Weighted L^p estimates for oscillating kernels

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0. Introduction

Let $0<a\neq 1$, $b\leqq 1$, $y\in\mathbf{R}$, and consider the kernels

$$K_{a,b+iy}(t) = \exp(i|t|^a)(1+|t|)^{-(b+iy)}, \quad t\in\mathbf{R}.$$

The convolution operators $K_{a,b+iy}*f$ and closely related weakly singular operators and multiplier operators have been studied by many authors. It is well known that these operators satisfy the following norm inequalities [5, 8, 10, 12, 13, 14]:

$$(0.1) \qquad \|K_{a,1}*f\|_p \leqq C\|f\|_p, \quad 1<p<\infty$$

and

$$(0.2) \qquad \|K_{a,b}*f\|_p \leqq C\|f\|_p, \quad b<1, \quad \frac{a}{2}+b\geqq 1, \quad \frac{a}{a+b-1}\leqq p\leqq \frac{a}{1-b}.$$

In addition, $K_{a,1}$ maps H^1 into L^1 and, by duality, L^∞ into BMO.

The purpose of this paper is to consider norm inequalities of the form

$$\|K_{a,b+iy}*f\|_{p,w} \leqq C\|f\|_{p,w},$$

where $\|f\|_{p,w}=(\int_{\mathbf{R}}|f(x)|^p w(x)\,dx)^{1/p}$. Our approach is to consider these operators as convolutions, although they can be treated as multipliers and many of our results originated from this latter point of view.

Our first result is

Theorem 1. *Let* $0<a\neq 1$, $b\leqq 1$, *and* $\frac{a}{2}+b\geqq 1$. *Let* $1<p<\infty$ *when* $b=1$ *and* $\frac{a}{a+b-1}\leqq p\leqq\frac{a}{1-b}$ *when* $b<1$. *Let* $w\in A_p$ *and define* $\alpha=a\left|\frac{1}{p}-\frac{1}{2}\right|+1-\frac{a}{2}$ *and* $\delta=\frac{b-\alpha}{1-\alpha}$. *Then,*

$$\|K_{a,b+iy}*f\|_{p,w^\delta} \leqq C(1+|y|)\|f\|_{p,w^\delta}.$$

This result is a consequence of the fact that the sharp function of $K_{a,1+iy}*f$ is controlled pointwise by the Hardy—Littlewood maximal function of f. This implies the L^p_w norm of $K_{a,1+iy}*f$ is bounded by the L^p_w norm of f^* for any A_∞ weight, w, and by the L^p_w norm of f for any A_p weight. The use of the sharp function is motivated by Theorem 1 of [6] where it is used to show that singular integral operators map L^∞ into BMO. See also [1, 4, 9]. The result extends to $b<1$ by interpolation of analytic families of operators.

Next, we consider weights of the form $w(x)=(1+|x|)^\alpha$. Such weights are particularly well suited for these kernels. These operators are shown to satisfy

Theorem 2. *Let* $a>1$, $b\leqq 1$, $\frac{a}{2}+b\geqq 1$, *and* $w(x)=(1+|x|)^\alpha$. *Let* $1<p<\infty$ *when* $b=1$ *and* $\frac{a}{a+b-1}\leqq p\leqq\frac{a}{1-b}$ *when* $b<1$. *Then,*

$$\|K_{a,b+iy}*f\|_{p,w}\leqq C(1+|y|)\|f\|_{p,w}$$

if and only if

$$\max[-a+p(1-b),\quad a-2+p(2-a-b)]\leqq\alpha$$

$$\leqq\min[a(p-1)-p(1-b),\quad bp+a-2].$$

The conditions on α reduce to $a-2+p(2-a-b)\leqq\alpha\leqq a(p-1)-p(1-b)$ for $p\leqq 2$ and $-a+p(1-b)\leqq\alpha\leqq bp+a-2$ for $p\geqq 2$. Theorem 2 is proved in several steps. A three parts proof is used to derive the weighted L^2 result for $K_{a,b+iy}$. The rest of the proof uses interpolation with change of measures, interpolation of analytic families of operators, and a three parts proof.

We wish to point out that the range on α in Theorem 2 is closed, which is quite unusual for a single weight problem. The weight $(1+|x|)^\alpha\in A_p$ if and only if $-1<\alpha<p-1$. When $b=1$ and $p=2$, the range on α in Theorem 2 is $-a\leqq\alpha\leqq a$. Thus, for $a>1$, there are weights $(1+|x|)^\alpha$ not in A_p for which $K_{a,1+iy}$ defines a bounded operator on L^p_w. This is not the case for powers of $|x|$; we only get norm inequalities for $|x|^\alpha\in A_p$. (See the end of Section 2.) Finally, Theorem 3.3 shows that for weights which are bounded away from 0, grow no faster than $|x|^{a+2b-2}$, and are essentially constant on annuli, $K_{a,b+iy}$ defines a bounded operator on L^2_w. We construct such a weight, neither in A_2 nor a power of $1+|x|$. In Section 4, Theorem 3.3 is generalized to other values of p if w satisfies, in addition, a smoothness condition.

We will consider the question of weak-type (1, 1) estimates for $K_{a,1}$ in a forthcoming paper. In particular, for $0<a\neq 1$ and $w\in A_1$, we will show

$$w(\{x:\ |(K_{a,1}*f)(x)|>\lambda\})\leqq\frac{C}{\lambda}\int|f(x)|w(x)\,dx.$$

The paper is divided into six parts. Section 1 contains background information. The proof of Theorem 1 is begun in Section 2 with A_p results for $K_{a,1+iy}$. The sufficiency of the range of α in Theorem 2 for $K_{a,1+iy}$ is contained in Sections 3 and 4. The results are extended to $K_{a,b+iy}$ in Section 5 and norm inequalities for related operators are mentioned. The necessity of the range of α is considered in Section 6.

We assume all functions and sets are measurable with respect to Lebesgue measure. By the letters B and C we denote constants which may vary from line to line but are independent of f and we define the conjugate of p, p', by $\frac{1}{p}+\frac{1}{p'}=1$.

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1. Preliminary results

In this section we wish to collect facts which will be useful in the sequel. We begin by discussing the space H^1 and complex interpolation.

A real-valued function $b(x)$ is called a $(1, 2)$-atom if

i) b is supported in an interval, I,

ii) $\int b^2(x)\,dx \leqq |I|^{-1}$,

iii) $\int b(x)\,dx=0$.

We say a function $f\in H^1$ if and only if there exist $(1, 2)$-atoms, $\{b_j\}$, and constants, $\{\lambda_j\}$, such that $f(x)\sim \Sigma\lambda_j b_j(x)$ and $\Sigma|\lambda_j|<+\infty$. Set $\|f\|_{H^1}=\inf \Sigma|\lambda_j|$, the infimum taken over all such decompositions of f (see [3]).

Let $D=\{z=x+iy\colon 0\leqq x\leqq 1\}$. We say a function $F(z)$ is of admissible growth if there is an $a<\pi$ such that $e^{-a|y|}\log|F(z)|$ is uniformly bounded on the interior of D. Suppose for each $z\in D$ we have a linear operator T_z such that $(T_z f)g$ is integrable whenever f and g are simple functions. The family $\{T_z\}$ is called an analytic family of operators if the function $F(z)=\int (T_z f)g$ is continuous on D, analytic on the interior, and of admissible growth (see [17]). The following is well-known [3].

Theorem 1.1. *Let T_z be an analytic family of linear operators. Suppose that, for all y,*

$$\text{i)}\quad \|T_{iy}f\|_1 \leqq B_1(y)\|f\|_{H^1}$$

$$\text{ii)}\quad \|T_{1+iy}f\|_2 \leqq B_2(y)\|f\|_2$$

where $\log B_j(y)\leqq Ce^{d|y|}, j=1, 2, C>0$ *and* $0<d<\pi$. *Then*

$$\|T_t f\|_p \leqq B\|f\|_p,$$

where $1/p=(1-t)+t/2$, *with* $0<t\leqq 1$.

Let T_z be an analytic family of operators and $w(x)$ be a positive function. Consider the family of operators

$$U_z f(x) = w(x)^z T_z(fw^{-z})(x). \tag{1.1}$$

If w is nice enough, then U_z is also an analytic family of operators. Our interest in U_z is that weighted norm inequalities for T_z follow from norm inequalities for U_z.

The idea of regular kernels, which has been considered by one of the authors [8], will be useful to us.

Definition 1.2. *A kernel K is called regular if it can be written as $K(t)=k(t)g(t)$ such that*

i) $|g(t)| \leqq C|g(x)|$ *for* $|x|/2 \leqq |t| \leqq 2|x|$,

ii) $\int_{\{|x|>2|t|\}} |k(x-t)-k(x)|\,|g(x)|\,dx \leqq C$ *for* $t \neq 0$,

iii) *K maps L^2 into L^2* (i.e., $\hat{K}\in L^\infty$).

It was shown in [8] that the kernels $K_{a,1}$, $0<a\neq 1$, are regular. If $0<a\neq 1$ and $u\geqq 0$, then $K_{a,1+iy}(t;u)=K_{a,1+iy}(t)\chi(\{|t|>u\})$ is also regular for all $y\in\mathbf{R}$. To see this, set $g(t)=\exp(i|t|^a)(1+|t|)^{-iy}$; (i) and (ii) are then trivial. We note that (0.1) and (0.2) can be generalized to

$$\|K_{a,b+iy}(\,\cdot\,;u)*f\|_p \leqq C(1+|y|)\,\|f\|_p. \tag{1.2}$$

In particular, the constant, C, does not depend on y and u. Applying the proof of Theorem 5 of [8] to the kernel $\Omega_z(t;u)=\exp(i|t|^a)(1+|t|)^{-(1+iy)-za/2}$, (1.2) follows. Note that $K_{a,b+iy}(t;0)=K_{a,b+iy}(t)$ and that p is restricted as in (0.1) and (0.2). For (iii), we use (1.2) with $b=1$ and $p=2$.

We recall the following result for later use.

Theorem 1.3. *Let $K=kg$ be a regular kernel and $w\in L^\infty$. Then*

$$\|K*(fw)\|_1 \leqq C\|f\|_{H^1}$$

if and only if there is a constant B such that

$$\int_{\{|t-\alpha|>2|I|\}} |k(t-\alpha)|\,|\{g*(wb)\}(t)|\,dt \leqq B$$

for all $(1,2)$-atoms b, with support $I=[\alpha,\beta]$.

Theorem 1.3 is a generalization of Theorem 1 in [8]. The proof is essentially the same. We note that $C\leqq B+c(K)$, where $c(K)$ depends only on the constants in the definition of regular kernels and the L^2 norm of K.

Our first results will deal with A_p weights.

Definition 1.4. *A non-negative, locally integrable function, w, is in A_p, $1<p<\infty$, if there exists a constant, C, such that for all intervals, I,*

$$\left(|I|^{-1}\int_I w(x)\,dx\right)\left(|I|^{-1}\int_I w(x)^{1-p'}\,dx\right)^{p-1}\leqq C.$$

For a discussion of A_p weights see [11]. We will need

Theorem 1.5. *Let $w\in A_p$, $p>1$. Then, there exists an r, $1<r<p$, such that $w\in A_{p/r}$.*

In proving results for A_p weights, we will use a generalization of the Hardy—Littlewood maximal function and the sharp function of C. Fefferman and Stein. These are defined as

$$f_r^*(x)=\sup_{x\in I}\left(\frac{1}{|I|}\int_I |f(y)|^r\,dy\right)^{1/r},\quad 1\leqq r<\infty,$$

and

$$f^\#(x)=\sup_{x\in I}\frac{1}{|I|}\int_I |f(y)-\mathrm{av}_I f|\,dy,\quad \mathrm{av}_I f=\frac{1}{|I|}\int_I f(y)\,dy.$$

The following theorems are easy consequences of results in [4, 11].

Theorem 1.6. *Let $0<r<p<\infty$ and $w\in A_{p/r}$. Then $\|f_r^*\|_{p,w}\leqq C\|f\|_{p,w}$.*

Theorem 1.7. *Let $w\in A_q$ for some q. For $1\leqq p<\infty$, if $f^*\in L_w^p$ then*

$$\|f\|_{p,w}\leqq C\|f^\#\|_{p,w}.$$

We will make repeated use of interpolation with change of measures [16]:

Theorem 1.8. *Let T be a sublinear operator and $1\leqq p\leqq q<\infty$. If*

$$\text{i)}\quad \|(Tf)u_1\|_p\leqq C\|fu_2\|_p$$

and

$$\text{ii)}\quad \|(Tf)v_1\|_q\leqq C\|fv_2\|_q,$$

then

$$\|(Tf)w_1\|_s\leqq C\|fw_2\|_s,$$

where $1/s=\theta/p+(1-\theta)/q$, for $0\leqq\theta\leqq 1$, and $w_i=u_i^\theta v_i^{1-\theta}$, $i=1,2$.

The following result is useful in obtaining the optimal range of α for weights of the form $(1+|x|)^\alpha$. Its statement and proof mimic those of a theorem of Hardy, Littlewood, and Paley.

Proposition 1.9. *Let $a>0$, $p\geqq 2$, and T be a linear operator satisfying*

$$\text{i)}\quad |x|^{(2-a)/p}|Tf(x)|\leqq C_1\|f\|_1$$

and

$$\text{ii)}\quad \|Tf\|_{2,|x|^{[1-(2/p)](a-2)}}\leqq C_2\|f\|_2.$$

Then, there is a constant, $C \leqq C_p \max\{C_1, C_2\}$, *such that*

$$\|Tf\|_p \leqq C\|f\|_{p,|x|^{p-2}}.$$

Proof. Consider the adjoint of T, T^*, defined by $\int gTf = \int (T^*g)f$. (The existence of T^* is guaranteed by (i) and the Radon—Nikodym Theorem.) From (i), we get

$$|T^*f(x)| \leqq C_1 \int |f(x)|\,|x|^{(a-2)/p}\,dx. \tag{1.3}$$

Notice that (ii) implies

$$\left\|T^*\left(g(\cdot)|t|^{[1-(2/p)][(a/2)-1]}\right)\right\|_2 \leqq C_2\|g\|_2;$$

setting $f(t)=g(t)|t|^{[1-(2/p)][(a/2)-1]}$, we have

$$\|T^*f\|_2 \leqq C_2\|f\|_{2,|x|^{[1-(2/p)](a-2)}}. \tag{1.4}$$

Fix $p \geqq 2$ and define $\Psi f(x) = |x|T^*\left(f(\cdot)|t|^{(a-2)/p'}\right)(x)$, $dv=|x|^{-2}dx$, and $d\mu=|x|^{a-2}dx$. Since $\|f\|_{1,d\mu}=\int\left(|f(x)|\,|x|^{(a-2)/p'}\right)|x|^{(a-2)/p}dx$, it follows from (1.3) that

$$v(\{x\colon |\Psi f(x)|>\lambda\}) \leqq v(\{x\colon C_1|x|\,\|f\|_{1,d\mu}>\lambda\})$$

$$\leqq 2\int_{\lambda/C_1\|f\|_{1,d\mu}}^{\infty} x^{-2}\,dx = (2C_1/\lambda)\|f\|_{1,d\mu}.$$

By (1.4),

$$\int\left|\left\{|x|T^*\left(f(\cdot)|t|^{(a-2)/p}\right)(x)\right\}\right|^2|x|^{-2}\,dx = \int\left|T^*\left(f(\cdot)|t|^{(a-2)/p'}\right)(x)\right|^2dx$$

$$\leqq C_2\int|f(x)|^2|x|^{2(a-2)/p'}|x|^{(1-2/p)(2-a)}\,dx = C_2\int|f(x)|^2|x|^{a-2}\,dx,$$

or

$$\|\Psi f\|_{2,dv} \leqq C_2\|f\|_{2,d\mu},$$

By the Marcinkiewicz Interpolation Theorem [17; II, p. 112],

$$\|\Psi f\|_{p',dv} \leqq C\|f\|_{p',d\mu},$$

with $1<p'<2$ and $C \leqq C_{p'}\max\{C_1, C_2\}$. Thus

$$\int\left|T^*\left(f(\cdot)|t|^{(a-2)/p'}\right)(x)\right|^{p'}|x|^{p'-2}\,dx \leqq C\int|f(x)|^{p'}|x|^{a-2}\,dx.$$

Setting $g(t)=f(t)|t|^{(a-2)/p'}$ yields

$$\int|T^*g(x)|^{p'}|x|^{p'-2}\,dx = C\int|g(x)|^{p'}\,dx$$

and the proof is completed by duality.

2. The kernels $K_{a,1+iy}(t)$ and A_p weights

In this section we will consider the kernels

$$K_{a,1+iy}(t) = \exp(i|t|^a)(1+|t|)^{-1-iy}.$$

We begin by deriving an estimate on the sharp function of $K_{a,1+iy}*f$, from which norm inequalities for A_p weights follow. We will see in the next section these kernels define bounded operators on weighted L^p spaces for weights in a class much larger than A_p.

To get the sharp function estimate we will need the following result which appeared in [8; Remark 2, p. 410].

Proposition 2.1. *Let* $\tilde{K}_a(t)=\exp(i|t|^a)(1+|t|)^{(a-2)/p'}$, *where* $0<a\neq 1$ *and* $1\leqq p\leqq 2$. *Then*

$$\|\tilde{K}_a*f\|_{p'} \leqq C\|f\|_p.$$

Notice that when $a=2$, $(\tilde{K}_2*f)(x)=(K_{2,0}*f)(x)=\exp(ix^2)\{\exp(it^2)f(t)\}\hat{}(x)$, where $\hat{f}(x)=\int_{\mathbf{R}}e^{-2ix\cdot t}f(t)\,dt$ denotes the Fourier transform of f. In this case Proposition 2.1 says that the Fourier transform maps L^p into $L^{p'}$ for $1\leqq p\leqq 2$. We can now prove

Lemma 2.2. *Let* $K_{a,1+iy}(t)=\exp(i|t|^a)(1+|t|)^{-1-iy}$, $0<a\neq 1$ *and* $1<r<\infty$. *There is a constant,* $C=C(r)$, *such that for any bounded function* f *with compact support and almost every* x,

$$(K_{a,1+iy}*f)^{\#}(x) \leqq C(1+|y|)f_r^*(x).$$

Proof. Without loss of generality, we may assume $x=0$ and $r\leqq 2$. Let $I=(-\delta,\delta)$, $\delta>0$. We want to show that there is a constant, c_I, such that

$$\frac{1}{|I|}\int_I |K_{a,1+iy}*f)(t)-c_I|\,dt \leqq C(1+|y|)f_r^*(0). \tag{2.1}$$

Set $f_1(x)=f(x)\chi(\{|x|<2\delta\})$ and $f_2=f-f_1$. By Hölder's inequality and (1.2), we get

$$\frac{1}{|I|}\int_I |K_{a,1+iy}*f_1)(t)|\,dt \leqq \left(\frac{1}{|I|}\int_I |(K_{a,1+iy}*f_1)(t)|^r\,dt\right)^{1/r} \tag{2.2}$$

$$\leqq C_r(1+|y|)\left(\frac{1}{|I|}\int_{\{|x|<2\delta\}} |f(x)|^r\,dx\right)^{1/r} \leqq C_r(1+|y|)f_r^*(0).$$

Next, write $K_{a,1+iy}(t)=k(t)h(t)$, where $k(t)=\exp(i|t|^a)(1+|t|)^{-\gamma}$ and $h(t)=(1+|t|)^{\gamma-1-iy}$, $\gamma<1$ to be chosen. If g is any function with support contained in $\{|x|\geqq 2\delta\}$,

$$(K_{a,1+iy}*g)(t) = \int k(x-t)\{h(x-t)-h(x)\}g(x)\,dx + \int k(x-t)h(x)g(x)\,dx = A+B. \tag{2.3}$$

If $t\in I$ and $x\in\operatorname{supp}(g)$, then $|t|<\delta$ and $|x|\geqq 2\delta$, so that

$$|h(x-t)-h(x)| \leqq C(1+|y|)|t|\,|x|^{\gamma-2} \quad\text{and}\quad |k(x-t)| \leqq |x|^{-\gamma}.$$

Therefore, if $E_k=\{2^k\delta\leqq|x|<2^{k+1}\delta\}$,

$$|A| \leqq \int C(1+|y|)|t|\,|x|^{-2}|g(x)|\,dx \leqq C(1+|y|)\delta\int_{\{|x|<2\delta\}}|g(x)|\,|x|^{-2}\,dx$$
$$\leqq C(1+|y|)\delta\sum_{k=1}^{\infty}(2^k\delta)^{-2}\int_{E_k}|g(x)|\,dx.$$

Since $(2^k\delta)^{-1}\int_{\{|x|<2^{k+1}\delta\}}|g(x)|\,dx\leqq Cg^*(0)$, $|A|\leqq C(1+|y|)(\sum_{k=1}^{\infty}2^{-k})g^*(0)$ and

$$\frac{1}{|I|}\int_I|A| \leqq C(1+|y|)\,g^*(0). \tag{2.4}$$

We consider four cases:

1) $a>1,\quad 1\leqq\delta,$

2) $a>1,\quad 0<\delta<1,$

3) $0<a<1,\quad 1\leqq\delta,$

4) $0<a<1,\quad 0<\delta<1.$

Case 1: $a>1$ and $1\leqq\delta$. Choose $\gamma=\dfrac{2-a}{r'}$. By (2.4) with $g=f_2$, we get

$$\frac{1}{|I|}\int_I|A| \leqq C(1+|y|)(f_2)^*(0) \leqq C(1+|y|)f^*(0).$$

Since $B=\tilde{K}_a*(hf_2)$, using Proposition 2.1, we get

$$\frac{1}{|I|}\int_I|B| \leqq \left(\frac{1}{|I|}\int_I|\{\tilde{K}_a*(hf_2)\}(t)|^{r'}\,dt\right)^{1/r'} \leqq C\delta^{-1/r'}\|hf_2\|_r$$
$$\leqq C\delta^{-1/r'}\left(\int_{\{|x|>2\delta\}}|f(x)|^r(1+|x|)^{-[1+(a-2)/r']r}\,dx\right)^{1/r}.$$

Note that $\left[1+\dfrac{a-2}{r'}\right]r=1+(a-1)(r-1)$, where $(a-1)(r-1)>0$ since $a>1$. Arguing as in the proof of (2.4), we see

$$\frac{1}{|I|}\int_I|B| \leqq C\delta^{-1/r'}\delta^{-(a-1)/r'}f_r^*(0) \leqq Cf_r^*(0)$$

since $\delta\geqq 1$. Combining (2.2) with the above estimates finishes the proof of Case 1.

Case 2: $a>1$ and $0<\delta<1$. Set $\beta=-\dfrac{1}{a-1}$ and define L by $2^L\delta\leqq\delta^\beta<2^{L+1}\delta$. Let $f_3(x)=f_2(x)\chi(\{\delta\leqq|x|<\delta^\beta\})$, $f_4=f_2-f_3$, and define $c_I=\int\exp(i|x|^a)(1+|x|)^{-1-iy}f_3(x)\,dx$. Then

$$|(K_{a,1+iy}*f_2)(t)-c_I|\leqq|(K_{a,1+iy}*f_3)(t)-c_I|+|(K_{a,1+iy}*f_4)(t)|.$$

By (2.3) with $\gamma=0$ and $g=f_3$,

$$|(K_{a,1+iy}*f_3)(t)-c_I|\leqq|A|+|B-c_I|\leqq C(1+|y|)(f_3)^*(0)+|B-c_I|,$$

where $B-c_I=\int\{\exp(i|x-t|^a)-\exp(i|x|^a)\}(1+|x|)^{-1-iy}f_3(x)\,dx$. By the Mean Value Theorem, $|\exp(i|x-t|^a)-\exp(i|x|^a)|\leqq c|t|\,|x|^{a-1}$, so

$$\begin{aligned}|B-c_I|&\leqq C\delta\int_{\{\delta<|x|<\delta^\beta\}}|x|^{a-2}|f(x)|\,dx\leqq C\delta\sum_{k=0}^{L}(2^k\delta)^{a-2}\int_{E_k}|f(x)|\,dx\\&\leqq C\delta\sum_{k=0}^{L}(2^k\delta)^{a-1}f^*(0)\leqq C\delta^{1+\beta(a-1)}f^*(0)\leqq Cf^*(0).\end{aligned}$$

Thus, $|(K_{a,1+iy}f_3)(t)-c_I|\leqq C(1+|y|)(f_3)^*(0)+Cf^*(0)\leqq C(1+|y|)f^*(0)$.

Setting $\gamma=\dfrac{2-a}{r'}$ in (2.4), $|(K_{a,1+iy}f_4)(t)|$ is handled as in Case 1. Putting these estimates together,

$$\frac{1}{|I|}\int_I|K_{a,1+iy}*f_2)(t)-c_I|\,dt\leqq C(1+|y|)f_r^*(0).$$

This completes Case 2 and proves (2.1) when $a>1$.

Case 3: $0<a<1$ and $\delta\geqq1$. Define β, L, f_3 and f_4 as before. Set $c_I=\int\exp(i|x|^a)(1+|x|)^{-1-iy}f_4(x)\,dx$. Then

$$|(K_{a,1+iy}*f_2)(t)-c_I|\leqq|(K_{a,1+iy}*f_3)(t)|+|(K_{a,1+iy}*f_4)(t)-c_I|.$$

The arguments here mimic those in Case 2, with the roles of f_3 and f_4 reversed.

Case 4: $0<a<1$ and $0<\delta<1$. In this case, set $c_I=\int\exp(i|x|^a)(1+|x|)^{-1-iy}f_2(x)\,dx$. We have

$$|(K_{a,1+iy}*f_2)(t)-c_I|\leqq|A|+\left|\int\{\exp(i|x-t|^a)-\exp(i|x|^a)\}(1+|x|^a)^{-1-iy}f_2(x)\,dx\right|.$$

$|A|$ is handled as before and the second term is estimated by a constant times $\delta^a f^*(0)\leqq f^*(0)$, since $\delta<1$. Adding all the estimates proves (2.1) and completes the proof of the lemma.

Using Lemma 2.2 we easily prove Theorem 1 for $K_{a,1+iy}$.

Theorem 2.3. *Let* $1<p<\infty$, $0<a\neq1$ *and* $w\in A_p$. *Then*

$$\|K_{a,1+iy}*f\|_{p,w}\leqq C(1+|y|)\|f\|_{p,w}.$$

Proof. By Theorem 1.5, there is an r, $1<r<p$, so that $w\in A_{p/r}$. Let f be a bounded function with compact support and apply Theorems 1.6 and 1.7 and the previous lemma, to get

$$\|K_{a,1+iy}*f\|_{p,w}\leqq C\|(K_{a,1+iy}*f)^{\#}\|_{p,w}\leqq C(1+|y|)\|f_r^*\|_{p,w}\leqq C(1+|y|)\|f\|_{p,w}.$$

Since bounded functions with compact support are dense in L_w^p, the result follows.

A consequence of Theorem 2.3 is that $K_{a,1+iy}$ defines a bounded operator on L_w^p for $w(x)=(1+|x|)^\alpha$, $-1<\alpha<p-1$. However, a much wider range of α is allowable when $a>1$, which we prove in the following sections. We note that these weights are peculiar to this kind of kernel (i.e., with denominator $(1+|x|)$) and that consideration of powers of $|x|$ does not lead one out of A_p. In particular, fix a, $0<a\neq 1$, and choose $\varepsilon<(2^{1/a}3)^{-1}$. Let $f(x)=\chi(\{\varepsilon<x<2\varepsilon\})$ and $|x|<\varepsilon$. Then

$$|(K_{a,1}*f)(x)|=\left|\int_\varepsilon^{2\varepsilon}\exp(i|x-t|^a)(1+|x-t|)^{-1}\,dt\right|$$
$$\geqq\left|\int_\varepsilon^{2\varepsilon}\cos(|x-t|^a)(1+|x-t|)^{-1}\,dt\right|\geqq\varepsilon\cos(1/2),$$

since $|x-t|^a\leqq(3\varepsilon)^a<1/2$. Thus, if $w(x)=|x|^\alpha$ with $\alpha\leqq-1, f\in L_w^p$ while $K_{a,1}*f\notin L_w^p$. By duality, we see that for $\alpha\geqq p-1, K_{a,1}$ does not define a bounded operator on L_w^p, $w(x)=|x|^\alpha$. Hence, for powers of $|x|$, one must have A_p.

3. The weights $w(x)=(1+|x|)^\alpha$ and L^2

This section contains results for the kernels $K_{a,1+iy}$ on L^2 with weights of the form $w(x)=(1+|x|)^\alpha$. Note that we only consider the case $a>1$, since when $0<a<1$, Theorem 2.3 gives a better result than we obtain here. As it involves no further complications, we will prove our results for $K_{a,b+iy}$.

Lemma 3.1. *Let* $u>0$, $a>1$ *and* $d\geqq 4^a$. *Set* $Q=\left[\frac{a}{d}u^{a-1}, da\,u^{a-1}\right]$ *and* $J=\mathbf{R}-Q$. *Then, there is a constant* $C=C(a,b)$ *such that*

$$\left|\int_{\{u/2\leqq|t|\leqq 4u\}}K_{a,b+iy}(t)e^{-itx}\,dt\right|\leqq C(1+|y|)u^{1-b}\{u^{-a/2}\chi_Q(x)+u^{-a}\chi_J(x)\}.$$

Proof. Let $x\in J$ and write

$$\Phi=\int_{\{u/2\leqq|t|\leqq 4u\}}K_{a,b+iy}(t)e^{-itx}\,dt$$
$$=\int_{\{u/2\leqq|t|\leqq 4u\}}\frac{e^{i|t|^a}e^{-itx}(a|t|^{a-1}-x)}{(1+|t|)^{b+iy}(a|t|^{a-1}-x)}\,dt.$$

Using integration by parts (i.e., integrating the numerator and differentiating the rest) and noting that for $u/2 \leqq |t| \leqq 4u$, $|a|t|^{a-1}-x|$ is equivalent to $|x|$ if $|x| \geqq dau^{a-1}$ and to u^{a-1} if $|x| \leqq \frac{a}{d} u^{a-1}$, we see that $|\Phi| \leqq C(1+|y|)u^{1-b-a}$. If $x \in Q$, we can use van der Corput's lemma to show $|\Phi| \leqq C(1+|y|)u^{1-b-(a/2)}$.

For $p=2$, Lemma 3.1 improves (1.2) to

$$\|K_{a,b+iy}(\cdot\,;\,u)*f\|_2 \leqq C(1+|y|)u^{1-b-(a/2)}\|f\|_2, \tag{3.1}$$

with C depending only on a and b.

Using Lemma 3.1, we prove

Theorem 3.2. *Let* $a>1$, $\frac{a}{2}+b \geqq 1$, $b \leqq 1$ *and* $w(x)=(1+|x|)^\alpha$, *with* $|\alpha| \leqq a+2b-2$. *Then*

$$\|K_{a,b+iy}*f\|_{2,w} \leqq C(1+|y|)\|f\|_{2,w}.$$

Proof. Let $E_k=\{2^k \leqq |x| < 2^{k+1}\}$, $k=0,1,2,\ldots$. Define $f_0(t)=f(t)\chi(\{|t|<1\})$ and $f_k(t)=f(t)\chi(\{t \in E_{k-1}\})$, $k=1,2,\ldots$; also, $\chi_{k,1}(t)=\chi(\{|t|<2^{k+1}\})$ and $\chi_{k,2}(\{|t| \geqq 2^{k+1}\})$, $k=0,1,2,\ldots$. Then $(K_{a,b+iy}*f)(x)=F_1(x)+F_2(x)$, where

$$F_j(x) = \sum_{k=0}^{\infty} \chi_{k,j}(x)(K_{a,b+iy}*f_k)(x), \quad j=1,2.$$

Consider F_1. If $|x| \leqq 1$, then $\chi_{k,1}(x)=1$ for all k so that $F_1(x)=(K_{a,b+iy}*f)(x)$. For any $\alpha>0$, by (1.2)

$$\begin{aligned}\int_{\{|x|\leqq 1\}} |F_1(x)|^2(1+|x|)^\alpha\,dx &\leqq C2^\alpha(1+|y|)^2\int |f(x)|^2\,dx \\ &\leqq C2^\alpha(1+|y|)^2\|f\|^2_{2,(1+|x|)^\alpha}.\end{aligned} \tag{3.2}$$

Suppose $x \in E_m$, $m \geqq 0$. Then, $\chi_{k,1}(x)=1$ if and only if $k \geqq m$; for such x, $F_1(x)=(K_{a,b+iy}*f^m)(x)$, where $f^m(x)=\sum_{k=m}^{\infty} f_k(x)=f(x)\chi(\{|x| \geqq 2^{m-1}\})$. Therefore,

$$\begin{aligned}\int_{\{|x|>1\}} |F_1(x)|^2(1+|x|)^\alpha\,dx &= \sum_{m=0}^{\infty}\int_{E_m} |F_1(x)|^2(1+|x|)^\alpha\,dx \\ \leqq C\sum_{m=0}^{\infty} 2^{m\alpha}\int |(K_{a,b+iy}*f^m)(x)|^2\,dx &\leqq C(1+|y|)^2\sum_{m=0}^{\infty} 2^{m\alpha}\int |f^m(x)|^2\,dx \\ &= C(1+|y|)^2\sum_{m=0}^{\infty} 2^{m\alpha}\int_{\{|x|\geqq 2^{m-1}\}} |f(x)|^2\,dx \\ \leqq C(1+|y|)^2\int_{\{|x|>1/2\}} |f(x)|^2\Big(\sum_{m=0}^{\ln|x|} 2^{m\alpha}\Big)\,dx &\leqq C(1+|y|)^2\int |f(x)|^2|x|^\alpha\,dx.\end{aligned}$$

Using (3.2), we see

$$\int |F_1(x)|^2(1+|x|)^\alpha\,dx \leqq C(1+|y|)^2\int |f(x)|^2(1+|x|)^\alpha\,dx. \tag{3.3}$$

Let $\alpha=a+2b-2$. Since the support of F_2 is contained in $\{|x| \geqq 1\}$, for $x \in E_m$, $m \geqq 0$, $\chi_{k,2}(x)=1$ if and only if $k<m$. Thus

$$\begin{aligned}\int |F_2(x)|^2(1+|x|)^\alpha\,dx &\leqq \sum_{m=0}^{\infty}\int_{E_m}\Big|\int_{\{|t|<2^{m-1}\}} K_{a,b+iy}(x-t)f(t)\,dt\Big|^2(1+|x|)^\alpha\,dx \\ &= \sum_{m=0}^{\infty} I_m.\end{aligned}$$

For m fixed, we can view the support of $K_{a,b+iy}$ as contained in the set $\{2^{m-1}\leqq|x|\leqq 2^{m+2}\}$. Let $K^{(m)}_{a,b+iy}(t)=K_{a,b+iy}(t;2^{m-1})-K_{a,b+iy}(t;2^{m+2})$. Note that

$$I_m \leqq 2\int_{E_m}\left|\int K^{(m)}_{a,b+iy}(x-t)f(t)\,dt\right|^2(1+|x|)^\alpha\,dx$$

$$+2\int_{E_m}\left|\int_{\{|t|\geqq 2^{m-1}\}}K^{(m)}_{a,b+iy}(x-t)f(t)\,dt\right|^2(1+|x|)^\alpha\,dx = I_{m,1}+I_{m,2},$$

where $I_{m,2}\leqq 2^{m\alpha}\int_{E_m}|F_1(x)|^2dx$ (with F_1 defined with respect to $K^{(m)}_{a,b+iy}$). By (1.2) and the argument preceeding (3.3),

$$\sum_{m=0}^{\infty} I_{m,2} \leqq C(1+|y|)^2\int|f(x)|^2(1+|x|)^\alpha\,dx. \tag{3.4}$$

Plancherel's Theorem and Lemma 3.1 (with $u=2^m$) imply

$$I_{m,1} \leqq C2^{m\alpha}\int|(K^{(m)}_{a,b+iy})^\wedge(x)|^2|\hat f(x)|^2\,dx$$

$$\leqq (C1+|y|)^2 2^{m\alpha}2^{2m(1-b)}\left\{2^{-ma}\int|\hat f(x)|^2\chi_Q(x)\,dx+2^{-2ma}\int|\hat f(x)|^2\chi_J(x)\,dx\right\},$$

where $Q=Q(m)=[4^{-a}a2^{m(a-1)},\ 4^a a2^{m(a-1)}]$ and $J=J(m)=\mathbf{R}-Q(m)$.

Adding up the estimates and using the bounded overlaps of the $Q(m)$'s and the definition of α yields

$$\sum_{m=0}^{\infty} I_{m,1} \tag{3.5}$$

$$\leqq C(1+|y|)^2\sum_{m=0}^{\infty}\left\{\int|\hat f(x)|^2\chi_{Q(m)}(x)\,dx+2^{-ma}\int|f(x)|^2\,dx\right\}$$

$$\leqq C(1+|y|)^2\left\{\int|\hat f(x)|^2\left(\sum_{m=0}^{\infty}\chi_{Q(m)}(x)\right)dx+\|f\|_2^2\right\}$$

$$\leqq C(1+|y|)^2\{C\|f\|_2^2+\|f\|_2^2\}\leqq C(1+|y|)^2\|f\|_2^2.$$

By (3.4) and (3.5), since α is non-negative,

$$\int|F_2(x)|^2(1+|x|)^\alpha\,dx\leqq C\sum_{m=0}^{\infty} I_m\leqq C(1+|y|)^2\|f\|^2_{2,(1+|x|)^\alpha}.$$

This completes the proof when $\alpha=a+2b-2$. We get the result when $\alpha=-(a+2b-2)$ by duality. Using interpolation with change of measures between these two end-point results completes the proof of the theorem.

Remarks. We note that because of (1.2) the estimate for F_1 works for any positive α and any p such that convolution with $K_{a,b+iy}$ or $K^{(m)}_{a,b+iy}$ defines a bounded operator on L^p. Theorem 3.2 is also true for $b<1$ when $0<a<1$. This result will follow from Theorem 5.2.

Before considering other values of p, we would like to make some observations about the proof of Theorem 3.2. The estimate (3.2) uses the facts that $w(x)$ is bounded above for $|x|\leqq 1$ and bounded below for all x. The rest of the estimates use the facts that $w(x)$ is essentially constant on annuli and bounded above by $|x|^{a+2b-2}$. This is obvious for F_2. Suppose that $w(x)\leqq Bw(y)$ for $|y|/2\leqq|x|\leqq 2|y|$, with $B>1$, and $w(x)\leqq c|x|^{a+2b-2}$. Assume $\frac{a}{2}+b>1$. Since w is constant on annuli, for F_1 we need only consider

$$I=\int_{E_m}\left|\int_{\{|t|\geqq 2^{m+2}\}}K_{a,b+iy}(x-t)f(t)\,dt\right|^2 w(x)\,dx$$
$$\leqq Bw(2^m)\left\{\int_{E_m}\left|\int_{\{2^{m+2}\leqq|t|\leqq 2^s\}}K_{a,b+iy}(x-t)f(t)\,dt\right|^2 dx\right.$$
$$\left.+\int_{E_m}\left|\int_{\{|t|>2^s\}}K_{a,b+iy}(x-t)f(t)\,dt\right|^2 dx\right\},$$

where $s=m+(ma/T\beta)$, with $\beta=\log_2 B$ and T chosen so that $S=[(T-1)a/T]+2b-2>0$, and $m\geqq 2T\beta/a$. By the condition on w,

$$\frac{w(2^m)}{w(2^s)}=\frac{w(2^m)}{w(2^{m+1})}\cdot\ldots\cdot\frac{w(2^{s-1})}{w(2^s)}\leqq B^{s-m},$$

so that $w(2^m)\leqq B^{s-m}w(x)$ for all x such that $2^m\leqq|x|\leqq 2^s$. Therefore by (3.1),

$$I\leqq C(1+|y|)^2 w(2^m)2^{-m(a+2b-2)}\int_{\{2^{m+2}\leqq|x|\leqq 2^s\}}|f(x)|^2\,dx$$
$$+C(1+|y|)^2 w(2^m)2^{-s(a+2b-2)}\int_{\{|x|>2^s\}}|f(x)|^2\,dx$$
$$\leqq C(1+|y|)^2(B^{s-m})2^{-m(a+b-2)}\|f\|_{2,w}^2+C(1+|y|)^2 2^{(m-s)(a+2b-2)}\|f\|_{2,w}^2$$
$$\leqq C(1+|y|)^2\{2^{-mS}+2^{-(ma)(a+2b-2)/T\beta}\}\|f\|_{2,w}^2.$$

We also note that Lemma 3.1 and Theorem 3.2 are valid for $0<a<1$. Thus, we have the following result for weights which are essentially constant on annuli:

Theorem 3.3. *Let* $0<a\neq 1$, $b\leqq 1$ *and* $\frac{a}{2}+b>1$. *Suppose* $w(x)\geqq 1$ *and satisfies*

i) $\quad w(x)\leqq C(1+|x|)^{a+2b-2}$, *for all* x, *and*

ii) $\quad w(x)\leqq Cw(t)$, *for* $|t|/2\leqq|x|\leqq 2|t|$.

Then

$$\int|(K_{a,b+iy}*f)(x)|^2 w(x)\,dx\leqq C(1+|y|)^2\int|f(x)|^2 w(x)\,dx.$$

We conclude this section with an example of a weight, not in A_2 and not of the form $(1+|x|)^\alpha$, which satisfies Theorem 3.3. Let $a>1$. Let $n(1)=1$ and $m(1)=2^a$, and for $k>1$, set $n(k)=2[n(k-1)+2m(k-1)+1]$ and $m(k)=2^{n(k)a}$. Define $w(x)=1$ for $|x|\leqq 4$ and on sets of the form $\{x\colon 2^{n(k)+1}\leqq|x|\leqq 2^{2[n(k)2m(k)+1]}\}$ by

$$w(x)=\begin{cases} j2^{ja} & 2^{n+j}\leqq|x|\leqq 2^{n+j+1}, \quad j=1,2,\ldots,m.\\ (1+|x|)^a, & 2^{n+m+1}\leqq|x|\leqq 2^{n+2m}\\ 2^{(n+2m)a-ja}, & 2^{n+2m+j}\leqq|x|\leqq 2^{n+2m+j+1}, \quad j=1,2\ldots,n+2m\end{cases}$$

where $n=n(k)$ and $m=m(k)$. It is easy to see that $1\leqq w(x)\leqq(1+|x|)^a$ for all

x and w is essentially constant on $\{R<|x|<2R\}$. Clearly, w is not of the form $(1+|x|)^\alpha$ for any α. Let $I=[2^{n(k)+m(k)+1}, 2^{n(k)+m(k)}]$. Then

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1}\,dx\right) \sim 2^{m(k)(a-1)}$$

which goes to ∞. Thus, $w\notin A_2$.

4. Weighted L^p for $p\neq 2$

We are interested in extending Theorem 3.2 to weighted L^p spaces for $p\neq 2$. Our main result in this section is

Theorem 4.1. *Let* $1<p<\infty$, $1<a$, *and* $w(x)=(1+|x|)^\alpha$. *If*

i) $$1<p\leqq 2 \quad and \quad a-2+(1-a)p\leqq\alpha\leqq a(p-1)$$

or

ii) $$2\leqq p<\infty \quad and \quad -a\leqq\alpha\leqq p+a-2$$

then

$$\|K_{a,1+iy}*f\|_{p,w}\leqq C(1+|y|)\|f\|_{p,w}.$$

We will prove this theorem in a series of steps. We begin with a result of a slightly different nature.

Lemma 4.2. *Let* $1<a$ *and define* $U_{i\gamma}$ *by*

$$U_{i\gamma}f(x)=\int K_{a,1+iy}(x-t)f(t)(1+|t|)^{-i\gamma}\,dt.$$

Then

$$\|U_{i\gamma}f\|_1\leqq C(1+|\gamma|)(1+|y|)\|f\|_{H^1}.$$

Proof. Let $w(t)=(1+|t|)^{-i\gamma}$, $g(t)=\exp(i|t|^a)(1+|t|)^{(a-2)/2}$, and $k(t)=(1+|t|)^{-a/2-i\gamma}$. Since w is bounded, by Theorem 1.3, it is enough to show that there is a constant, B, such that for any (1.2)-atom b, supported in $I=[\alpha,\beta]$,

$$\int_{\{|t-\alpha|>2\delta\}}|k(t-\alpha)|\,|\{g*(wb)\}(t)|\,dt\leqq B, \tag{4.1}$$

where $\delta=|I|$. Denote the left hand side of (4.1) by Ψ and consider two cases.

Case 1: $\delta\geqq 1$. Using the notation of Proposition 2.1 with $p=2$, notice that for $f\in L^2$, $g*f=\tilde{K}_a*f$. Thus, by Schwarz's inequality and Proposition 2.1, we get

$$\begin{aligned}\Psi &\leqq \left(\int_{\{|t-\alpha|>2\delta\}}|k(t-\alpha)|^2\right)^{1/2}\|g*(wb)\|_2\\ &\leqq C\delta^{(1-a)/2}\|wb\|_2\leqq C\delta^{-a/2}\leqq C,\end{aligned}$$

since b is a (1.2)-atom and $\delta\geqq 1$.

Case 2: $0<\delta<1$. Let $\Delta=\delta^{-1/(a-1)}$ and write

$$\Psi = \int_{\{2\delta\leqq|t-\alpha|\leqq 2\Delta\}} |k(t-\alpha)|\,|\{g*(wb)\}(t)|\,dt + \int_{\{|t-\alpha|>2\Delta\}} |k(t-\alpha)|\,|\{g*(wb)\}(t)|\,dt = \Psi_1+\Psi_2.$$

Arguing as above, we get $\Psi_2\leqq C\Delta^{(1-a)/2}|wb|_2\leqq C\delta^{1/2}\delta^{-1/2}\leqq C$. To handle Ψ_1, in the inner integral (i.e., $g*(wb)$) add and subtract $(1+|\alpha|)^{-i\gamma}$ and then use the fact that $\int b=0$ to get

$$\Psi_1 = \int_{\{2\delta\leqq|t-\alpha|\leqq 2\Delta\}} |k(t-\alpha)|\,\Big|\int_I \{g(x-t)[(1+|x|)^{-i\gamma}-(1+|\alpha|)^{-i\gamma}] + (1+|\alpha|)^{-i\gamma}[g(x-t)-g(\alpha-t)]\}\,b(x)\,dx\Big|\,dt.$$

By the Mean Value Theorem and the fact that $|t-\alpha|\geqq 2|x-\alpha|$, we get

$$\big|(1+|x|)^{-i\gamma}-(1+|\alpha|)^{-i\gamma}\big| \leqq |\gamma|\,|x-\alpha|$$

and

$$|g(x-t)-g(\alpha-t)| \leqq C|t-\alpha|^{a-1}(1+|t-\alpha|)^{(a/2)-1}|x-\alpha|.$$

Since $|x-\alpha|\leqq\delta\leqq 1$,

$$\Psi_1 \leqq C\delta\left(\int_{\{2\delta\leqq|t-\alpha|\leqq 2\Delta\}} \frac{(|\gamma|+|t-\alpha|^{a-1})(1+|t-\alpha|)^{(a-2)/2}}{(1+|t-\alpha|)^{a/2}}\,dt\right)\|b\|_1$$

$$\leqq C\delta\left(|\gamma|\ln\frac{1}{\delta}+\frac{1}{\delta}\right) \leqq C(1+|\gamma|).$$

Thus, B in (4.1) is bounded by $C(1+|\gamma|)$. By the statement following Theorem 1.3, this proves the theorem.

Remark. Suppose $w(x)$ is a real-valued function. Clearly, $w(x)^{-i\gamma}$ is then bounded, so we may replace $(1+|t|)$ in the definition of $U_{i\gamma}$ by $w(x)$ if w satisfies the following condition:

$$|w(x)^{-i\gamma}-w(\alpha)^{-i\gamma}| \leqq C|\gamma|\,|x-\alpha|.$$

This is used to estimate the first inner integral in Ψ_1. The rest of the argument is the same.

Lemma 4.3. *Let* $a>1$, $1<p\leqq 2$, *and* $w(x)=(1+|x|)^\alpha$. *If*

$$-a(p-1) \leqq \alpha \leqq a(p-1), \quad \textit{then}$$

$$\|K_{a,1+i\gamma}*f\|_{p,w} \leqq C(1+|\gamma|)\|f\|_{p,w}.$$

Proof. By interpolation with change of measures (with p fixed), it is enough to prove the result for $\alpha=\pm a(p-1)$. We use the ideas developed in Section 1 for analytic families of operators.

In order to prove the result for $\alpha=a(p-1)$, define

$$T_z f(x) = (1+|x|)^{az/2}(K_{a,1+iy}*f)(x)$$

and

$$U_z f(x) = (1+|x|)^{az/2}\int K_{a,1+iy}(x-t)f(t)(1+|t|)^{-az/2}\,dt.$$

Since $1+|x|\geqq 1$, if f and g are simple functions, $F(z)=\int(U_z f)g$ is continuous on D and analytic on the interior. Further,

$$|F(z)| \leqq \|K_{a,1+iy}\|_\infty \|f\|_1 \int(1+|x|)^{a/2}|g(x)|\,dx \leqq C$$

since g has compact support. Thus, F is of admissible growth and both T_z and U_z are analytic families of operators.

By Theorem 3.2 and Lemma 4.2,

$$\|U_{1+i\gamma}f\|_2 \leqq C(1+|\gamma|)(1+|y|)\|f\|_2 \quad \text{and} \quad \|U_{i\gamma}f\|_1 \leqq C(1+|\gamma|)(1+|y|)\|f\|_{H^1}.$$

Thus, by Theorem 1.1,

$$\|T_\theta f\|_p \leqq C(1+|y|)\|f\|_{p,(1+|x|)^{(a/2)\theta p}}, \quad \text{where} \quad 1/p = 1-\theta/2.$$

Since $T_\theta f=K_{a,1+iy}*f$ (for any θ) and $\theta=2(p-1)/p$, the proof for $\alpha=a(p-1)$ is completed by the definition of T_θ.

To prove the result for $\alpha=-a(p-1)$, we set $V_z f=U_{-z}f$, $z\in D$, and repeat the argument above. This completes the proof of Theorem 4.1.

Notice that Lemma 4.3 is true for $0<a<1$, and, by duality, for $p>2$ if $-a\leqq\alpha\leqq a$. If we use the remark following Lemma 4.2 and Theorem 3.3 (instead of Theorem 3.2), we have

Corollary 4.4. *Let* $0<a\neq 1$ *and* $1<p<\infty$. *Set* $\beta=\min(a(p-1), a)$. *Suppose*

i) $$1 \leqq w(x) \leqq C(1+|x|)^\beta, \quad \textit{for all } x;$$

ii) $$w(x) \leqq Cw(t), \quad \textit{for} \quad |t|/2 \leqq |x| \leqq 2|t|;$$

iii) $$|w(x)^{-i\gamma}-w(t)^{-i\gamma}| \leqq C|\gamma|\,|x-t|, \quad \textit{for all } \gamma \textit{ and } \quad |x-t| \leqq 1.$$

Then

$$\|K_{a,1+iy}*f\|_{p,w} \leqq C(1+|y|)\|f\|_{p,w}.$$

Lemma 4.3 dealt with L^p for $1<p\leqq 2$. We now consider $p>2$ and prove

Lemma 4.5. *Let* $1<a, b\leqq 1$, *and* $\dfrac{a}{2}+b\geqq 1$. *Suppose* $2<p\leqq\dfrac{a}{1-b}$ (*or* $2<p<\infty$ *when* $b=1$) *and* $0\leqq\alpha\leqq bp-2+a$. *Then*

$$\|K_{a,b+iy}*f\|_{p,(1+|x|)^\alpha} \leqq C(1+|y|)\|f\|_{p,(1+|x|)^\alpha}.$$

Proof. Using the notation of the proof of Theorem 3.2, we write

$$(K_{a,b+iy}*f)(x) = F_1(x)+F_2(x).$$

By the remark following the proof of Theorem 3.2, for any $\alpha>0$ and p satisfying $\frac{a}{a+b-1}\leqq p\leqq\frac{a}{1-b}$ $(1<p<\infty$ for $b=1)$,

$$\int |F_1(x)|^p(1+|x|)^\alpha\, dx \leqq C(1+|y|)^p\int |f(x)|^p(1+|x|)^\alpha\, dx. \tag{4.2}$$

For F_2 we have

$$\int |F_2(x)|^p(1+|x|)^\alpha\, dx = \sum_{m=0}^\infty I_m,$$

where

$$\begin{aligned} I_m &= \int_{E_m}\left|\int_{\{|t|<2^{m-1}\}} K_{a,b+iy}(x-t)f(t)\,dt\right|^p|(1+|x|)^\alpha\, dx \\ &\leqq 2^{p-1}\int_{E_m}\left|\int K^{(m)}_{a,b+iy}(x-t)f(t)\,dt\right|^p(1+|x|)^\alpha\, dx \\ &+2^{p-1}\int_{E_m}\left|\int_{\{|t|\geqq 2^{m-1}\}} K^{(m)}_{a,b+iy}(x-t)f(t)\,dt\right|^p(1+|x|)^\alpha\, dx = I_{m,1}+I_{m,2}. \end{aligned}$$

The argument leading to (3.4) now yields

$$\sum_{m=0}^\infty I_{m,2} \leqq C(1+|y|)^p\int |f(x)|^p(1+|x|)^\alpha\, dx. \tag{4.3}$$

Thus, we need only consider

$$\sum_{m=0}^\infty I_{m,1} = \sum_{m=0}^\infty 2^{p-1}\int_{E_m}\left|\int K^{(m)}_{a,b+iy}(x-t)f(t)\,dt\right|^p(1+|x|)^\alpha\, dx.$$

Fix $a, b,$ and p satisfying the hypothesis and set $\alpha=bp+a-2$. Define

$$Tf(x) = (1+|x|)^{b+(a-2)/p}\sum_{m=0}^\infty \chi(\{x\in E_m\})(K^{(m)}_{a,b+iy}*f)(x).$$

For $x\in\operatorname{supp} Tf\subset\{|x|\geqq 1\}$, say $2^m\leqq|x|\leqq 2^{m+1}$, we get

$$\begin{aligned} |x|^{(2-a)/p}|Tf(x)| &= |x|^{(2-a)/p}(1+|x|)^{b+(2-a)/p}|(K^{(m)}_{a,b+iy}*f)(x)| \\ &\leqq C\int_{\{2^{m-1}\leqq|x-t|\leqq 2^{m+2}\}}(1+|x|)^b(1+|x-t|)^{-b}|f(t)|\,dt \leqq C\int |f(t)|\,dt. \end{aligned}$$

Consider now

$$\begin{aligned} &\int |x|^{[1-(2/p)](a-2)}|Tf(x)|^2\,dx \\ &= \sum_{m=0}^\infty\int_{E_m}|x|^{[1-(2/p)]}(1+|x|)^{2b+2(a-2)/p}|(K^{(m)}_{a,b+iy}*f)(x)|^2\,dx \\ &\leqq C\sum_{m=0}^\infty\int |(K^{(m)}_{a,b+iy}*f)(x)|^2(1+|x|)^{a+2b-2}\,dx \end{aligned}$$

By (3.5), we get $(\alpha=a+2b-2)$

$$\int |x|^{[1-(2/p)](a-2)}|Tf(x)|^2\,dx \leqq C(1+|y|)^2\int |f(x)|^2\,dx.$$

Thus, T satisfies Proposition 1.9, so that

$$\|Tf\|_p \leqq C(1+|y|)\|f\|_{p,(1+|x|)^{p-2}}.$$

Rewriting, we get

$$\sum_{m=0}^{\infty} I_{m,1} = 2^{p-1}\|Tf\|_p^p \leqq C(1+|y|)^p \int |f(x)|(1+|x|)^{p-2}\,dx. \tag{4.4}$$

Since $p-2 \leqq bp+a-2$ for $p \leqq \dfrac{a}{1-b}$, combining (4.2), (4.3), and (4.4) we get

$$\|K_{a,b+iy} * f\|_{f,(1+|x|)^\alpha} \leqq C(1+|y|)\|f\|_{p,(1+|x|)^\alpha}.$$

With $a, b,$ and p fixed, using interpolation with change of measures between this result and (1.2) completes the proof.

Remark. We note that if $0<a<1$ and $b=1$, or if $p=2$, this result is a consequence of Theorems 2.3 or 3.2.

Proof of Theorem 4.1. The positive values for α are contained in Lemmas 4.3 and 4.5. The negative values follow by duality.

There are examples of weights which satisfy Corollary 4.4 and are not in A_p nor of the form $(1+|x|)^\alpha$. Such examples can be constructed in a manner similar to the one at the end of Section 3, but must be smoothed out to satisfy the last condition.

5. Applications to $K_{a,b+iy}$ and related operators

Using the results of the previous sections, we now consider the kernels $K_{a,b+iy}$. We will also show that weighted norm inequalities for similar kernels and related multiplier operators follow from results for $K_{a,b}$. We begin with a result for A_p weights.

Theorem 5.1. *Let* $a\neq 1$, $b<1$, $\dfrac{a}{2}+b \geqq 1$, *and* $\dfrac{a}{a+b-1} \leqq p \leqq \dfrac{a}{1-b}$. *If*

$$w \in A_p \quad and \quad \delta = \frac{b-\alpha}{1-\alpha}, \quad with \quad \alpha = a\left|\frac{1}{p}-\frac{1}{2}\right|+1-\frac{a}{2},$$

then

$$\|K_{a,b+iy} * f\|_{p,w^\delta} \leqq C(1+|y|)\|f\|_{p,w^\delta}.$$

Note that this result is true when $b=1$ and $1<p<\infty$. This is then Theorem 2.3. Thus, proving Theorem 5.1 will complete the proof of Theorem 1.

Proof. Let $K_z(t) = \exp(i|t|^a)(1+|t|)^{-(\alpha+iy+(1-\alpha)z)}$ and define $U_z f(x) = w^z(x)\int K_z(x-t)f(t)w^{-z}(t)\,dt$ By the definition of α, p equals either $\dfrac{a}{a+\alpha-1}$ or $\dfrac{a}{1-\alpha}$. By (1.2),

$$\|U_{i\gamma} f\|_p \leqq C(1+|y|+|\gamma|)\|f\|_p \leqq C(1+|y|)(1+|\gamma|)\|f\|_p.$$

Since $w \in A_p$, Theorem 2.3 implies

$$\|U_{1+i\gamma}f\|_p \leqq C(1+|y|+|\gamma|)\|f_p\| \leqq C(1+|y|)(1+|\gamma|)\|f\|_p.$$

Using interpolation of analytic families of operators, we get

$$\|U_\theta f\|_p \leqq C(1+|y|)\|f\|_p \quad \text{with} \quad 0 \leqq \theta \leqq 1.$$

Rewriting, this becomes

$$\|K_{a,\alpha+iy+(1-\alpha)\theta} * f\|_{p,w^\theta} \leqq C(1+|y|)\|f\|_{p,w^\theta},$$

and solving $\alpha+(1-\alpha)\theta=b$ yields $\theta=\dfrac{b-\alpha}{1-\alpha}=\delta$.

Note that $\alpha \leqq b \leqq 1$ so that $0 \leqq \delta \leqq 1$, and $w \in A_p$ implies that $w^\delta \in A_p$. Thus, Theorem 5.1 implies norm inequalities for kernels and multipliers studied in [5, 8, 10, 12, 13, 14]. Consider first the convolution kernel $\bar{K}_{a,b}(t)=\exp(i|t|^a)|t|^{-b}$, $0<b<1$. Since

$$\bar{K}_{a,b}(t) = K_{a,b}(t)+\exp(i|t|^a)[|t|^{-b}-(1+|t|)^{-b}],$$

with the last term on the right having a radial majorant in L^1, we have that

$$|\bar{K}_{a,b} * f(x)| \leqq |K_{a,b} * f(x)|+c|f^*(x)|.$$

The norm inequality follows. For $\bar{K}_{a,b}$ with $a \geqq 2$ and $1-\dfrac{a}{2} \leqq b \leqq 0$, use interpolation with change of measures between the known unweighted result for $\bar{K}_{a,1-(a/2)}$ and the previous weighted result for any positive b. This gives the appropriate value of δ.

Similarly, let $\theta(\xi)$ be a smooth function, $\theta(\xi) \equiv 0$ for $|\xi| \leqq 1/2$, and $\theta(\xi) \equiv 1$ for $|\xi| \geqq 1$. Let $m(\xi)=m_{\alpha,\beta}(\xi)=\theta(\xi)\exp(i|\xi|^\alpha)|\xi|^{-\beta}$, with $\alpha>1$ and $\beta+\dfrac{\alpha}{2}>1$. Define the multiplier operator Tf by $(Tf)\hat{}=m\hat{f}$. It is shown in [14] that $Tf=K*f$ where $K(t)=K_{a,b}(t)+h(t)$, with $a=\dfrac{\alpha}{\alpha-1}$, $b=\dfrac{\beta-1+\alpha/2}{\alpha-1}$, and $|h(t)| \leqq c(1+|t|)^{-(1+\varepsilon)}$. As above,

$$|Tf(x)| = |K*f(x)| \leqq |K_{a,b} * f(x)|+c|f^*(x)|.$$

If $m(\xi)=m_{\alpha,\beta}(\xi)=\theta\left[\dfrac{1}{\xi}\right]\exp(i|\xi|^\alpha)|\xi|^{-\beta}$, with $\alpha<0$ and $\beta+\dfrac{\alpha}{2}<1$, it can be shown that $K(t)=K_{a,b}(t)+h(t)$, with a, b, and $h(t)$ defined as above. Thus, Theorem 5.1 extends to these operators. For the multipliers, we need $0 \leqq \beta \leqq \alpha/2$ when $\alpha>1$ and $\alpha/2 \leqq \beta \leqq 0$ when $\alpha<0$ so that $b \leqq 1$ and $a/2+b \geqq 1$. Notice that we can only guarantee norm inequalities for A_p weights because of the presence of the maximal function.

Proposition 5.2. *Let* $a \neq 1$, $b<1$, $\frac{a}{2}+b \geqq 1$, $\frac{a}{a+b-1} \leqq p \leqq \frac{a}{1-b}$ *and* $\alpha = p(b-1)/a+\min[p-1, 1]$. *Suppose* w *is a non-negative weight such that for any* $y \in \mathbf{R}$,

$$\|K_{a,1+iy} * f\|_{2,w} \leqq B(1+|y|)\|f\|_{2,w}.$$

Then

$$\|K_{a,b+iy} * f\|_{p,w^\alpha} \leqq C(1+|y|)\|f\|_{p,w^\alpha}.$$

Proof. Let $K_z(t) = \exp(i|t|^a)(1+|t|)^{-(1+iy+a(z-1)/2)}$ and set $U_z f(x) = w^{z/2}(x)\int K_z(x-t)w^{-z/2}(t)f(t)\,dt$. By Lemma 1 of [8], the Fourier transform of $K_{i\gamma}(t)$ is bounded by $B(1+|y|+|\gamma|)$, with B independent of γ. This implies that for all γ,

$$\|U_{i\gamma} f\|_2 \leqq B(1+|y|)(1+|\gamma|)\|f\|_2.$$

By hypothesis,

$$\|U_{1+i\gamma} f\|_2 \leqq C(1+|y|)(1+|\gamma|)\|f\|_2,$$

so that by interpolation of analytic families of operators, we get

$$\|U_\theta f\|_2 \leqq C(1+|y|)\|f\|_2, \quad 0 \leqq \theta \leqq 1. \tag{5.1}$$

Setting $b = 1+a(\theta-1)/2$ implies $\theta = 2(b-1)/a+1 = \alpha$. Rewriting (5.1) and using the definition of U_θ gives

$$\|K_{a,b+iy} * f\|_{2,w^\alpha} \leqq C(1+|y|)\|f\|_{2,w^\alpha},$$

which proves the result for $p=2$.

To handle arbitrary p, $\frac{a}{a+b-1} \leqq p \leqq 2$, we use interpolation with change of measures between the weighted L^2 case above and (1.2) with $p = \frac{a}{a+b-1}$. This implies

$$\|K_{a,b+iy} * f\|_{p,w^\alpha} \leqq C(1+|y|)\|f\|_{p,w^\alpha}$$

with $\frac{a}{a+b-1} \leqq p \leqq 2$ and $\alpha = [2(b-1)/a+1]\frac{1}{p}\left[\frac{a-p(a+b-1)}{a-2(a+b-1)}\right]p = p(b-1)/a+p-1$. This completes the proof of the theorem for $p \leqq 2$. The remainder of the theorem follows the same argument using (1.2) with $p = a/(1-b)$.

We now consider weights of the form $w(x) = (1+|x|)^\alpha$ and prove

Theorem 5.3. *Let* $a>1$, $b<1$, $\frac{a}{2}+b \geqq 1$, $\frac{a}{a+b-1} \leqq p \leqq \frac{a}{1-b}$, *and* $w(x) = (1+|x|)^\alpha$. *If*

i) $\frac{a}{a+b-1} \leqq p \leqq 2$ *and* $a-2+(2-a-b)p \leqq \alpha \leqq a(p-1)+p(b-1)$

or

ii) $$2 \leqq p \leqq \frac{a}{1-b} \quad \textit{and} \quad -a+p(1-b) \leqq \alpha \leqq bp+a-2$$

then

$$\|K_{a,b+iy}*f\|_{p,w} \leqq C(1+|y|)\|f\|_{p,w}.$$

Proof. The ranges on α in i) and ii) are dual to each other. By interpolation with change of measures, it is enough to prove the result for $\alpha=a(p-1)+p(b-1)$, and $\alpha=bp+a-2$. By Theorem 3.2, we can set $w(x)=(1+|x|)^a$ in Proposition 5.2 which proves the result for $\alpha=a(p-1)+p(b-1)$. The other value of α is contained in Lemma 4.5.

Theorems 4.1 and 5.3 contain the sufficiency of the range of α in Theorem 2. The necessity of the range is shown in the next section.

6. Necessary conditions for Theorem 2

Our first result shows that for these kernels a weight which is zero on a set of positive measure is zero almost everywhere.

Lemma 6.1. *Let $a>0$ and $b\leqq 1$. Set $E=\{x: w(x)=0\}$. If $|E|\neq 0$, then either $w(x)=0$ almost everywhere or $K_{a,b+iy}$ does not define a bounded operator on L^p_w for any p.*

Proof. Assume $|E^c|\neq 0$ ($E^c=\{x\notin E\}$). Let $B_R=\{|x|<R\}$. We need only consider two cases:

1) There is an $R>0$ such that $|E\cap B_R|\neq 0$ and $|E^c\cap(B_{2R})^c|\neq 0$;

2) There exist $\varepsilon, R>0$ such that $|E\cap B_\varepsilon|=0$ and $w(x)=0$ for almost every $|x|>R$.

To see this, let $R_0=\inf\{R: |E\cap B_R|\neq 0\}$. If $R_0=0$, then $|E\cap B_R|\neq 0$ for all $R>0$. Since $|E^c\cap(B_{2R})^c|$ increases to $|E^c|$ as R approaches 0, we can choose an R satisfying 1). Next, suppose $R_0>0$ and no R satisfies 1). Then, there is an R such that $w(x)=0$ for almost every $|x|>R$. If $\varepsilon<R_0$, $|E\cap B_\varepsilon|=0$. Thus, R and ε satisfy 2).

Case 1. Without loss of generality, assume $|\{x>2R: w(x)\neq 0\}|\neq 0$. Choose $N>2R$ such that for $\mu=\max(4aN^{a-1}, 2a(2N+1)^{a-1}, 2|y|, 1)$ we have $|\{N<x<N+(1/\mu): w(x)\neq 0\}|\neq 0$. Let

$$f(x) = \chi(\{x\in E\cap B_R\})\exp(-i|N-x|^a)(1+|N-x|)^{iy}.$$

Then, for $N<x<N+(1/\mu)$,

$$|(K_{a,b+iy}*f)(x)| = \left|\int_{E\cap B_R} \frac{e^{i(x-t)^a}e^{-i(N-t)^a}(1+|N-t|)^{iy}}{(1+|x-t|)^{b+iy}}\,dt\right|$$

$$= \left|\int_{E\cap B_R} \frac{e^{i[\{(x-t)^a - y\ln(1+x-t)\}-\{(N-t)^a-y\ln(1+N-t)\}]}}{(1+|x-t|)^b}\,dt\right|.$$

Applying the Mean Value Theorem to the exponent of e and taking the real part of the integral, we get

$$|(K_{a,b+iy}*f)(x)| \geqq \left|\int_{E\cap B_R} \frac{\cos[\{a\xi^{a-1}-y/(1+\xi)\}(x-N)]}{(1+|x-t|)^b}\,dt\right|,$$

for some ξ, $N-t<\xi<x-t$. Since $|t|<R<N/2$ and $N<x<N+(1/\mu)$, we have $|x-t|<2N+1$ and $N/2<\xi<2N+1$. If $a<1$,

$$|a\xi^{a-1}(x-N)| \leqq a(N/2)^{a-1}(1/\mu) \leqq (2aN^{a-1})/(4aN^{a-1}) = 1/2;$$

if $a>1$,

$$|a\xi^{a-1}(x-N)| \leqq a(2N+1)^{a-1}(1/\mu) \leqq \big(a(2N+1)^{a-1}\big)/\big(2a(2N+1)^{a-1}\big) = 1/2.$$

Finally, if $y\neq 0$,

$$\left|\frac{y}{1+\xi}(x-N)\right| \leqq |y|\frac{1}{\mu} \leqq |y|\frac{1}{2|y|} = 1/2.$$

Thus, for $N<x<N+(1/\mu)$, with $M=\inf\{(1+|x-t|)^{-b}\colon t\in E\cap B_R\}>0$,

$$|(K_{a,b+iy}*f)(x)| \geqq M\cos(1)|E\cap B_R| > 0.$$

By the definition of f, $\|f\|_{p,w}=0$ while $\|(K_{a,b+iy}*f)\|_{p,w}\neq 0$.

Case 2. Choose $R>2$ such that $R<|x|<2R$ implies $w(x)=0$ almost everywhere. Let $f(x)=\chi(\{R<|x|<2R\})\exp(-i|x|^a)(1+|x|)^{iy}$ and fix $\varepsilon<(2\cdot 3^{|a-1|}aR^{a-1}+2|y|+1)^{-1}$. If $|x|<\varepsilon$, arguing as above,

$$|(K_{a,b+iy}*f)(x)| = \left|\int_{\{R<|t|<2R\}} \frac{e^{i|x-t|^a-i|t|^a}(1+|t|)^{iy}}{(1+|x-t|)^{b+iy}}\,dt\right|$$

$$\geqq \left|\int_{\{R<|t|<2R\}} \frac{\cos[\{a|\xi-t|^{a-1}-y/(1+|\xi-t|)\}|x|]}{(1+|x-t|)^b}\,dt\right|,$$

for some ξ, $|\xi|<|x|<\varepsilon$. Since $\varepsilon<1$ and $|t|$ is equivalent to R, $R/3<|t-\xi|<3R$ and $a|t-\xi|^{a-1}|x|\leqq a\cdot 3^{|a-1|}R^{a-1}/(2\cdot 3^{|a-1|}aR^{a-1})=1/2$ and $\big|y/(1+|\xi-t|)\big|\,|x|\leqq |y|/2|y|=1/2$. For $|x|<\varepsilon$, with $M=\inf\{(1+|x-t|)^{-b}\colon R<|t|<2R\}$,

$$|(K_{a,b+iy}*f)(x)| \geqq M\cos(1)2R > 0.$$

The proof of Case 2 is completed as above. This proves the lemma.

In the previous proof, the main steps were to choose a function of the form $f(x)=\chi_E(x)\exp(-i|N-x|^a)(1+|N-x|)^{iy}$, for an appropriate set E and real number N, use the Mean Value Theorem on the exponent of e in the integral defining $K_{a,b+iy}*f$, and take the real part of the integral. Changing the argument slightly, we prove

Lemma 6.2. *Let* $1\leqq p<\infty$, $0<a$, *and* $w(x)$ *positive and locally integrable. Suppose* $K_{a,b+iy}$ *defines a bounded operator on* L^p_w. *There exists a constant* $d=d(a,|y|)\leqq 1$ *such that for any positive* R *and* δ *satisfying* $R^{1-a}/2\leqq\delta\leqq R/2$, *if* $S(R,\delta)=\{R\leqq|x|\leqq R+(dR^{2-a}/\delta)\}$ *then*

$$\delta^p\int_{S(R,\delta)} w(x)(1+|x|)^{-bp}\,dx \leqq C\int_{\{|x|\leqq\delta\}} w(x)\,dx$$

Proof. Fix R and δ as above. Assume $y=0$ and consider $S^+=\{x\in S(R,\delta),\ x>0\}$. Set $f(x)=\chi(\{|x|\leqq\delta\})\exp(-i|R-x|^a)$. Then

$$\|K_{a,b}*f\|^p_{p,w}=\int\left|\int_{\{|t|\leqq\delta\}}\frac{e^{i|x-t|^a-i|R-t|^a}}{(1+|x-t|)^b}\,dt\right|^p w(x)\,dx \tag{6.1}$$

$$\geqq\int_{S^+}\left|\int_{\{|t|\leqq\delta\}}\frac{e^{iF(t)}}{(1+|x-t|)^b}\,dt\right|^p w(x)\,dx,$$

where $F(t)=|x-t|^a-|R-t|^a$.

Since $|t|\leqq\delta$ and $|x|>2\delta$, F is a differentiable function near the origin and $F(t)=F(0)+F'(\xi)t$ where ξ is between 0 and t. By assumption, $dR^{2-a}/\delta\leqq 2R$. It follows that $|x-\xi|$ and $|R-\xi|$ are bounded above by $6R$ and below by $R/2$. Thus

$$|F'(\xi)t| \leqq |a(a-1)\int_{R-\xi}^{x-\xi} s^{a-2}\,ds|\delta \leqq B(a)R^{a-2}(x-R)\delta \leqq B(a)\,d. \tag{6.2}$$

Choosing $d\leqq 1$ sufficiently small, $|F'(\xi)t|\leqq 1$.

By the above estimate, since $e^{iF(0)}$ is independent of t and has modulus 1, taking the real part of the inner integral in (6.1) yields

$$\|K_{a,b}*f\|^p_{p,w}\geqq\int_{S^+}\left|\int_{\{|t|\leqq\delta\}}\cos(1)(1+|x|)^{-b}\,dt\right|^p w(x)\,dx$$

$$\geqq C\delta^p\int_{S^+} w(x)(1+|x|)^{-bp}\,dx.$$

For $\{x\in S(R,\delta),\ x<0\}$, set $f(x)=\chi(\{|x|\leqq\delta\})\exp(-i|R+x|^a)$ and repeat the argument. The norm inequality for $K_{a,b}$ implies

$$\delta^p\int_{S(R,\delta)} w(x)(1+|x|)^{-bp}\,dx \leqq C\int_{\{|x|\leqq\delta\}} w(x)\,dx,$$

as we wished to show. For $y\neq 0$, multiply f by $(1+|R\pm x|)^{iy}$ and argue as above.

Let $a>1$ and $w(x)=(1+|x|)^\alpha$. A consequence of Lemma 6.2 is that for $\alpha>\min[a(p-1)-p(1-b), bp-2+a]$, $K_{a,b+iy}$ does not define a bounded operator on L^p_w. The case $\alpha>a(p-1)-p(1-b)$ is excluded by setting $\delta=R^{1-a}$ and letting R approach infinity; $\alpha>bp-2+a$ by setting $\delta=1$ and letting R approach infinity. By duality, $K_{a,b+iy}$ does not define a bounded operator on L^p_w if $\alpha<\max[-a+p(1-b), a-2+(2-a-b)p]$. This completes the proof of Theorem 2.

The previous result is invariant under translation. Repeating the argument and using the fact that for $a<1$ the integral in (6.2) is convergent at infinity, we have

Lemma 6.3. *Let* $1\leqq p<\infty$, $0<a<1$, *and* $w(x)$ *positive and locally integrable. Suppose* $K_{a,b+iy}$ *defines a bounded operator on* L^p_w. *If* I *is an interval with center* x_I *and* $\tilde{I}=\{|x-x_I|<|I|^{1/(1-a)}\}$, *then*

$$|I|^p\int_{\mathbf{R}-\tilde{I}} w(x)(1+|x-x_I|)^{-bp}\,dx \leqq C\int_I w(x)\,dx.$$

Taking $I=[-1,1]$ and $b=1$, it follows that $K_{a,1+iy}$ defines a bounded operator on $L^p_{(1+|x|)^\alpha}$ if and only if $-1<\alpha<p-1$. That is, for $0<a<1$ and $b=1$, the range on α for the weights $(1+|x|)^\alpha$ is exactly the A_p range.

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