $\mathbf{P}^1$ associated to V, and let $\mathcal{H}$ on $\bar{V}$ be the line bundle defining the map $\bar{V}\to V$ which contracts a $\mathbf{P}^1$ to the vertex. Let $\bar{S}$ be the strict transform of S. If, by abuse of notation, $\mathcal{F}$ also denotes the pullback of $\mathcal{F}$ to $\bar{V}$, then $\bar{S}=(2\mathcal{H}-\mathcal{F})\cap(2\mathcal{H}-\mathcal{F})$ on $\bar{V}$. The canonical divisor on $\bar{V}$ is $-4\mathcal{H}+\mathcal{F}$. Thus, by adjunction, $K_{\bar{S}}=-\mathcal{F}_{\bar{S}}$. Since both quadrics $\mathcal{Q}_i$ pass through the vertex, the $\mathbf{P}^1$ lying over the vertex is contained in $\bar{S}$. Clearly, it meets every $\mathbf{P}^3$ of $\bar{V}$ in a point. Therefore this curve, call it E, on $\bar{S}$ has intersection $E\cdot K_{\bar{S}}=-1$. Hence E is an exceptional curve of the first kind on $\bar{S}$, which is blown down on S. Thus $|-K_S|$ is a pencil of elliptic curves with one base point at the vertex of V. It follows that S is rational of degree 8, sectional genus 3 and $K_S^2=1$. The adjunction $|\mathcal{H}_S+K_S|$ maps the surface birationally to $\mathbf{P}^2$, so the surface is $\mathrm{Bl}_8(\mathbf{P}^2)$ with $\mathcal{H}_S=4l-\sum_{i=1}^{8}E_i$. It is straightforward to check with this linear system that the surface cannot contain lines with selfintersection ≤ -4. These surfaces are defined by six quadrics in $\mathbf{P}^6$.

Example 1.13. (*Two families of surfaces of degree* 9.) Let $\bar{V}$ be a $\mathbf{P}^3$-bundle of degree 3 over $\mathbf{P}^1$ with ruling $\mathcal{F}$ and let V be its image rational normal 4-fold of degree 3 in $\mathbf{P}^6$ under the map defined by $\mathcal{H}$ as in Example 1.5. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be general quadrics with no common point in the vertex of V and which contain a smooth cubic surface $S_3=V\cap\mathbf{P}^4$, for some general $\mathbf{P}^4\subset\mathbf{P}^6$. Then $V\cap\mathcal{Q}_1\cap\mathcal{Q}_2=S_3\cup S$, where S is a smooth surface of degree 9. The curve of intersection $C=S_3\cap S$ is then a curve of degree 6 represented in S_3 by the divisor $2\mathcal{H}_{S_3}$. Thus C has genus 2. Moreover, S cannot have trisecants since C has no trisecant lines.

Now, S_3 meets each ruling of V in a line, so the surface S, which is linked to S_3 on V in two quadrics, meets each ruling of V in a twisted cubic curve. Therefore S is rational. Furthermore, $K_{\overline{V}}=-4\mathcal{H}+\mathcal{F}$, so $K_S=(-4\mathcal{H}+\mathcal{F}+4\mathcal{H})|_S-S_3|_S=-C+\mathcal{F}_S$, by adjunction. Thus $K_S\cdot\mathcal{H}_S=(\mathcal{F}_S-C)\cdot\mathcal{H}_S=-3$ and S has sectional genus 4. Let $D=\mathcal{H}_S-C$, then D has degree 3 and moves in a pencil, so it must be a twisted cubic curve, with $D^2=0$. By adjunction

$$6=\mathcal{H}_S(\mathcal{H}_S+K_S)=C(C+K_S)+D(D+K_S)+2C\cdot D=2-2+2C\cdot D,$$

since C has genus 2, hence $C\cdot D=3$. Therefore, $C^2=3$, while $C\cdot\mathcal{F}_S=2$ and $K_S^2=(\mathcal{F}_S-C)^2=-1$.

The cubic scroll V is a cone with vertex a point or a line. Thus we get two types of surfaces S, of which the latter is a degeneration of the former. Both are rational surfaces of degree 9, sectional genus 4 and $K_S^2=-1$. The adjoint linear system $|\mathcal{H}+K_S|$ maps the surface birationally to a smooth quadric surface in $\mathbf{P}^3$, so the surface is $\mathrm{Bl}_9(\mathbf{P}^1\times\mathbf{P}^1)$ with $\mathcal{H}_S=(3,3)-\sum_{i=1}^9 E_i$. In both cases S has, in fact, two pencils of twisted cubic curves. The above construction applies to either pencil. If the vertex of the cubic 4-fold scroll V is a point, then any two of the curves in a pencil span $\mathbf{P}^6$, while any two of them span a $\mathbf{P}^5$ if the vertex of V is a line. Intrinsically the two cases correspond to whether the 9 points on the quadric lies on a rational quartic curve of type $(3,1)$ (or $(1,3)$) or not. From the linear system it is straightforward to check that the surface cannot contain lines with selfintersection ≤ -4. These surfaces are defined by six quadrics in $\mathbf{P}^6$.

One may also construct these surfaces by linkage. Consider a conic bundle of degree 7 as in Example 1.10. This surface is cut out by eight quadrics in $\mathbf{P}^6$, and in a complete intersection $(2,2,2,2)$ it is linked to a rational surface of the above type.

Example 1.14. (*A family of surfaces of degree* 10.) Consider the Del Pezzo surface S_6 of degree 6 in $\mathbf{P}^6$, and 4 general quadrics containing it, $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ and $\mathcal{Q}_4$. Then $\mathcal{Q}_1\cap\mathcal{Q}_2\cap\mathcal{Q}_3\cap\mathcal{Q}_4=S\cup S_6$, where S is a smooth surface of degree 10 in $\mathbf{P}^6$. Let V be the complete intersection of a general set of three quadrics in $\langle\mathcal{Q}_1,\dots,\mathcal{Q}_4\rangle$. By adjunction we see that $C=S\cap S_6=\mathcal{H}_V|_{S_6}-K_{S_6}=2\mathcal{H}_{S_6}$. Therefore S has no proper trisecant lines. Furthermore, $K_S=(\mathcal{H}_V-S_6)_S$ and $K_S\cdot\mathcal{H}_S=((\mathcal{H}_V-S_6)_S)\cdot\mathcal{H}_S=10-12=-2$, so the sectional genus is 5. The exact sequence

$$(1)\qquad 0\longrightarrow\mathcal{O}_V(\mathcal{H}_V-S_6)\longrightarrow\mathcal{O}_V(\mathcal{H}_V)\longrightarrow\mathcal{O}_{S_6}(\mathcal{H}_{S_6})\longrightarrow 0,$$

and the fact that $h^0(\mathcal{O}_{S_4}(\mathcal{H}_{S_4}))=7$, give $h^1(\mathcal{O}_V(\mathcal{H}_V-S_6))=h^2(\mathcal{O}_V(\mathcal{H}_V-S_6))=0$ and $h^0(\mathcal{O}_V(\mathcal{H}_V-S_6))=0$. Plugging those values into the long exact cohomology sequence of

$$(2) \qquad 0 \longrightarrow \mathcal{O}_V(-\mathcal{H}_V) \longrightarrow \mathcal{O}_V(\mathcal{H}_V-S_6) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow 0,$$

we get $p_g(S)=0$ and $q=0$ and thus $\chi(S)=1$. Since $K_S\cdot\mathcal{H}_S=-2$, no pluricanonical divisor is effective, so S is rational. If L is a line on S with $L^2\leq-4$, then, by adjunction $-2-L^2=L\cdot K_S=L\cdot(\mathcal{H}_S-C)=1-L\cdot C$. Therefore $L\cdot C=3+L^2\leq-1$, and L is a component of $C=S\cap S_6$. Thus L must be a (-1)-line on S_6. This is certainly possible. One may show that in this case the line L is a bisecant to a conic section on S, so that S has a 2-dimensional family of trisecants in this case.

The general surfaces S, however, are defined by five quadrics in $\mathbf{P}^6$. The adjoints of the surfaces in Section 2.2 below are of this type.

1.2.2. Nonrational surfaces

There are also nonrational surfaces without proper trisecant lines.

Example 1.15. (*Elliptic scrolls.*) The elliptic normal scrolls of degree 7 for which the minimal selfintersection of a section is 1, are defined by seven quadrics (cf. [11]).

Example 1.16. (*A family of nonminimal $K3$-surfaces.*) Consider an inner projection of a general nontrigonal and nontetragonal $K3$-surface $\bar{S}$ of degree 12 in $\mathbf{P}^7$ (cf. [14]). Let S be the projection from a point $p\in\bar{S}$, $\pi_p\colon\bar{S}\to S$. Then S is a $K3$-surface of degree 11 in $\mathbf{P}^6$ with one line, i.e. the exceptional line over p. Any trisecant of S will come from a trisecant of $\bar{S}$ or from a 4-secant $\mathbf{P}^2$ to $\bar{S}$ through p. But a normally embedded $K3$-surface with a trisecant is trigonal, and with a 4-secant plane is tetragonal which is avoided by assumption. So S has no trisecant. These surfaces are defined by five quadrics in $\mathbf{P}^6$, and it is easy to check, like in Example 1.14, that they are linked to Del Pezzo surfaces of degree 5 in the complete intersection of four quadrics. Since $\bar{S}$ is $K3$, any line on S has selfintersection ≥-3.

2. Surfaces not defined by quadrics

In this section we conclude the construction of examples by considering three particular families of surfaces which do not form the intersection of quadric hypersurfaces. They have no trisecant lines and contain no lines.

2.1. Nonminimal $K3$-surfaces of degree 12

Consider four general quadrics $\mathcal{Q}_1, \dots, \mathcal{Q}_4 \subset \mathbf{P}^6$ containing the Veronese surface $S_4 \subset \mathbf{P}^5$ and let S be the residual surface of degree 12 in $\mathbf{P}^6$. Let V be the complete intersection of a general set of three quadrics in $\langle \mathcal{Q}_1, \dots, \mathcal{Q}_4 \rangle$. By adjunction we see that $C = S \cap S_4 = \mathcal{H}_V|_{S_4} - K_{S_4} = 5l$, where l is the generator of $\operatorname{Pic}(S_4)$. Then $K_S = (\mathcal{H}_V - S_4)_S$ and

$$K_S \cdot \mathcal{H}_S = (\mathcal{H}_V + S_4)_S \cdot \mathcal{H}_S = 12 - 10 = 2.$$

The exact sequence

$$(3) \qquad 0 \longrightarrow \mathcal{O}_V(\mathcal{H}_V - S_4) \longrightarrow \mathcal{O}_V(\mathcal{H}_V) \longrightarrow \mathcal{O}_{S_4}(\mathcal{H}_{S_4}) \longrightarrow 0$$

and the fact that $h^0(\mathcal{O}_{S_4}(\mathcal{H}_{S_4})) = 6$ gives $h^0(\mathcal{O}_V(\mathcal{H}_V - S_4)) = 1$ and

$$h^1(\mathcal{O}_V(\mathcal{H}_V - S_4)) = h^2(\mathcal{O}_V(\mathcal{H}_V - S_4)) = 0.$$

Plugging those values into the long exact cohomology sequence of

$$(4) \qquad 0 \longrightarrow \mathcal{O}_V(-\mathcal{H}_V) \longrightarrow \mathcal{O}_V(\mathcal{H}_V - S_4) \longrightarrow \mathcal{O}_S(K_S) \longrightarrow 0,$$

we get $p_g(S) = 1$ and $q = 0$ and thus $\chi(S) = 2$. Thus the canonical curve is a (-1)-conic section or two disjoint (-1)-lines. It is the residual to the intersection C of S with S_4 in a hyperplane section of S, so each component intersects the curve C in at least two points. If the canonical curve is two disjoint lines these would be secants to S_4. But a secant line to S_4 is a secant line to a unique conic section on S_4. Any quadric which contains a secant line must therefore contain the plane of this conic section. Therefore S contains no secant line to S_4 as soon as S is irreducible. We conclude that the canonical curve is a (-1)-conic section. The surface S is a blown up $K3$-surface and $K_S^2 = -1$. We now show that S contains no lines. Assume the contrary and let L be a line on S. First assume that the line does not intersect S_4. Let G be the Grassmannian of 4-dimensional subspaces of quadrics in $\mathbf{P}^6$ which contain S_4. Consider the incidence variety

$$I_e = \left\{ (L, U) \in \operatorname{Gr}(2,7) \times G \;\middle|\; L \subset \bigcap_{\mathcal{Q} \in U} \mathcal{Q},\ L \cap S_4 = \emptyset \right\}$$

and let $p: I_e \to \operatorname{Gr}(2,7)$ and $q: I_e \to G$ be the two projections. Since the lines in $\mathbf{P}^6$ form a 10-dimensional family and a line imposes three conditions on quadrics, it is clear that $\operatorname{codim} q(p^{-1}(L)) = 12$ and that $\operatorname{codim} q(p^{-1}(\operatorname{Gr}(2,7))) \geq 12 - 10 = 2$. This means that we can choose U general enough so that there is no line $L\subset\bigcap_{Q\in U} Q$ which is disjoint from S_4.

Similarly consider

$$I_{ne}=\left\{(L,U)\in \mathrm{Gr}(2,7)\times G \,\middle|\, L\subset \bigcap_{Q\in U} Q,\ L\cap S_4\neq\emptyset\right\}$$

with projections p and q, where

$$p\colon I_{ne}\longrightarrow \overline{G}=\{L\in \mathrm{Gr}(2,7)\mid L\cap S_4\neq\emptyset\}.$$

The set of lines which intersect a given surface in $\mathbf{P}^6$ has codimension 3 in $\mathrm{Gr}(2,7)$ so $\dim \overline{G}=7$. A line L through a point on S_4 imposes two conditions on quadrics through S_4, so $\operatorname{codim} q(p^{-1}(L))=8$ in this case. But then $\operatorname{codim} q(p^{-1}(\overline{G}))\geq 8-7=1$, so for a general U there are no lines on S intersecting S_4 in one point. We are left to examine the case when L meets S_4 in at least two points, i.e. when L is a secant line to the Veronese surface S_4. But as above this is impossible as long as S is irreducible.

Thus S has no lines on it and is the blow up of a $K3$ surface in one point. Moreover $\mathcal{H}_S=p^*(\overline{\mathcal{H}})-2E$, where $p\colon S\to\overline{S}$ is the blow up map and $\overline{\mathcal{H}}$ is a line bundle on $\overline{S}$ of degree 16. Again S has no trisecant line because such a line would necessarily be a line in the Veronese surface. Notice that the conic sections on the Veronese surface each meet S in five points. In fact, the Veronese surface is the union of the 5-secant conic sections to S and is therefore contained in any quadric which contains S. It is straightforward to check that the surfaces S are defined by four quadrics and three cubics in $\mathbf{P}^6$.

2.2. A family of rational surfaces of degree 12

Let $S=\mathrm{Bl}_{11}(\mathbf{P}^2)$ be polarized by the line bundle $H=9l-\sum_{i=1}^{5}3E_i-\sum_{j=6}^{11}2E_j$. Assume that the eleven points blown up are in general position. More precisely we require that the following linear systems are empty for all possible sets of distinct indices:

$$|E_i-E_j|,\quad |l-E_i-E_j-E_k|,\quad \left|2l-\sum_{k=1}^{6}E_{i_k}\right|,\quad \left|3l-\sum_{k=1}^{10}E_{i_k}\right|,$$

$$\left|3l-2E_{i_0}-\sum_{k=1}^{7}E_{i_k}\right|,\quad \left|4l-2\sum_{k=1}^{2}E_{i_k}-\sum_{k=3}^{11}E_{i_k}\right|,\quad \left|4l-2\sum_{k=1}^{3}E_{i_k}-\sum_{k=4}^{10}E_{i_k}\right|,$$

$$\left|5l-2\sum_{k=1}^{5}E_{i_k}-\sum_{k=6}^{11}E_{i_k}\right|,\quad \left|6l-3E_{i_0}-2\sum_{k=1}^{6}E_{i_k}-\sum_{k=7}^{10}E_{i_k}\right|,\quad \left|6l-2\sum_{k=1}^{9}E_{i_k}-E_{i_{10}}\right|.$$

Lemma 2.1. *The line bundle H is very ample on S.*

Proof. The line bundle H is shown to be very ample in [13]. We report here a different short proof which relies on the following lemma.

Lemma 2.2. (Alexander, [15, Lemma 0.15]) *If H has a decomposition*

$$H=C+D,$$

where C and D are curves on S such that $\dim|C|\geq 1$, and if the restriction maps $H^0(\mathcal{O}_S(H))\to H^0(\mathcal{O}_D(H))$ and $H^0(\mathcal{O}_S(H))\to H^0(\mathcal{O}_C(H))$ are surjective, and $|H|$ restricts to a very ample linear system on D and on every C in $|C|$, then $|H|$ is very ample on S.

Consider the reducible hyperplane section

$$H=D_1+D_2=\left(3l-\sum_{i=1}^{8}E_i\right)+\left(6l-2\sum_{i=1}^{5}E_i-\sum_{j=6}^{8}E_j-2\sum_{k=9}^{11}E_k\right).$$

Then D_1 is embedded as a degree 6 elliptic curve and D_2 as a sextic curve of genus 2. Moreover, a general element of $|D_2|$ is irreducible by the choice of points in general position, and all the elements in the pencil $|D_1|$ are irreducible. It follows that H_{D_1} and H_{D_2} are very ample. The fact that both the maps $H^0(S,H)\to H^0(D_1,H_{D_1})$ and $H^0(S,H)\to H^0(D_2,H_{D_2})$ are surjective concludes the argument. □

Lemma 2.3. *There are no lines on S.*

Proof. Assume $L=al-\sum_{i=1}^{5}a_iE_i-\sum_{j=6}^{11}b_jE_j$ is a line on S. Then looking at the intersection of L with the twisted cubics E_i, $i=1,\dots,5$, and the intersection of L with the conics E_j, $j=6,\dots,11$, we derive the bounds $0\leq a_i\leq 2$ and $0\leq b_i\leq 2$. This implies that $1=H\cdot L=9a-\sum_{i=1}^{5}3a_i-\sum_{i=6}^{11}2b_i\geq 9a-30-24$, i.e. $a\leq 6$. The only numerical possibilities are

(1) $L=E_i-E_j$;
(2) $|5l-2\sum_{k=1}^{5}E_k-2E_{i_6}-\sum_{k=7}^{11}E_{i_k}|$, $\{i_6,\dots,i_{11}\}=\{6,\dots,11\}$;
(3) $|4l-2\sum_{k=1}^{3}E_{i_k}-\sum_{k=4}^{11}E_{i_k}|$, $\{i_1,\dots,i_{11}\}=\{1,\dots,11\}$, $i_1<i_2\leq 5<i_3$;
(4) $L=3l-2E_{i_1}-\sum_{k=2}^{9}E_{i_k}$, $\{i_1,\dots,i_5\}=\{1,\dots,5\}$, $6\leq i_6<\dots<i_9\leq 11$;
(5) $L=2l-\sum_{i=1}^{5}E_i-E_j$, $6\leq j\leq 11$;
(6) $L=l-E_i-E_j-E_k$, $1\leq i<j\leq 5<k\leq 11$.

But those are empty linear systems by the general position hypothesis. □

Proposition 2.4. *The surface S in $\mathbf{P}^6$ has no trisecant lines.*

Proof. Consider the reducible hyperplane section $H=F_{i,j}+F^{i,j}$, where

(1) $F_{i,j}=-K_S+E_i+E_j$ $i,j=6,\dots,11$, i.e. an elliptic quartic curve;

(2) $F^{i,j}=H-F_{i,j}=6l-2\sum_{k=1}^{5}E_k-\sum_{k=6}^{11}E_k-E_i-E_j$, i.e. a curve of degree 8 of genus 3.

Fix a curve $F_{i,j}$. Any trisecant line L would together with $F_{i,j}$ span a hyperplane, so there is some reducible hyperplane section $H=F_{i,j}+F^{i,j}$ for which L is a trisecant. Since neither $F^{i,j}$ nor $F_{i,j}$ have trisecants, L must in fact intersect both these curves, so $F_{i,j}$ and L span at most a $\mathbf{P}^4$. This means that we can always find a curve $C\in|F^{i,j}|$ passing through the points in $L\cap F_{i,j}$, and such that L is a trisecant to $C\cup F_{i,j}$. If C is irreducible this is impossible since C has no trisecant lines. Assume it is reducible and write $C=A+B$, where A and B are irreducible with $\deg A\leq\deg B$. Then the following cases could occur:

(a) A is a plane conic;

(b) A is a plane cubic or a twisted cubic;

(c) A is a quartic curve.

Let $A=\alpha l-\sum_{k=1}^{5}\alpha_k E_k-\sum_{k=6}^{11}\beta_k E_k$.

Assume $\alpha=0$. If $A=E_k$ for $k\in\{1,\dots,5,\ i,\ j\}$ then $B=F^{i,j}-E_k$, which is impossible by the general position hypothesis. If $A=E_k$ for $k\in\{6,\dots,11\}\setminus\{i,j\}$ then $B=F^{i,j}-E_k$, this possibility will be analyzed more closely below.

Assume now $\alpha>0$, i.e. $A\neq E_k$, then by intersection properties and the assumption that A and B are effective divisors, $0\leq\alpha_k\leq 2$, $0\leq\beta_k\leq 1$ and $1\leq\alpha\leq 6$.

(a) Going over the possibilities for α, α_k and β_k gives no result by the general position hypothesis.

(b) Examining the possible choices for α, α_k and β_k we get:

(1) $A=l-E_m-E_n$ residual to $B=5l-2\sum_{k=1}^{5}E_k+E_m+E_n-\sum_{k=6}^{11}E_k-E_i-E_j$, $1\leq m<n\leq 5$;

(2) $A=2l-\sum_{k=1}^{5}E_k$ residual to $B=4l-\sum_{k=1}^{11}E_k-E_i-E_j$;

(3) $A=3l-\sum_{k=1}^{11}E_k+E_m$ and $B=3l-\sum_{k=1}^{5}E_k-E_m-E_i-E_j$, $1\leq m\leq 5$;

(4) $A=4l-\sum_{k=1}^{11}E_k-E_m-E_n$, $1\leq m<n\leq 5$;

(5) $A=5l-2\sum_{k=1}^{5}E_k-\sum_{k=6}^{11}E_k$.

In the first two cases the residual curve B does not exist by the general position hypothesis. Likewise the curve A does not exist in the last cases.

(c) Similar computations lead to

(1) $A=l-E_m-E_n$, $1\leq m\leq 5$ and $6\leq n\leq 11$, whose residual curve does not exist;

(2) $A=2l-\sum_{k=1}^{5}E_k+E_m-E_n$, $1\leq m\leq 5<n\leq 11$, whose residual curve does not exist;

(3) $A=F_{k,l}$, $B=F_{m,n}$ with $\{i,j,k,l,m,n\}=\{6,\dots,11\}$.

We are left with the cases $C=E_k+(6l-2\sum_{m=1}^{5}E_m-\sum_{n=6}^{11}E_n-E_i-E_j-E_k)$ or $C=F_{k,l}+F_{m,n}$.

Notice that in both cases the projective spaces spanned by the two components, $\langle A\rangle,\langle B\rangle$, intersect in a line $L=\langle A\rangle\cap\langle B\rangle$. Moreover, neither A nor B admit trisecant lines and $A\cap B=2$. It follows that $A\cap\langle B\rangle=A\cap B=B\cap\langle A\rangle$. Any trisecant line L to C, must meet A (or B) in two points and thus it is contained in $\langle A\rangle$, which implies $L\cap B\subset A\cap B$. But this means that L is a trisecant line for A, which is impossible. □

These surfaces are defined by three quadrics and four cubics in $\mathbf{P}^6$.

2.3. Abelian surfaces

Recently Bauer–Szemberg [4] have proved that the general $(1,7)$-polarized abelian surface in $\mathbf{P}^6$ does not have any trisecant lines. The argument uses a generalization of Reider's criterion to higher order embeddings. These surfaces are not contained in any quadrics.

3. Complete list of surfaces with no lines

In the remaining sections we prove Theorem 0.1 and Proposition 0.2.

Throughout this section we will assume that S is a surface embedded in $\mathbf{P}^6$ by the line bundle $\mathcal{H}_S$, with no lines on it and no trisecant lines. By C we will denote the general smooth hyperplane section of S.

3.1. Numerical relations

By naive dimension count arguments one expects that a surface in $\mathbf{P}^6$ has a 1-dimensional family of trisecant lines. Le Barz makes this count rigorous, by defining a cycle in the Hilbert scheme of aligned length 3 subschemes of $\mathbf{P}^6$ (cf. [12]). There is a natural map from this Hilbert scheme to the Grassmannian of lines, and the corresponding cycle map defines a 1-dimensional cycle of trisecant lines. This 1-dimensional cycle is determined by the numerical invariants of the surface. These are

$$n=\deg S\quad k=K_S^2,\quad c=c_2(S),\quad \chi=\chi(\mathcal{O}_S)\quad\text{and}\quad e=K_S\cdot\mathcal{H}.$$

Noether's formula (cf. [10, Appendix A]),

$$c_2(S)+K_S^2=12\chi(\mathcal{O}_S),\quad\text{i.e. } c+k=12\chi,$$

gives a relation between these surface invariants. Moreover, the adjunction formula $2p(C)-2=C^2+K_S\cdot C=n+e$ gives a relation between these invariants and the arithmetic genus $p(C)$ of a hyperplane section C [10, Chapter V, Proposition 1.5]. The formula of Le Barz for the number of trisecant lines meeting a fixed $\mathbf{P}^4\subset\mathbf{P}^6$ is (cf. [12])

$$D_3=\tfrac{1}{6}(2n^3-42n^2+196n-k(3n-28)+c(3n-20)-e(18n-132)), \tag{5}$$

and the formula for the number of lines in $\mathbf{P}^6$ which are tangential trisecants, i.e. tangent lines that meet the surface in a scheme of length at least 3, is

$$T_3=6n^2-84n+k(n-28)-c(n-20)+e(4n-84). \tag{6}$$

Finally he computes a formula for the number of trisecant lines to a smooth surface in $\mathbf{P}^7$,

$$S_3=\tfrac{1}{6}(n^3-30n^2+224n+c(3n-40)-k(3n-56)-e(15n-192)), \tag{7}$$

which we shall need towards the end of the next section.

The first formula is enumerative unless the family of trisecants is at least 2-dimensional or the surface contains infinitely many lines, i.e. is a scroll. The same holds for the second and third formula with the additional assumption that lines L on the surface contribute with multiplicity $4\binom{3+L^2}{2}$ to the formula (6) and with multiplicity $-\binom{4+L^2}{3}$ to the formula (7) (cf. [12]). In particular (-1)-lines contribute with multiplicity 4 and -1, respectively, while lines with selfintersection $-3\le L^2\le -2$ do not contribute at all.

With our hypothesis in this section, this means that the formulas (5) and (6) above are zero.

We shall combine these formulas with several bounds for the sectional genus $p(C)$ in order to find the possible invariants of S.

The first one is Castelnuovo's bound for the arithmetic genus $p(C)$ of a reduced and irreducible curve C of degree n in $\mathbf{P}^N$. Let $[x]$ denote the greatest integer $\le x$, then (cf. [1])

$$p(C)\le p(N)=\left[\frac{n-2}{N-1}\right]\left(n-N-\left(\left[\frac{n-2}{N-1}\right]-1\right)\frac{N-1}{2}\right).$$

Setting $N=5$ and using the adjunction formula we get the following bound for e.

$$e\le\frac{n^2-10n+\varepsilon}{4},\quad\text{where } \varepsilon=\begin{cases}0, & \text{if } n\equiv 0,2 \pmod 4,\\ -3, & \text{if } n\equiv 1 \pmod 4,\\ 1, & \text{if } n\equiv 3 \pmod 4).\end{cases} \tag{8}$$

The next ones are refined versions of it given by a theorem of Harris on the genus of a curve that does not lie on a surface of minimal degree. We state the theorem for curves in $\mathbf{P}^5$.

Theorem 3.1. ([9], [6, Theorem 3.4]) *Let C be a reduced, irreducible curve in $\mathbf{P}^5$ of degree n and arithmetic genus p.*

(a) *If*

$$p > p_1 = \frac{n^2 - 5n + \varepsilon}{10}, \quad \text{where } \varepsilon = \begin{cases} 10, & \text{if } n \equiv 0 \pmod 5, \\ 4, & \text{if } n \equiv 1,4 \pmod 5, \\ 6, & \text{if } n \equiv 2,3 \pmod 5, \end{cases}$$

and $n \geq 11$, then C lies on a surface of degree 4 in $\mathbf{P}^5$.

(b) *If furthermore, $n \geq 13$ and $p = p_1$, then C lies on a surface of degree ≤ 5.*

In addition to these bounds on the genus of C, we shall apply the Hodge index theorem (cf. [10, Chapter V, Theorem 1.9]) to $\mathcal{H}_S$ and K_S:

$$K_S^2 \cdot \mathcal{H}_S^2 \leq (K_S \cdot \mathcal{H}_S)^2, \quad \text{i.e. } kn \leq e^2. \tag{9}$$

3.2. Surfaces of degree $n \leq 10$

We first apply the above formulas to surfaces of small degree. Notice that surfaces in $\mathbf{P}^3$ and in $\mathbf{P}^4$ have trisecant lines or contain lines. Furthermore, the only surfaces in $\mathbf{P}^5$ which do not contain lines or have trisecants are the Veronese surfaces and the general complete intersections $(2,2,2)$, i.e. the general nontrigonal $K3$-surfaces of degree 8 (cf. [3]). Therefore we may assume that S spans $\mathbf{P}^6$.

Eliminating and solving from (5) and (6) with respect to $\chi = \frac{1}{12}(c+k)$ we get

$$\chi = \frac{-36n^3 - 1344n - 576e + 108en + 404n^2 - 3en^2 + n^4}{48n}. \tag{10}$$

If $p(C)=0$, i.e. $e=-n-2$, then S is a scroll or a Veronese surface. Otherwise $e \geq -n$. Furthermore S spans at least $\mathbf{P}^6$, so there is the upper bound (8). Since, by adjunction, n and e have the same parity, we get the list given in Table 2 for $n \leq 10$.

Table 2.

n	χ	Bounds for e	Numerical possibilities
5	$\frac{1}{80}(-165-37e)$	$e=-7$	none
6	$-\frac{1}{8}e$	$e=-6$	none
7	$\frac{1}{112}(147+11e)$	$-7 \leq e \leq -5$	none
8	$\frac{1}{4}(8+e)$	$-8 \leq e \leq -4$	$e=-8,-4$
9	$\frac{1}{144}(315+40e)$	$-9 \leq e \leq -3$	none
10	$\frac{1}{40}(80+17e)$	$-10 \leq e \leq 0$	$e=0$

In addition we get the numerical possibilities $(n,e)=(4,-6),(8,0)$ corresponding to the two families of surfaces in $\mathbf{P}^5$ mentioned above. When $n=8$ and $e=-8$, then $\chi=0$ and the sectional genus $p(C)=1$, so S is an elliptic scroll, contrary to our assumption that S contains no lines. Thus we are left with the invariants given in Table 3.

Table 3.

	n	e	k	χ
(1)	4	−6	9	1
(2)	8	−4	2	1
(3)	8	0	0	2
(4)	10	0	0	2

Examples 1.1–1.4 in Section 1 have these invariants, and it is easy to see that these are the only ones. In the first two cases any smooth surface in the family would have no lines, but for $K3$-surfaces it is easy to construct degenerations to smooth surfaces with one or several lines. These lines would be (-2)-lines on the $K3$-surface. Therefore we have the following four cases:

(1) $(S,\mathcal{H})=(\mathbf{P}^2,\mathcal{O}_{\mathbf{P}^2}(2))$ is a Veronese surface in $\mathbf{P}^5$;
(2) $(S,\mathcal{H})=(Bl_7(\mathbf{P}^2),-2K_S)$ is a Del Pezzo surface in $\mathbf{P}^6$;
(3) S is a general nontrigonal $K3$ surface of degree 8 in $\mathbf{P}^5$;
(4) S is a general nontrigonal $K3$ surface of degree 10 in $\mathbf{P}^6$.

3.3. Surfaces of degree $n\geq 11$

When $n\geq 11$, Theorem 3.1 applies, so we may use the refined genus bound $p(C)\leq p_1$, unless C lies on a surface of degree 4 in $\mathbf{P}^5$. The surfaces of degree 4 in $\mathbf{P}^5$ are the rational normal scrolls and the Veronese surfaces. With the assumption that C has no trisecant we may easily treat these cases first.

If C is contained in a rational normal scroll S_4 of degree 4 in $\mathbf{P}^5$ and has no trisecant, then C is rational or hyperelliptic. If C is rational, then S is a scroll, so this is impossible. If C is hyperelliptic, then $C=2\mathcal{H}_{S_4}+(n-8)\mathcal{F}_{S_4}$ on S_4, where $\mathcal{H}_{S_4}$ denotes a hyperplane section and $\mathcal{F}_{S_4}$ a ruling of S_4. The canonical divisor on S_4 is $K_{S_4}=-2\mathcal{H}_{S_4}+2\mathcal{F}_{S_4}$, so adjunction on S_4 gives $2p(C)-2=2n-12$. By adjunction on S this means that $K_S\cdot\mathcal{H}_S=e=n-12$. Surfaces with hyperelliptic hyperplane sections are conic bundles when $n\geq 9$ (cf. [18]), therefore the adjoint linear system is composed with a pencil and $(K_S+\mathcal{H})^2=0$ which gives $K_S^2=24-3n$.

This implies the existence of $3n-16$ singular fibers, i.e. (-1)-lines in S, contrary to our assumption that there should be no lines on S.

Assume next that the general hyperplane section C is contained in a Veronese surface S_4. Then the surface S itself is contained in a cone V over S_4: In fact, C is linearly normal and so is therefore also S. It follows that every quadric hypersurface which contains C is the restriction of a quadric which contains S. For degree reasons, the quadrics which contain C are precisely the quadrics which define the Veronese surface S_4. The quadrics through S therefore define a threefold, which clearly must be a cone over S_4. Let $\overline{V}$ be the blow up of V in the vertex, then $\overline{V}$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^2$. In this case $S\in|2\mathcal{H}+b\mathcal{F}|$ on $\overline{V}$, where $\mathcal{F}$ is the pullback of a line from $\mathbf{P}^2$. Consider $(\mathcal{H}-2\mathcal{F})^2\cdot S=-2b$, where $\mathcal{H}-2\mathcal{F}$ is the contracted divisor. Since S is smooth, $b=0$ and S is a Del Pezzo surface of degree $n=8<11$ in $\mathbf{P}^6$ (cf. Example 1.2).

We may therefore assume that C is not contained in a surface of minimal degree in $\mathbf{P}^5$. First we set the formulas (5) and (6) equal to 0, eliminate c and get

$$nk=\tfrac{1}{8}[n^4-32n^3+332n^2-1120n-e(3n^2-80n+480)].$$

The Hodge index theorem (9) reduces to the inequality

$$(11)\qquad 8(e^2-nk)=8e^2+e(3n^2-80n+480)-(n^4-32n^3+332n^2-1120n)\geq 0.$$

Now S is not a scroll, so $p(C)\geq 1$, i.e. $e\geq -n$. Thus $e+3n^2-80n+480\geq 0$ when $n\geq 27$. On the other hand, the upper bound $p(C)\leq p_1$ of Theorem 3.1 implies $e<\frac{1}{5}n^2-2n$, so

$$\begin{aligned}8\big(\tfrac{1}{5}n^2-2n\big)^2+\big(\tfrac{1}{5}n^2-2n\big)(3n^2-80n+480)-(n^4-32n^3+332n^2-1120n)\\ =-\tfrac{2}{25}n^4+\tfrac{18}{5}n^3-44n^2+160n\geq 0.\end{aligned}$$

But this is never true when $n\geq 28$, so we are left to treat $n\leq 27$. Now, χ can be easily computed from (10).

In the second column of Table 4 we give the values for $n\leq 27$. In the third column we put the bounds for e, coming from the bounds $1\leq p(C)\leq p_1$, and, where necessary, the inequality (11) from the Hodge index theorem. In the last column the possible numerical solutions from the first two columns are indicated.

If $\chi\geq 2$, then S contains some effective canonical curve, therefore $e>0$. This excludes the cases $(n,e)=(26,0),(25,-25),(22,0)$. For $n=12$, the assumption that S is not a conic bundle i.e. $n+2e+k>0$ reduces to $e>-4$. Similarly for $n=16$ we get $e\geq -12$. For $e=-12$, the sectional genus $p=3$, while $\chi=-6$, i.e. S is birationally

Table 4.

n	χ	Bounds for e	Numerical possibilities
11	$\frac{25}{16}+\frac{83}{176}e$	$-11\leq e\leq 1$	none
12	$1+\frac{1}{2}e$	$-12\leq e\leq 4$, $e^2-6e+24\geq 0$	$e=-12,-10,\dots,4$
13	$\frac{7}{16}+\frac{107}{208}e$	$-13\leq e\leq 7$	none
14	$\frac{29}{56}e$	$-14\leq e\leq 10$	$e=0$
15	$-\frac{3}{16}+\frac{41}{80}e$	$-15\leq e\leq 15$	none
16	$\frac{1}{2}e$	$-16\leq e\leq 18$, $e^2-4e-192\geq 0$	$e=-16,-14,-12,16,18$
17	$\frac{11}{16}+\frac{131}{272}e$	$-17\leq e\leq 23$	none
18	$2+\frac{11}{24}e$	$-18\leq e\leq 28$, $2e^2+3e-1440\geq 0$	none
19	$\frac{65}{8}+\frac{131}{304}e$	$-19\leq e\leq 33$	none
20	$7+\frac{2}{5}e$	$-20\leq e\leq 40$, $e^2+10e-1800\geq 0$	$e=40$
21	$\frac{175}{16}+\frac{41}{112}e$	$-21\leq e\leq 45$	none
22	$16+\frac{29}{88}e$	$-22\leq e\leq 52$	$e=0$
23	$\frac{357}{16}+\frac{107}{368}e$	$-23\leq e\leq 59$	none
24	$30+\frac{1}{4}e$	$-24\leq e\leq 66$, $e^2+36e-6720\geq 0$	none
25	$\frac{627}{16}+\frac{83}{400}e$	$-25\leq e\leq 75$	$e=-25$
26	$50+\frac{17}{104}e$	$-26\leq e\leq 82$	$e=0$
27	$\frac{1001}{16}+\frac{17}{144}e$	$-27\leq e\leq 91$, $8e^2+507e-113373\geq 0$	none

ruled over a curve of genus 7. But this is absurd. In case $(n,e)=(16,18)$, the sectional genus $p(C)=18$. Thus Theorem 3.1(b) applies. If C does not lie on a surface of degree 4, it must therefore lie on a Del Pezzo surface of degree 5 or a cone over an elliptic curve of degree 5. In either case it is easy to check that C has trisecant lines, contrary to our assumption. Therefore we are left with the possibilities given in Table 5. Let us examine the various cases.

Case (a). Since $\chi=0$, the surface is an elliptic ruled surface, an abelian surface or a finite quotient of an abelian surface. In the latter two cases there are effective

Table 5.

Case	n	e	k	χ
(a)	12	-2	-3	0
(b)	12	0	-2	1
(c)	12	2	-1	2
(d)	12	4	0	3
(e)	14	0	0	0
(f)	16	16	16	8
(g)	20	40	70	23

pluricanonical curves, but $e=\mathcal{H}_S\cdot K_S=-2$ so this is impossible. Furthermore, $k=K_S^2=-3$, so the surface S is the blow up of an elliptic ruled surface in three points.

Since S has no lines the exceptional curves have degree at least 2. Now we have $(K_S+\mathcal{H}_S)^2=h^0(K_S+\mathcal{H}_S)=5$, so by Reider's criterion [16], S is embedded in $\mathbf{P}^4$ via $|K_S+\mathcal{H}_S|$. But there are no nonminimal elliptic ruled surfaces of degree 5 in $\mathbf{P}^4$, so this case does not occur.

Case (b). Any nonrational surface with $\chi=1$ and $K_S^2=k<0$ has effective pluricanonical curves. Since $e=K_S\cdot\mathcal{H}_S=0$, the surface S must therefore be rational. Since the surface has no (-1)-lines by assumption, it follows from Reider's criterion [16] that the adjoint linear system $|K_S+\mathcal{H}_S|$ embeds S as a surface of degree 10 and genus 5 in $\mathbf{P}^6$. Furthermore, the linear system $|2K_S+\mathcal{H}_S|$ blows down (-1)-conics and embeds the blown down surface as a Del Pezzo surface of degree 4 in $\mathbf{P}^4$. Therefore $K_S+\mathcal{H}_S=6l-\sum_{i=1}^5 2E_i-\sum_{i=6}^{11} E_i$ and $\mathcal{H}_S=9l-\sum_{i=1}^5 3E_i-\sum_{i=6}^{11} 2E_i$, where l is the pullback to S of a line in $\mathbf{P}^2$, while the E_i are exceptional curves. This is Example 2.2.

Case (c). In this case $\chi=2$ and any canonical curve has degree $e=K_S\cdot\mathcal{H}_S=2$. Since we assume that S has no lines, this must be a conic section, in fact a (-1)-conic section on S. Furthermore, $k=K_S^2=-1$, so S must be a $K3$-surface blown up in one point. Let $\pi\colon S\to\bar{S}$ denote the blow up map, then $\mathcal{H}_S=\pi^*(\mathcal{H}_{\bar{S}})-2E$ where $\mathcal{H}_{\bar{S}}^2=16$. This is Example 2.1.

Case (d). Since $\chi=3$ and $k=K_S^2=0$, this is an elliptic surface of degree 12 and every canonical curve has degree $e=K_S\cdot\mathcal{H}_S=4$. Now $K_S=mF+\sum_{i=1}^k(m_i-1)F$, where F is the general fiber of the elliptic fibration $S\to B$ and m_iF, $i=1,\dots,k$, the multiple fibers. Furthermore, $m=\chi(\mathcal{O}_S)+2g(B)-2\geq 1$ so for degree reasons alone $m=1$, $k=0$, $g(B)=0$ and $K_S=F$. Thus $|K_S|$ defines an elliptic fibration over $\mathbf{P}^1$ with no multiple fibers. The canonical curves are elliptic quartic curves, they each span a $\mathbf{P}^3$. These $\mathbf{P}^3$'s generate a rational scroll. This is a cubic scroll, otherwise the surface is not linearly normal. But $|H_S - K|$ is a linear system of curves of genus 5 and degree 8 which spans at most a $\mathbf{P}^4$ while the residual curves $|K|$ move in a pencil, so this is not possible. The surface has degree 12 and the canonical pencil has no basepoints so S must be complete intersection of the scroll and two quadric hypersurfaces. This is Example 1.5.

Case (e). Since $\chi=0$ the surface S must be minimal abelian or bielliptic. Following Serrano' s analysis, cf. [17], of ample divisors on bielliptic surfaces, one sees that any minimal bielliptic surface of degree 14 in $\mathbf{P}^6$ has an elliptic pencil of plane cubic curves, i.e. it has a 3-dimensional family of trisecants. The general abelian surface, however, has no trisecant (cf. [4]). This is Example 2.3.

Case (f). The surface must be the complete intersection of 4 quadric hypersurfaces in $\mathbf{P}^6$, Example 1.6.

Case (g). In this case $p(C)=p_1=31$ and thus by Theorem 3.1(b), the general hyperplane section C is contained in a surface of degree 4 or 5. In the former case C must lie on a rational scroll, and it has no trisecants only if it is a bisection. Thus S must be a conic bundle, contradicting the fact that $(\mathcal{H}_S+K_S)^2=170\neq 0$. In the latter case C lies on a surface S_5 of degree 5 in $\mathbf{P}^5$, i.e. an anticanonically embedded Del Pezzo surface or the cone over an elliptic quintic curve in $\mathbf{P}^4$. In either case the sectional genus of C implies that C is the intersection of S_5 with a quartic hypersurface, and each line on S_5 will be a 4-secant line to C. This excludes Case (g).

The results of Sections 3.2 and 3.3 can be summarized as follows.

Theorem 3.2. *Let S be a smooth surface embedded in $\mathbf{P}^6$ with no lines. Then S has no trisecants if and only if it belongs to one of the cases listed in Table* 6.

4. List of surfaces with lines

For surfaces with lines the trisecant formulas do not a priori have any enumerative significance, as explained in Section 3.1. The extreme case is the scrolls; surfaces with infinitely many lines on them. But also for surfaces with finitely many lines on them the use of the formulas requires a careful argument. The computation of Le Barz shows that only (-1)-lines and lines with selfintersection ≤ -4 contribute to the trisecant formulas ([12]).

Table 6.

Surface	Degree	Linear system	Example
$\mathbf{P}^2$	4	$\mathcal{O}_{\mathbf{P}^2}(2)$	1.1
$\mathrm{Bl}_7(\mathbf{P}^2)$	8	$6l-\sum_{i=1}^{7}2E_i$	1.2
$K3$	8		1.3
$K3$	10		1.4
$\mathrm{Bl}_{11}(\mathbf{P}^2)$	12	$9l-\sum_{i=1}^{5}3E_j-\sum_{j=6}^{11}2E_j$	2.2
$\mathrm{Bl}_1(K3)$	12	$p^*(\overline{\mathcal{H}})-2E$	2.1
Regular elliptic, $p_g=2$	12		1.5
Abelian	14	$(1,7)$-polarization	2.3
General type	16		1.6

We divide our analysis according to the occurrence of lines. We deal with the following cases:

(1) S is a scroll;

(2) S contains an isolated (-1)-line, a line that does not intersect other lines on S of negative selfintersection;

(3) S contains a (-1)-line which intersects some other (-1)-line;

(4) S contains a (-1)-line that can be contracted so that it is the exceptional line of an inner projection from $\mathbf{P}^7$.

Notice that the first three cases are mutually exclusive when S spans $\mathbf{P}^6$, while the last one may overlap the two previous cases. Furthermore there may be examples which do not fall into any of these four cases. These would be surfaces with lines of selfintersection ≤ -4 or a finite number of (-1)-lines on them, each intersected by some other line of selfintersection at most -2, such that the surface is no inner projection of a surface in $\mathbf{P}^7$ (cf. Theorem 0.1). The above cases suggest different strategies of dealing with surfaces with lines. Together they will recover all our examples. Each approach aims at finding some new relation replacing the relation from (6) which we lose when the surface S is a scroll or contains a (-1)-line.

The first approach is projection to $\mathbf{P}^4$ from a line on S.

Lemma 4.1. *Let S be a smooth surface in $\mathbf{P}^6$ with no proper trisecant lines. Assume that $L\subset S$ is a line on the surface, and let $\pi_L\colon S\to\mathbf{P}^4$ be the projection of S from L, i.e. the morphism defined by $|\mathcal{H}_S-L|$. Then π_L is the composition of the contraction of any line on S which meets L and an embedding. In particular, if S has finitely many lines, $\pi_L(S)$ is smooth unless there is some line L_1 on S meeting L with $L_1^2\leq -2$.*

Proof. Let P be a plane in $\mathbf{P}^6$ which contains L. Let Z_P be residual to L in $S\cap P$. If Z_P is finite, its degree is at most 1, otherwise S would have a trisecant in P. If Z_P contains a curve, this curve would have to be a line L_1 which would coincide with Z_P, since again S has no trisecants. Since S has finitely many lines, $L_1^2 \le -1$. This line is clearly contracted by π_L to a singular point if and only if $L_1^2 \le -2$. □

Let us use this lemma to examine the case of surfaces with at least a one dimensional family of lines.

Proposition 4.2. *Let S be a scroll in $\mathbf{P}^6$. Then S has no proper trisecant lines if and only if S is*

(1) *a rational normal scroll, or*

(2) *an elliptic normal scroll, with minimal selfintersection of a section $E_0^2=1$.*

Proof. If S is a rational normal scroll or elliptic normal scroll with $e=1$, then it is defined by quadrics and therefore has no trisecant lines (cf. Example 1.15).

Assume now that S is a scroll with no trisecant lines and let L be a line on it. Consider the projection of S from L:

$$\pi_L: S \longrightarrow \mathbf{P}^4.$$

If there is no line on S meeting L then π_L is an embedding by Lemma 4.1. Since the only smooth scrolls in $\mathbf{P}^4$ are the rational cubic scrolls and the elliptic quintic scrolls, S must be as in the statement.

If π_L is not a finite map, i.e. there is a line section L_0, then S must be rational and normal. Assume, in fact, that S is not normal, i.e. $S=\mathbf{P}(\mathcal{O}(1)\oplus\mathcal{O}(b))$ with $b\ge 5$, and consider the projection

$$\pi_{L_0}: S \longrightarrow \mathbf{P}^4.$$

The image curve $\pi_{L_0}(S)$ is a rational nonnormal curve in $\mathbf{P}^4$ and therefore it has a trisecant line, L_t. Then the linear span $\mathbf{P}^3=\langle L_t, L_0\rangle$ contains three rulings of S, and therefore also a pencil of trisecant lines to S. □

If S is not a scroll, it has only a finite number of lines. We first look at the case of an isolated line.

4.1. Surfaces with at least one isolated (-1)-line

Assume that S contains a finite number of (-1)-lines, and that at least one of them, say L, is isolated, i.e. it does not intersect any other line on S. Then the projection

$$\pi_L: S \longrightarrow \mathbf{P}^4$$

is an embedding by Lemma 4.1. Let $\pi_L^*(\mathcal{H})+L=\mathcal{H}_S$, and notice that $K_{\pi_L(S)}=K_S$. Using the same invariants n, e, k, c for S we have that

$$\deg \pi_L(S)=n-3, \quad K_{\pi_L(S)}\cdot\mathcal{H}=e+1, \quad K^2_{\pi_L(S)}=k \quad \text{and} \quad c_2(\pi_L(S))=c.$$

Now the double point formula for surfaces in $\mathbf{P}^4$ ([10, Appendix A]) applies to $\pi_L(S)$, giving a new numerical relation

$$(n-3)(n-13)-5e-k+c-5=0. \tag{12}$$

Eliminating c from (5) and (12) and solving with respect to k and similarly for $\chi=c+k$ we get

$$k=\tfrac{1}{8}(n^3-26n^2+226n-680+3ne-32e)$$

and

$$\chi=\tfrac{1}{48}(n^3-30n^2+290n-816+36ne-12e).$$

The Hodge index theorem reduces to

$$nk-e^2=\tfrac{1}{8}(n^4-26n^3+226n^2-680n+e(3n^2-32n-8e))\le 0.$$

Now $e(3n^2-32n-8e)$ clearly attains its minimal value at an endpoint of the range of permissible values of e, i.e. at $e=-n$ or at the Castelnuovo bound $e=\frac{1}{4}(n^2-5n)$. In the former case we get the inequality

$$n^4-29n^3+250n^2-680n\le 0.$$

In the latter case we get

$$5n^4-131n^3+1014n^2-2720n\le 0.$$

But the latter is never true while the former is true only for $n\le 11$.

The only possibilities are easily collected in Table 7.

Table 7.

n	χ	Bounds for e	Numerical possibilities
7	$\frac{1}{16}(29+3e)$	$-7\le e\le -3$	none
8	$\frac{1}{4}(8+e)$	$-8\le e\le -2$	$e=-4,-8$
9	$\frac{1}{16}(31+5e)$	$-9\le e\le -1$	$e=-3$
10	$\frac{1}{8}(14+3e)$	$-10\le e\le 0$	$e=-2$
11	$\frac{1}{16}(25+7e)$	$-11\le e\le 3$	$e=1$

Table 8.

Case	n	e	k	χ
(a)	8	−8	5	0
(b)	8	−4	1	1
(c)	9	−3	−1	1
(d)	10	−2	−2	1
(e)	11	1	−1	2

Table 7 leaves us with the possible invariants given in Table 8.

Case (a). We have $\chi=0$, so $0\geq K_S^2=k=5$, which is absurd.

Cases (b)–(d). Since $\mathcal{H}_S\cdot K_S=e\leq-2$ the surface S cannot have effective pluricanonical divisors. Since $\chi=1$, the surface S is in each case rational. The adjunction morphism defined by $|\mathcal{H}_S+K_S|$ is birational and maps S onto $\mathbf{P}^2$, a quadric in $\mathbf{P}^3$ and a Del Pezzo surface of degree 4 in $\mathbf{P}^4$, respectively, in the three cases, (cf. [18]). Thus we recover the surfaces constructed in Examples 1.12–1.14.

Case (e). Since $\mathcal{H}_S\cdot K_S=e=1$ any canonical curve must be a line. Since $\chi=2$ the surface S must be a $K3$-surface blown up in one point. Thus we recover the inner projection of the general $K3$-surface of degree 12 in $\mathbf{P}^7$ described in Example 1.16. The line L is the only exceptional line on the surface.

We have then proved the following result.

Proposition 4.3. *Let S be a surface in $\mathbf{P}^6$ with no trisecant lines and with at least one isolated (-1)-line. Then S belongs to the following list:*

(1) *a rational surface of degree* 8 *and genus* 3, *as in Example* 1.12;
(2) *a rational surface of degree* 9 *and genus* 4, *as in Example* 1.13;
(3) *a rational surface of degree* 10 *and genus* 5, *as in Example* 1.14;
(4) *a nonminimal $K3$-surface of degree* 11 *and genus* 7, *as in Example* 1.16.

4.2. Conic bundles

Next we assume that S has at least two (-1)-lines L_1 and L_2 which meet. Then $(L_1+L_2)^2=0$ and L_1+L_2 or some multiple of it moves in an algebraic pencil of conic sections. Thus S is a conic bundle.

If there is a line L of selfintersection $L^2\leq-2$ intersecting L_1, then this line intersects all the members of the pencil and therefore is mapped onto the base curve. This would imply that S is a rational conic bundle in $\mathbf{P}^6$. Now, let F be a general fiber of the conic bundle and let $D=L+L_1+F$. Then D has arithmetic genus 0 and selfintersection $D^2=1$. By Riemann–Roch $|D|$ maps S birationally to $\mathbf{P}^2$. Since D has degree 4, the rational map from $\mathbf{P}^2$ to S is defined by curves of degree 4. Therefore the sectional genus $p(C)=p$ is at most 3. Since S is not a scroll, the sectional genus is at least 1, so we get $e=-n+2p-2$ with $1\leq p\leq 3$. For a conic bundle $(K_S+\mathcal{H}_S)^2=n+2e+k=0$, so we get correspondingly: $k=n-4p+4$ and $c=8-n+4p$. Substituting those values in the trisecant formula of Le Barz, (5), we get

$$n^3-15n^2+80n-156-6pn+36p=0.$$

For $p=1$ this is $(n-6)(n-5)(n-4)=0$, for $p=2$ it becomes $(n-6)(n-7)(n-1)=0$, while $p=3$ implies $(n-6)(n-8)(n-1)=0$.

The solutions except $(n,p)=(1,2),(1,3),(6,3)$ correspond precisely to the Del Pezzo surfaces of Example 1.8 and the conic bundles constructed in Examples 1.9–1.11. The first two of the remaining three cases, when $n=1$, are clearly impossible, while a surface with $(n,p)=(6,3)$ can span only a $\mathbf{P}^4$ and would therefore have trisecants.

Let us assume that there is a line L, in addition to L_2, of selfintersection $L^2=-1$, which intersects L_1. Then L meets all members of the pencil of conic sections, so as above S is rational. Furthermore $D=L+L_1+L_2$ has arithmetic genus 0 or 1 and degree 3, so like above S must be a Del Pezzo surface.

Finally, we assume that L_2 is the only line of selfintersection $L^2\leq -1$ intersecting L_1. Then the projection

$$\pi_{L_1}: S \longrightarrow \mathbf{P}^4$$

will be a composition of the contraction of L_2 and an embedding by Lemma 4.1. The conic bundle structure is of course preserved. Therefore the image surface in $\mathbf{P}^4$ is rational of degree 4 or 5, or it is an elliptic conic bundle of degree 8 cf. [5], [8]. It is clear that the two first cases come from the surfaces of type Examples 1.10 and 1.11. The elliptic conic bundle has a plane quartic curve on it in $\mathbf{P}^4$ which is a bisection on the surface, cf. [5]. Its preimage on S would have degree 4, 5 or 6 depending on the intersection with L_1. But this is a curve of genus 3 so the degree upstairs must be 4 or 6 if S is smooth. A curve of degree 4 or 6 and genus 3 has trisecants, so this excludes this case. We have therefore shown the following result.

Proposition 4.4. *Let S be a conic bundle in $\mathbf{P}^6$ with no trisecant lines, then S is rational and belongs to the following list:*

(1) *a Del Pezzo surface of degree* 4, 5 *or* 6 *and genus* 1, *as in Example* 1.8;
(2) *a surface of degree* 6 *and sectional genus* 2, *as in Example* 1.9;
(3) *a surface of degree* 7 *and sectional genus* 2, *as in Example* 1.10;
(4) *a surface of degree* 8 *and sectional genus* 3, *as in Example* 1.11.

4.3. Inner projections from $\mathbf{P}^7$

Our last approach to surfaces with lines on them is the case of inner projections from $\mathbf{P}^7$, i.e. surfaces S obtained by projecting a smooth surface $\bar{S}$ in $\mathbf{P}^7$ from a point x on the surface. To simplify slightly the argument we may assume that S is not a conic bundle, since these are all treated above. Thus we assume that the surface has r pairwise disjoint (-1)-lines. Furthermore, we assume that it has no lines of selfintersection ≤ -4. In other words, S has r exceptional curves of the first kind, which can therefore be contracted. We assume that we can contract at least one of them, E, down to $x \in \bar{S}$ such that x is not a base point for $\bar{\mathcal{H}}$, where $\mathcal{H} = p^*(\bar{\mathcal{H}}) - E$ and p is the projection map $p: S \to \bar{S}$. Thus $\bar{\mathcal{H}}$ embeds $\bar{S}$ in $\mathbf{P}^7$ with no trisecant lines and with $(r-1)$ disjoint (-1)-lines.

Notice that the existence of r disjoint (-1)-lines and no lines of selfintersection ≤ -4 on S gives a contribution of $4r$ in the number of tangential trisecants (cf. [12]), which in terms of the formula as stated in (6) means that

$$T_3 = 4r.$$

Moreover, the existence of $(r-1)$ disjoint (-1)-lines and no lines of selfintersection ≤ -5 on $\bar{S}$ gives a contribution of $-(r-1)$ in the formula for trisecant lines of surfaces in $\mathbf{P}^7$ (cf. [12]). Using the same invariants n, e, k, c for S, as in Section 3, we have

$$\deg \bar{S} = n+1, \quad K_{\bar{S}} \cdot \bar{\mathcal{H}} = e-1, \quad K_{\bar{S}}^2 = k+1 \quad \text{and} \quad c_2(\bar{S}) = c-1.$$

Plugging those invariants in the formula of Le Barz for the number of trisecants to a smooth surface in $\mathbf{P}^7$, we get (cf. (7))

$$6S_3 = n^3 - 27n^2 + 176n + 108 + c(3n-37) - k(3n-53) - e(15n-177) = -6r + 6.$$

In addition we have from (5) and (6) that

$$\begin{aligned} 6D_3 &= 2n^3 - 42n^2 + 196n - k(3n-28) + c(3n-20) - e(18n-132) = 0, \\ T_3 &= 6n^2 - 84n + k(n-28) - c(n-20) + e(4n-84) = 4r. \end{aligned}$$

Thus

$$\begin{aligned} -2(6r-6) = 2S_3 & \\ &= 6D_3 - 12n^2 + 156n + 108 - k(3n-78) + c(3n-54) - e(12n-222) \\ &= 6D_3 - 3T_3 + 6n^2 - 96n + 216 - 6k + 6c - 30e \\ &= -12r + 6n^2 - 96n + 216 - 6k + 6c - 30e \end{aligned}$$

and hence

$$n^2-16n+34-k+c-5e=0. \tag{13}$$

Now, we assume that the number r of (-1)-lines on S is positive, thus

$$T_3=6n^2-84n+k(n-28)-c(n-20)+e(4n-84)=4r>0.$$

The two relations (5) and (13) yield the following expressions for k and c in terms of e and n:

$$k=\tfrac{1}{8}(n^3-26n^2+226n-680+3ne-32e),$$
$$c=\tfrac{1}{8}(n^3-34n^2+354n-238+3ne+8e).$$

Substituted into the inequality $T_3>0$ they yield

$$-4n^2+411n-4ne+412e>0,$$

which simplifies to

$$n(11-n)>e(n-12).$$

When $n\geq 13$ this means that $e<-n$, and therefore that S has sectional genus 0. But this means that S is a scroll, which is absurd. By assumption, S spans $\mathbf{P}^6$ and is not a scroll, so $n\geq 6$. For $n=6$ the surface is either a scroll or a Del Pezzo surface, so we may assume that $n\geq 7$, and $e\geq -n$.

For $7\leq n\leq 12$ we get the list given in Table 9.

Table 9.

n	χ	Bounds for e	Numerical possibilities
7	$\frac{1}{16}(25+3e)$	$-7\leq e\leq -3$	$e=-3$
8	$\frac{1}{4}(8+e)$	$-8\leq e\leq -2$	$e=-4,-8$
9	$\frac{1}{16}(31+5e)$	$-9\leq e\leq -1$	$e=-3$
10	$\frac{1}{8}(14+3e)$	$-8\leq e\leq 0$	$e=-2$
11	$\frac{1}{16}(25+7e)$	$-9\leq e\leq 1$	$e=1$
12	$\frac{1}{2}(3+e)$	$-12\leq e\leq 4$	none

Table 10.

n	e	k	c	r
8	−4	1	11	8
9	−3	−1	13	9
10	−2	−2	14	6
11	1	−1	25	1

The case $(n,e)=(7,-3)$ corresponds to a surface which spans only $\mathbf{P}^5$, it has trisecants since this is the case for every general hyperplane section. In the case $(n,e)=(8,-8)$, the invariant $\chi=0$, so the surface is an elliptic scroll, which is impossible by Proposition 4.2. Thus we are left with the numerical possibilities given in Table 10.

One can immediately see that we have recovered Examples 1.12–1.14 and 1.16, respectively.

The results of this section can be summarized as follows.

Proposition 4.5. *Let S be a surface embedded in $\mathbf{P}^6$ with no trisecant lines and having r (-1)-lines and no lines of selfintersection ≤ -4 on it. Assume that S is the inner projection of a smooth surface $\bar{S}\subset\mathbf{P}^7$, then S is as in Proposition* 4.3.

5. Conclusion

5.1. Proof of Theorem 0.1 and Proposition 0.2

Clearly, Theorem 3.2 together with Propositions 4.3–4.5 prove Proposition 0.2.

Furthermore, Theorem 0.1 follows as soon as we establish that a surface defined by quadrics belongs to the list of the theorem. For this, consider the complete intersection T of four general quadrics in the ideal of S. Then T has degree 16 and $T=S\cup S'$, for some smooth surface S', unless $S=T$ and is a complete intersection itself (Example 1.6). By construction, the quadrics containing S' cut out a surface. Furthermore, if S' is a plane or a scroll, then its lines or rulings will be trisecants to S. It follows easily that the only possibilities for S' with $\deg S'\leq 7$ are Veronese surfaces, the Del Pezzo surfaces of degree 4, 5 and 6, and the conic bundles of degree 6 and 7 (Examples 1.9 and 1.10). The corresponding linked surfaces S are the Examples 2.1, 1.5, 1.16, 1.14, 1.4 and 1.13, respectively. The remaining cases $\deg S\leq 8$ follow from a straightforward analysis combining the numerical relations (5), (8) and (9). □

We have made computations as in Section 3 fixing the number r of (-1)-lines on the surface. Checking up to $r=100$ gives no new possibilities compared to the list we have produced.

This numerical observation and the fact that the examples constructed in Section 1 cover all the cases listed in Section 4 lead us to make the following conjecture.

Conjecture 5.2. *Let S be a surface in $\mathbf{P}^6$ with no trisecant line, then the surface belongs to the list of examples in Sections* 1 *and* 2.

References

1. ARBARELLO, E., CORNALBA, M., GRIFFITHS, P. and HARRIS, J., *Geometry of Algebraic Curves, Volume I*, Springer-Verlag, Berlin–Heidelberg–New York, 1985.
2. AURE, A. B., The smooth surfaces in $\mathbf{P}^4$ without apparent triple points, *Duke Math. J.* **57** (1988), 423–430.
3. BAUER, I., Inner projections of algebraic surfaces: a finiteness result, *J. Reine Angew. Math.* **460** (1995), 1–13.
4. BAUER, T. and SZEMBERG, T., Primitive higher order embeddings of Abelian surfaces, *Trans. Amer. Math. Soc.* **349** (1997), 1675–1683.
5. BRAUN, R. and RANESTAD, K., Conic bundles in projective fourspace, in *Algebraic Geometry* (Newstead, P. E., ed.), pp. 331–339, Marcel Dekker, New York, 1998.
6. CILIBERTO, C., Hilbert function on finite sets of points and the genus of a curve in a projective space, in *Space Curves* (Ghione, F., Peskine, C. and Sernesi, E., eds.), Lecture Notes in Math. **1266**, pp. 24–73, Springer-Verlag, Berlin–Heidelberg, 1987.
7. DI ROCCO, S., k-very ample line bundles on Del Pezzo surfaces, *Math. Nachr.* **179** (1996), 47–56.
8. ELLIA, P. and SACCHIERO, G., Smooth surfaces of $\mathbf{P}^4$ ruled in conics, in *Algebraic Geometry* (Newstead, P. E., ed.), pp. 49–62, Marcel Dekker, New York, 1998.
9. HARRIS, J., *Curves in Projective Space*, Presses de l'Université de Montréal, Montreal, Que., 1982.
10. HARTSHORNE, R., *Algebraic Geometry*, Springer-Verlag, New York, 1977.
11. HULEK, K., KATZ, S. and SCHREYER, F. -O., Cremona transformations and syzygies, *Math. Z.* **209** (1992), 419–443.
12. LE BARZ, P., Formules pour les trisecantes des surfaces algebriques, *Enseign. Math.* **33** (1987), 1–66.
13. LIVORNI, E. L., On the existence of some surfaces, in *Algebraic Geometry* (Sommese, A. J., Biancofiore, A. and Livorni, E. L., eds.), Lecture Notes in Math. **1417**, pp. 155–179, Springer-Verlag, Berlin–Heidelberg, 1990.
14. MUKAI, S., Curves, $K3$ surfaces and Fano 3-folds of genus $g\leq 10$, in *Algebraic Geometry and Commutative Algebra. Vol. I* (Hijikata, H., Hironaka, H., Maruyama, M., Matsumura, H., Miyanishi, M., Oda, T. and Ueno, K., eds.), Kinokuniya, Tokyo, 1988.

15. POPESCU, S. and RANESTAD, K., Surfaces of degree 10 in the projective fourspace via linear systems and linkage, *J. Algebraic Geom.* **5** (1996), 13–76.
16. REIDER, I., Vector bundles of rank 2 and linear systems on algebraic surfaces, *Ann. of Math.* **127** (1988), 309–316.
17. SERRANO, F., Divisors of bielleptic surfaces and embeddings in $\mathbf{P}^4$, *Math. Z.* **203** (1990), 527–533.
18. SOMMESE, A. J. and VAN DE VEN, A., On the adjunction mapping, *Math. Ann.* **278** (1987), 593–603.

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