Determination of invariant convex cones in simple Lie algebras

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1. Introduction

In this paper, the determination of all convex cones in a real simple Lie algebra invariant under the adjoint group will be essentially reduced to an abelian problem. Specifically, if $\mathbf{h}$ is a compact Cartan subalgebra in the simple Lie algebra $\mathbf{g}$, the mapping $C \to C \cap \mathbf{h}$ is shown one-to-one from the class of (open or closed) invariant convex cones C in $\mathbf{g}$, onto an explicitly described class of cones in $\mathbf{h}$ invariant under the Weyl group. All such cones C in $\mathbf{g}$ have open dense interiors, and each such interior element, whether regular or not, is contained in a unique maximal compact subalgebra. The orthogonal projection of the orbit of such an interior element onto a compact Cartan subalgebra is determined explicitly, extending to these noncompact orbits known results for projections of compact group orbits. The above correspondence $C \to C \cap \mathbf{h}$ is shown to preserve the duality relation between cones, and the class of self-dual cones in the classical algebras corresponding to convex quadratic cones in the compact Cartan subalgebras is determined.

It is well known that the Poincaré group, the symmetry group of Minkowski space, contains a four-dimensional invariant semigroup, that of all vector displacements into the "future". This semigroup is the exponential of a corresponding invariant convex cone in the Lie algebra, which is precisely the cone of generators that are carried into nonnegative self-adjoint operators by infinitesimal unitary representations of "positive energy", such as those associated with certain hyperbolic partial differential equations (for example, Maxwell's equations).

This situation is not peculiar to the Poincaré group, but is applicable to a variety of other groups. For example, the universal cover $\tilde{G}$ (locally identical

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to $O(2, 4)$) of the conformal group of Minkowski space M, acts on the universal cover $\tilde{M}$ of conformal space (i.e. conformally compactified Minkowski space), extending the usual action of the Poincaré group on M. Conformally invariant wave equations extend naturally to $\tilde{M}$. It may be seen that the semigroup within $\tilde{G}$ of all future displacements of $\tilde{M}$ contains the $\tilde{G}$-invariant semigroup generated by the above positive-energy semigroup; otherwise the relation between the two semigroups is not at all clear a priori. The same situation develops for the conformal group, locally $O(2, n+1)$, of a vector space with a flat metric of signature $(1, n)$.

Thus the general study of invariant convex cones in Lie algebras appears of interest from a theoretical physical standpoint. This work determines the structure of invariant convex cones in the Lie algebras of all simple Lie groups. (According to a result of Kostant cited by Segal in [14], only those simple Lie groups of hermitian symmetric type can admit a nontrivial such cone.) The classification is reduced to the determination of certain convex cones in an abelian subalgebra that are invariant under a finite reflection group (Weyl group). It was noted independently by Vinberg [15], and earlier in [14], that there are always unique (up to sign) minimal and maximal causal cones; all others lie between these. These extremal cones and others were determined more explicitly for the classical matrix algebras in [12].

It follows, for example, that in the above cases of conformal groups, the future-displacement semigroups and the above "positive-energy" semigroup are indeed distinct, in fact maximally so, being generated by the maximal and minimal, respectively, invariant convex cones in the Lie algebra.* Another corollary to this work, is that the "cone of positivity" of a unitary representation of a simple Lie group, defined to be all elements of the Lie algebra carried to nonnegative self-adjoint operators by the infinitesimal representation, is determined straightforwardly by the restriction of the representation to any maximal essentially compact subgroup.

On the other hand, the symplectic Lie algebras have unique (up to sign) closed invariant convex cones (cf. also [15]). The structure of these symplectic cones (especially a generalization of the present Theorem 4 to the infinite-dimensional case) has recently been utilized to determine a unique probabilistic quantization (i.e. a unique invariant vacuum state) for a wide class of time-dependent wave equations [11], [13]. Another important ingredient in this development is an adaptation of the stability theory of M. G. Krein and collaborators [8].

Some of the results presented here (essentially Lemma 6 and parts of section 3)

* In this direction, it is interesting to note that the spin 0, 1/2, and 1 essentially conventional mass 0 representations of the universal cover of the conformal group of four-dimensional space-time, carry every generator of the forward-displacement semigroup into a positive self-adjoint operator, but that for higher spins, this is the case only for generators in smaller cones dependent on the spin [16].

were earlier obtained by Vinberg [15]. In addition, a classification theorem equivalent to the present Theorem 2 has recently been obtained by Olshansky [10], by a completely different method of proof involving the holomorphic discrete series. I thank Gert Heckman for his suggestions towards the "$\supseteq$" inclusion in Theorem 1.

2. Notation and preliminary lemmas

Let $\mathbf{g}$ be a simple hermitian symmetric Lie algebra with adjoint group G and Killing form $B(\cdot,\ \cdot)$. Let $\mathbf{g}=\mathbf{k}+\mathbf{p}$ be a fixed Cartan decomposition with Cartan involution θ, and define the positive definite inner product

$$\langle\cdot,\ \cdot\rangle = -B(\cdot, \theta\cdot)$$

on $\mathbf{g}$. Let K be the subgroup corresponding to $\mathbf{k}$.

Let $\mathbf{h}$ be a maximal abelian subalgebra of $\mathbf{k}$, and let $\mathbf{g}^{\mathbf{C}}$, $\mathbf{h}^{\mathbf{C}}$ be the complexifications of $\mathbf{g}, \mathbf{h}$. Let Δ be the roots of $\mathbf{g}^{\mathbf{C}}$ with respect to $\mathbf{h}^{\mathbf{C}}$, and choose root vectors $H_\alpha \in i\mathbf{h}$ for $\alpha\in\Delta$ as usual, so that $\alpha(H_\alpha)=2$. Also for each $\alpha\in\Delta$ choose a vector E_α in the root space corresponding to α so that $[E_\alpha, E_{-\alpha}]=H_\alpha$ and $i(E_\alpha+E_{-\alpha})$, $E_\alpha-E_{-\alpha}\in\mathbf{k}+i\mathbf{p}$. Given $\alpha\in\Delta$ let σ_α be the associated reflection in $\mathbf{h}$.

Let $\mathbf{c}$ be the one-dimensional center of $\mathbf{k}$; we have $\mathbf{c}\subseteq\mathbf{h}$. Choose $Z\in\mathbf{c}$ and compatible orderings in the duals of $i\mathbf{c}$ and $i\mathbf{h}$ so that, as usual, a root α is positive and noncompact (compact) if and only if $\alpha(iZ)=1$ (resp. $\alpha(Z)=0$). Let W_K be the group generated by the σ_α with α compact. Let Q_+ be the set of positive noncompact roots, and define for $\alpha\in Q_+$

$$h_\alpha = -iH_\alpha \in \mathbf{h},\quad X_\alpha = E_\alpha + E_{-\alpha},\quad Y_\alpha = -i(E_\alpha - E_{-\alpha}),$$

so that

$$[Z, X_\alpha] = Y_\alpha,\quad [Z, Y_\alpha] = -X_\alpha,\quad [X_\alpha, Y_\alpha] = -2h_\alpha,$$
$$[H, X_\alpha] = i\alpha(H)Y_\alpha,\quad [H, Y_\alpha] = -i\alpha(H)X_\alpha$$

for all $H\in\mathbf{h}$. We note also $[X_\alpha, h_\alpha\pm Y_\alpha]=\mp 2(h_\alpha\pm Y_\alpha)$.

Now choose some maximal strongly orthogonal system $\Sigma_0\subseteq Q_+$. Let $Z_0=\frac{1}{2}\sum_{\alpha\in\Sigma_0} h_\alpha$, and let $\mathbf{h}^-$ and $\mathbf{a}$ be the real spans of the h_α and X_α, respectively, where $\alpha\in\Sigma_0$. Then $Z-Z_0$ is orthogonal to $\mathbf{h}^-$ [6], and if $X=\sum_{\alpha\in\Sigma_0} t^\alpha X_\alpha \in \mathbf{a}$, then

$$\text{(1)}\qquad \operatorname{Ad}(\exp X)Z = Z - Z_0 + \tfrac{1}{2}\textstyle\sum_{\alpha\in\Sigma_0}\big((\cosh 2t^\alpha)h_\alpha - (\sinh 2t^\alpha)Y_\alpha\big).$$

Recall that there are at most two root lengths in Δ.

Lemma 1. *All $\alpha\in\Sigma_0$ are long, and any short $\beta\in Q_+$ is the average of two elements of Σ_0.*

Proof. By Theorem 2(b) in [9], all $\alpha\in\Sigma_0$ have the same length; cf. also [1]. Now take any $\beta\in Q_+$ such that $\langle\beta,\beta\rangle\neq\langle\alpha,\alpha\rangle$ for all $\alpha\in\Sigma_0$. By Lemma 11 of [3] there exists $\alpha\in\Sigma_0$ such that $\langle\beta,\alpha\rangle>0$. If $\langle\beta,\beta\rangle>\langle\alpha,\alpha\rangle$, then $\beta-\alpha$ and $\beta-2\alpha$ are roots, which contradicts Lemma 12 in [3]: for any roots $\delta\in\Delta, \alpha\in\Sigma_0, \delta\pm\alpha$ cannot both be roots.

Thus all $\alpha\in\Sigma_0$ are long, β is short, and $\alpha-2\beta$ is a root. By Lemma 16 in [3], the restriction of β to $\mathbf{h}^-$ is either $\frac{1}{2}\alpha$ or $\sigma=\frac{1}{2}\alpha+\frac{1}{2}\gamma$ for some $\gamma\in\Sigma_0$. However, the former cannot occur (otherwise $\alpha-2\beta\in -Q_+$ would restrict to 0 on $\mathbf{h}^-$), and it follows that $\beta=\sigma$ since β cannot be any longer than σ. Q.e.d.

Lemma 2. *All long roots in Q_+ are conjugate under W_K.*

Proof. Given $\alpha\in Q_+$ long, there exists $\beta\in\Sigma_0$ as above such that $\langle\alpha,\beta\rangle>0$. If $\alpha\neq\beta$ then $\gamma=\alpha-\beta$ is compact and $\sigma_\gamma(\alpha)=\beta$. Q.e.d.

3. Minimal and maximal cones

Take any long $\alpha\in Q_+$, and let

$$\mathcal{O}_+ = \mathrm{Ad}(G)(h_\alpha+X_\alpha),$$

an orbit of a "highest weight vector". By Lemma 2, $\mathcal{O}_+$ is independent of the particular α chosen, and $\mathcal{O}_+=\mathrm{Ad}(G)(h_\alpha+rX_\alpha+sY_\alpha)$ for any r, s such that $r^2+s^2=1$. We have not defined the restricted roots, but it is clear that there exist subgroups A, N of G as usual so that $G=KAN$, $\mathrm{Ad}(AN)(h_\alpha+X_\alpha)=\mathbf{R}^+(h_\alpha+X_\alpha)$, and $\mathrm{Ad}(G)(h_\alpha+X_\alpha)=\mathbf{R}^+\mathrm{Ad}(K)(h_\alpha+X_\alpha)$ has closure $\mathcal{O}_+\cup\{0\}$.

Definitions. Let

$$c_m = \left\{\textstyle\sum_{\alpha\in Q_+} t^\alpha h_\alpha\colon\ t^\alpha \geqq 0\right\},$$

$$c_M = \{X\in\mathbf{h}\colon \langle X, h_\alpha\rangle \geqq 0 \quad \text{for all} \quad \alpha\in\mathcal{O}_+\},$$

$$C_{\mathrm{Min}} = \text{the closed convex cone generated by } \mathrm{Ad}(G)Z,$$

and

$$C_{\mathrm{Max}} = \{X\in\mathbf{g}\colon \langle X, Y\rangle \geqq 0 \quad \text{for all} \quad Y\in\mathcal{O}_+\}.$$

It follows from Kostant's theorem in [14] (mentioned in § 1) that C_{Min} is a proper cone in $\mathbf{g}$, and that any invariant closed convex cone in $\mathbf{g}$ contains either Z or $-Z$. The linear span of c_m is $\mathbf{h}$, and $c_m\subseteqq c_M$ because $\alpha+\beta\notin\Delta$ for all $\alpha, \beta\in Q_+$. Since $\mathcal{O}_+$ is the set of rays generated by a compact set,

$$(C_{\mathrm{Max}})^0 = \{X\in\mathbf{g}\colon \langle X, Y\rangle > 0 \quad \text{for all} \quad Y\in\mathcal{O}_+\}.$$

We recall some general facts about convex sets [2]. Any convex cone C has a nonempty interior C^0 relative to the space V it spans, and we have $\overline{C^0}=\overline{C}$,

$(\bar{C})^0 = C^0$, hence $C^0 + \bar{C} = C^0$, where $\bar{C}$ denotes the closure of C. (Note that since $\mathbf{g}$ is simple, the space spanned by any nonzero G-invariant convex cone equals $\mathbf{g}$.) Finally, $(C^*)^* = \bar{C}$, where $C^* = \{f \in V' : f(x) \geqq 0$ for all $x \in C\}$ is the dual cone to C.

Lemma 3. *$Z \in (c_m)^0$ (interior relative to $\mathbf{h}$), and the convex cone generated by $\mathrm{Ad}(G)Z$ is open in $\mathbf{g}$.*

Proof. If $H \in \mathbf{h}$,

$$\langle Z, H\rangle = -\mathrm{Tr}(\mathrm{ad}\, H\, \mathrm{ad}\, Z) = 2\sum_{\alpha \in Q_+} i\alpha(H) = \sum_{\alpha \in Q_+} c_\alpha \langle h_\alpha, H\rangle \quad \text{where} \quad c_\alpha > 0,$$

$$\text{so } Z = \sum_{\alpha \in Q_+} c_\alpha h_\alpha.$$

Since the h_α, $\alpha \in Q_+$ span $\mathbf{h}$, $Z \in (c_m)^0$.

For the second statement, let $\mathscr{L}$ denote the convex cone generated by $\mathrm{Ad}(G)Z$. It suffices to show that $Z \in \mathscr{L}^0$. By Lemmas 1, 2 and (1), $\mathscr{L}$ includes $Z + c_m$; since $Z \in (c_m)^0$, $\mathscr{L}$ also contains a neighborhood of Z in $\mathbf{h}$, hence a neighborhood of Z in $\mathbf{k}$. Finally, the tangent space of $\mathrm{Ad}(G)Z$ at Z is $\mathbf{p}$, and it follows that $\mathscr{L}$ is a neighborhood of Z in $\mathbf{g}$. Q.e.d.

Lemma 4. (a) *$(C_{\mathrm{Min}})^0$ is the convex cone generated by* $\mathrm{Ad}(G)Z$,
(b) *$C_{\mathrm{Min}} - \{0\}$ is the convex hull of $\mathcal{O}_+$, and*
(c) $C_{\mathrm{Min}} \subseteqq C_{\mathrm{Max}}$.

Proof. (a) follows from $(\bar{\mathscr{L}})^0 = \mathscr{L}^0$ and Lemma 3. By the earlier remarks on $\mathcal{O}_+$, for (b) it suffices to show that $Z \in$ convex hull of $\mathcal{O}_+$, and $h_\alpha + Y_\alpha \in C_{\mathrm{Min}}$ if $\alpha \in \Sigma_0$. Now we showed in Lemma 3 that $Z \in c_m$, and the convex hull of $\mathcal{O}_+$ contains all h_α, $\alpha \in Q_+$ by Lemmas 1, 2. On the other hand, by (1)

$$2 \lim_{t \to -\infty} (\cosh 2t)^{-1} \mathrm{Ad}(\exp tX_\alpha) Z = h_\alpha + Y_\alpha.$$

Finally, $Z \in C_{\mathrm{Max}}$ because if $\beta \in \Sigma_0$ and $a = \exp \sum_{\alpha \in \Sigma_0} t^\alpha X_\alpha$,

$$\left\langle Z, \mathrm{Ad}(a^{-1})(h_\beta + Y_\beta)\right\rangle = \left\langle (\cosh 2t^\beta)\tfrac{1}{2} h_\beta - (\sinh 2t^\beta)\tfrac{1}{2} Y_\beta,\, h_\beta + Y_\beta\right\rangle$$
$$= \langle h_\beta, h_\beta\rangle \exp(-2t^\beta) > 0. \qquad \text{Q.e.d.}$$

Corollary 1. *If C is a G-invariant convex cone, $(C \cap \mathbf{h})^0 = C^0 \cap \mathbf{h}$.*

Proof. $\supseteqq$ is immediate. If $X \in (C \cap \mathbf{h})^0$, then (we may assume) $X - \varepsilon Z \in C$ for some $\varepsilon > 0$, and $Z \in C^0$ by Lemma 4(a). Thus $X \in C + C^0 \subseteqq C^0$. Q.e.d.

The proof of Lemma 4 and duality also show

Corollary 2. $c_M \subseteqq C_{\mathrm{Max}} \cap \mathbf{h}$ *and* $c_m \subseteqq C_{\mathrm{Min}} \cap \mathbf{h}$, *and*

Corollary 3. *C_{Min} and C_{Max} are, respectively, minimal and maximal closed invariant convex cones in* $\mathbf{g}$.

4. Ellipticity of the open cones and a projection lemma

In fact the inclusions in Corollary 2 are equalities, but this fact requires requires Kostant's generalization [7] of a result of Horn [5], listed here as $[H-K]$. To state this result and introduce notation for later use, we let $\mathbf{h}(X), \mathbf{k}(X), \mathbf{p}(X)$ be the $\langle\cdot,\cdot\rangle$-orthogonal projections of X onto $\mathbf{h}, \mathbf{k}, \mathbf{p}$, respectively, and write $|p(X)|=\langle\mathbf{p}(X), \mathbf{p}(X)\rangle^{1/2}$. In these terms $[H-K]$ states that given $H\in\mathbf{h}$, $\{\mathbf{h}(\mathrm{Ad}(k)H)\colon k\in K\}$ is equal to the convex hull of the W_K-orbit of H.

Corollary 4. *The inclusions in Corollary 2 are equalities.*

Proof. The remaining two inclusions are equivalent to $\langle\mathrm{Ad}(k)X, h_\alpha+Y_\alpha\rangle\geqq 0$ for all $k\in K$, $\alpha\in\Sigma_0$, and $X\in c_M$, which follows from $[H-K]$. Q.e.d.

Definitions. Let $X, Y\in\mathbf{h}$. Say X is in the *noncompact convex hull* of Y if $X=L+V$, where $L\in c_m$ and V is in the convex hull of the W_K-orbit of Y.

Lemma 5. *If* $X\in\mathbf{g}-\mathbf{k}$ *and* $\mathbf{k}(X)$ *is conjugate under* K *to* $H\in(c_M)^0$, *then there exists* Y *in the G-orbit of* X *such that* (a) $|\mathbf{p}(Y)|<|\mathbf{p}(X)|$, *and* (b) $\mathbf{h}(X)$ *is in the noncompact convex hull of* $\mathbf{h}(Y)$.

Proof. There exists $k\in K$ such that $\mathbf{k}(\mathrm{Ad}\,(k)X)\in h$, and by $[H-K]$ it suffices to consider the case where $\mathbf{k}(X)\in\mathbf{h}$. Therefore let

$$X = H+\textstyle\sum_{\alpha\in Q_+} c_\alpha X_\alpha + d_\alpha Y_\alpha$$

where $\alpha(iH)>0$ for all $\alpha\in Q_+$.

Take $U=\sum_{\beta\in Q_+} e_\beta X_\beta+f_\beta Y_\beta\in\mathbf{p}$, and compute

$$[U, X] = \textstyle\sum_{\beta\in Q_+} i\beta(H)(f_\beta X_\beta-e_\beta Y_\beta)+M+\sum_{\alpha\in Q_+}(e_\alpha d_\alpha-c_\alpha f_\alpha)2iH_\alpha,$$

where $M\in\mathbf{k}$ and $\mathbf{h}(M)=0$. Taking now $e_\beta=d_\beta/(i\beta(H))$, $f_\beta=-c_\beta/(i\beta(H))$, we have

$$[U, X] = -\textstyle\sum_{\alpha\in Q_+}(c_\alpha X_\alpha+d_\alpha Y_\alpha)+M-W,$$

where

$$W = \textstyle\sum_{\alpha\in Q_+} 2(i\alpha(H))^{-1}(c_\alpha^2+d_\alpha^2)\, h_\alpha\in c_m.$$

It follows that

$$\mathrm{Ad}\,(\exp\varepsilon U)X = H+(1-\varepsilon)\textstyle\sum_{\alpha\in Q_+}(c_\alpha X_\alpha+d_\alpha Y_\alpha)+\varepsilon M-\varepsilon W+O(\varepsilon^2).$$

Thus (a) and (b) will follow (taking $Y=\mathrm{Ad}\,(\exp\varepsilon U)X$, $\varepsilon>0$ small), provided we can show that

$$H-\mathbf{h}(\mathrm{Ad}\,(\exp\varepsilon U)X)\in c_m \tag{2}$$

for $\varepsilon>0$ sufficiently small.

c_m is generated by finitely many vectors, so the same must be true of the dual cone c_M. Thus there exist $V_j\in\mathbf{h}$, $j=1, \ldots, n$, such that $X\in c_m$ if and only if

$\langle X, V_j\rangle \geqq 0$, $j=1, \ldots, n$. Recall the nonzero $W \in c_m$ earlier, and suppose that $\langle W, V_j\rangle = 0$ for $j=1, \ldots, r$, where $r<n$. Then $j \leqq r$ if and only if (for all $\alpha \in Q_+$) $c_\alpha = d_\alpha = 0$ and/or $\alpha(V_j)=0$, which in turn happens if and only if $[U, V_j]=0$.

Thus if $j \leqq r$,

(3) $$\langle H - \mathbf{h}(\mathrm{Ad}(\exp \varepsilon U)X),\ V_j\rangle$$
$$= -B(X, V_j) + B(\mathrm{Ad}(\exp \varepsilon U)X, V_j) = -B(X, V_j) + B(X, \mathrm{Ad}(\exp - \varepsilon U)V_j) = 0$$

identically in ε. If $j>r$ then (3) is $p_j\varepsilon + O(\varepsilon^2)$ for $p_j>0$, which proves (2). Q.e.d.

Lemma 6. *Each orbit in* $(C_{\mathrm{Max}})^0$ *is closed and intersects* $(c_M)^0$.

Proof. If $\langle X, Y\rangle > 0$ for all $Y \in O_+$, then there exists $k>0$ such that

(4) $$\langle X, Y\rangle \geqq k\langle Y, Y\rangle^{1/2} \quad \text{for all} \quad Y \in O_+.$$

If now

(5) $$\mathrm{Ad}(g_m)X \to Q \in \mathbf{g} \quad \text{as} \quad m \to \infty,$$

we write $g_m = k_m a_m t_m$ according to $G = K\bar{A}^+K$ so that $\mathrm{Ad}(g_m)X = \mathrm{Ad}(k_m)\mathrm{Ad}(a_m)X_m$, where X_m also satisfies (4). We will show that (5) implies that the a_m remain bounded so that some subsequence of $\{g_m\}$ converges.

$$\text{Since} \quad a_m \in \bar{A}^+, \quad \text{let} \quad a_m = \textstyle\prod_{\alpha \in \Sigma_0} \exp t_m^\alpha X_\alpha, \quad \text{where} \quad t_m^\alpha \geqq 0.$$

Now take any fixed $\alpha \in \Sigma_0$, and let $X_m = c_m h_\alpha + d_m X_\alpha + e_m Y_\alpha + W_m$ where W_m is perpendicular to $h_\alpha, X_\alpha, Y_\alpha$. (4) implies that

$$c_m - (d_m^2 + e_m^2)^{\frac{1}{2}} \geqq k_1 > 0 \quad \text{for all } m.$$

One computes that

$$\mathrm{Ad}(a_m)X_m = [(\cosh 2t_m^\alpha)e_m - (\sinh 2t_m^\alpha)c_m]Y_\alpha + U_m$$

where $\langle U_m, Y_\alpha\rangle = 0$, and (using $t_m^\alpha \geqq 0$) estimates

(6) $$(\cosh 2t_m^\alpha)e_m - (\sinh 2t_m^\alpha)c_m$$
$$\leqq (\sinh 2t_m^\alpha)e_m + |e_m| - (\sinh 2t_m^\alpha)c_m \leqq -k_1 \sinh 2t_m^\alpha + |e_m|.$$

Now $|e_m|$ and (6) must remain bounded as $m \to \infty$ (since the X_m and $\mathrm{Ad}(a_m)X_m$ are uniformly bounded) so all t_m^α, hence a_m, must remain bounded.

To complete the proof, let $X \in (C_{\mathrm{Max}})^0$, and partially order $\mathrm{Ad}(G)X$ by:

$Y_1 \geqq Y_2$ if and only if

(a) $|\mathbf{p}(Y_1)| \leqq |\mathbf{p}(Y_2)|$ and

(b) $\mathbf{h}(Y_2)$ is in the noncompact convex hull of $\mathbf{h}(Y_1)$.

Lemma 5 implies that $\geqq$-maximal elements of $\mathfrak{m} = \{Y \in \mathrm{Ad}(G)X \colon Y \geqq X\}$ must be in $\mathbf{k}$, and such exist by Zorn's lemma. ($\mathfrak{m}$ is compact since $\mathrm{Ad}(G)X$ is closed.) Thus there exists $H \in \mathbf{h}$ in the orbit of X. Finally, $H \in (c_M)^0$ by Cors. 1 and 4. Q.e.d.

Lemma 7. *For all* $H\in(c_M)^0$ *and* $g\in G$, $\mathbf{h}(\mathrm{Ad}(g)H)$ *is in the noncompact convex hull of* H.

Proof. As noted above, $(c_M)^0=(C_{\mathrm{Max}})^0\cap\mathbf{h}$. The argument of Lemma 6 implies that $\mathbf{h}(\mathrm{Ad}(g)H)$ is in the noncompact convex hull of some $H_1\in(c_M)^0$ to which H is conjugate under G. But then $\sigma(H)=H_1$ for some $\sigma\in W_K$. Q.e.d.

Let $\mathrm{Nch}(H)$ denote the noncompact convex hull of an $H\in\mathbf{h}$. Lemma 7 shows that the $\langle\cdot,\cdot\rangle$-orthogonal projection of the $\mathrm{Ad}(G)$-orbit of an $H\in(c_M)^0$ is contained in $\mathrm{Nch}(H)$. In fact, the reverse inclusion holds (Theorem 1), but this fact will not be needed to classify the invariant cones (Theorem 2). We remark also, that simple examples show that the conclusion of Lemma 7 fails generically if the $H\in\mathbf{h}$ there is in an open Weyl chamber not contained in $(c_M)^0$.

Theorem 1. *If* $H\in(c_M)^0$, *then*

$$\{\mathbf{h}(\mathrm{Ad}(g)H):\ g\in G\}=\mathrm{Nch}(H).$$

Proof. By Lemma 7, it suffices to prove "$\supseteq$". Let $L\in\mathrm{Nch}(H)$, so that $L=M+X$, for some $X\in c_m$ and M in the convex hull of $W_K(H)$. By [H−K] there exists $k\in K$ such that $Y=\mathrm{Ad}(k)H$ satisfies $\mathbf{h}(Y)=M$. To obtain a $Y_1\in\mathrm{Ad}(G)H$ such that $h(Y_1)=M+X$, we make certain successive applications of $\mathrm{Ad}(e^{tX_\alpha})$ (for $t\geqq 0$, $\alpha\in Q_+$) and $\mathrm{Ad}(e^{tZ})$ (recall that Z spans the center of $\mathbf{k}$) to Y, using the identity (for $\alpha\in Q_+$, $H\in h$)

$$\mathrm{Ad}(e^{tX_\alpha})H=H+(i\alpha(H)/2)(\cosh 2t-1)h_\alpha -(i\alpha(H)/2)(\sinh 2t)Y_\alpha, \tag{7}$$

which are determined in the next paragraph. However, we first make an observation about the adjoint action, namely that if $\alpha\in Q_+$ and $Y_1\in\mathbf{g}$ where $\mathbf{h}(Y_1)=0$, then there exists a real number s such that

$$\mathbf{h}(\mathrm{Ad}(e^{tX_\alpha})\mathrm{Ad}(e^{sZ})Y_1)=\mathbf{h}(\mathrm{Ad}(e^{sZ})Y_1)=\mathbf{h}(Y_1)$$

for all t. This follows from the fact that $\mathbf{h}(\mathrm{Ad}(e^{tX_\alpha})Y_1)$ depends only on the Y_α-component of Y_1 (with respect to the basis $X_\beta, Y_\beta, \beta\in Q_+$, of $\mathbf{p}$), and that an initial application of some $\mathrm{Ad}(e^{sZ})$ can rotate a linear combination $rX_\alpha+uY_\alpha$ into $(r^2+u^2)^{1/2}X_\alpha$, which is fixed under $\mathrm{Ad}(e^{tX_\alpha})$.

Now let $X=\sum_{\alpha\in Q_+}r_\alpha h_\alpha$, where all $r_\alpha\geqq 0$. Take any $\alpha\in Q_+$, and determine t so that $\mathbf{h}(\mathrm{Ad}(e^{tX_\alpha})Y)=M+r_\alpha h_\alpha$, using (7) and $i\alpha(M)>0$. Next take another $\beta\in Q_+$, and apply the observation at the end of the previous paragraph to the $\mathrm{Ad}(e^{uX_\beta})$-action, to obtain u, s such that

$$\mathbf{h}(\mathrm{Ad}(e^{uX_\beta})\mathrm{Ad}(e^{sZ})\mathrm{Ad}(e^{tX_\alpha})Y)=M+r_\alpha h_\alpha+r_\beta h_\beta,$$

using also the fact that $i\beta(h_\alpha)\geqq 0$. Continuing in this way with the remaining roots in Q_+, obtain finally a $Y_1\in\mathrm{Ad}(G)H$ such that $\mathbf{h}(Y_1)=M+X=L$. Q.e.d.

5. Classification of the open and closed cones and further pupperties

We define some collections of convex cones.

Definitions. Let
$\omega_+{}^0 = \{C\subset\mathbf{h}\colon C$ is an open W_K-invariant convex cone, $(c_m)^0\subseteqq C\subseteqq(c_M)^0\}$,
$\omega_+{}^c = \{C\subset\mathbf{h}\colon C$ is a closed W_K-invariant convex cone, $c_m\subseteqq C\subseteqq c_M\}$,
$\Omega_+{}^0 = \{C\subset\mathbf{g}\colon C$ is an open G-invariant convex cone containing $Z\}$,
$\Omega_+{}^c = \{C\subset\mathbf{g}\colon C$ is a closed G-invariant convex cone containing $Z\}$.
For convenience we record the following corollary of Lemma 7.

Corollary 5. *If* $H\in C$ *for some* $C\in\omega_+{}^c$, *then* $\mathbf{h}(\mathrm{Ad}\,(g)H)\in C$ *for all* $g\in G$.

Our main result on the classification of cones is

Theorem 2. (a) $C\cap\mathbf{h}\in\omega_+{}^0$ *for all* $C\in\Omega_+{}^0$, *and*

(8) $$\Omega_+{}^0 \to \omega_+{}^0\colon C \to C\cap\mathbf{h}$$

is a bijection. For all $C\in\Omega_+{}^0$,

(9) $$C=\{X\in\mathbf{g}\colon \mathrm{Ad}(g)X\in C\cap\mathbf{h}\ \ \textit{for some}\ \ g\in G\}$$
$$=\{X\ \textit{elliptic}\colon \mathbf{h}(\mathrm{Ad}(g)X)\in C\cap\mathbf{h}\ \ \textit{for all}\ \ g\in G\}.$$

(b) $C\cap\mathbf{h}\in\omega_+{}^c$ *for all* $C\in\Omega_+{}^c$, *and*

(10) $$\Omega_+{}^c\to\omega_+{}^c\colon C\to C\cap\mathbf{h}$$

is a bijection. For all $C\in\Omega_+{}^c$,

(11) $$\{X\in\mathbf{g}\colon \mathrm{Ad}(g)X\in C\cap\mathbf{h}\ \textit{for some}\ \ g\in G\}$$
$$=\{X\ \textit{elliptic}\colon\ \mathbf{h}(\mathrm{Ad}(g)X)\in C\cap\mathbf{h}\ \textit{for all}\ \ g\in G\}$$

and

(12) $$C=\{X\in\mathbf{g}\colon\ (\mathrm{Ad}(g)X)\in C\cap\mathbf{h}\ \textit{for all}\ \ g\in G\}$$
$$=\{X\in\mathbf{g}\colon\ \textit{for all}\ \ \varepsilon>0,\ \mathrm{Ad}(g)(X+\varepsilon Z)\in(C\cap\mathbf{h})^0\ \textit{for some}\ \ g\in G\}.$$

Proof. (a) Let $C\in\Omega_+{}^0$. $C\cap\mathbf{h}$ is clearly an open W_K-invariant convex cone, and then $C\cap\mathbf{h}\in\omega_+{}^0$ by Corollaries 1, 3, 4. The first equality in (9) (which establishes injectivity of (8)) follows from Lemma 6 and $C\subseteqq(C_{\mathrm{Max}})^0$, and the second follows from Lemma 7 and $C\cap\mathbf{h}=C\cap\mathbf{h}+\overline{C\cap\mathbf{h}}$.

It remains to show that (8) is surjective. Given $C_1\in\omega_+{}^0$, define $C_2=\{X\in\mathbf{g}: X$ is G-conjugate to some $H\in C_1\}$, and note that $C_2=\{X\in(C_{\mathrm{Max}})^0\colon \mathbf{h}(\mathrm{Ad}\,(g)X)\in C_1$ for all $g\in G\}$, again by Lemmas 6, 7. C_2 is open by Corollary 1, so $C_2\in\Omega_+{}^0$. Finally, $C_2\cap\mathbf{h}=C_1$ again by Lemma 7.

(b) Let $C \in \Omega_+{}^c$. $C \cap \mathbf{h} \in \omega_+{}^c$ is clear from Cors. 3, 4. (11) follows directly from Corollary 5. Proceeding to (12), we note that if $X \in C$ and $g \in G$, then for all $\varepsilon > 0$ there exists $g_\varepsilon \in G$ such that $\mathrm{Ad}(g_\varepsilon)(X+\varepsilon Z) \in \mathbf{h} \cap C$. Thus $\mathbf{h}\,(\mathrm{Ad}(g)(X+\varepsilon Z)) \in \mathbf{h} \cap C$ by Corollary 5, so

(13) $$C \subseteqq \{X \in \mathbf{g} \colon \mathbf{h}(\mathrm{Ad}\,(g)X) \in C \cap \mathbf{h} \quad \text{for all} \quad g \in G\}.$$

Conversely, the r.h.s. of (13) defines an element $C_1 \in \Omega_+{}^c$. If $X \in C_1$ then $X+\varepsilon Z \in (C_{\mathrm{Max}})^0$ for all $\varepsilon > 0$, so $X+\varepsilon Z \in C$ by Lemma 6, hence $X \in C$ since C is closed. The last equality in (12) follows from Corollary 1 and (8).

As in (a) it remains to show that (10) is surjective. Given $C_1 \in \omega_+{}^c$, clearly $C_2 \equiv \{X \in \mathbf{g} \colon \mathbf{h}(\mathrm{Ad}(g)X) \in C_1$ for all $g \in G\} \in \Omega_+{}^c$, and $C_2 \cap \mathbf{h} = C_1$ is immediate from Corollary 5. Q.e.d.

Theorem 3. *For all* $C \in \Omega_+{}^c$,

$$C^* \cap \mathbf{h} = (C \cap \mathbf{h})^*,$$

the duals being taken in $\mathbf{g}$ *and* $\mathbf{h}$, *respectively.*

Proof. $\subseteqq$ is clear. Conversely, assume that $H \in \mathbf{h}$ and

(14) $$\langle H, Y \rangle \geqq 0 \quad \text{for all} \quad Y \in C \cap \mathbf{h}.$$

We want to conclude that $\langle H, Y \rangle \geqq 0$ for all $Y \in C$. But $\langle H, Y \rangle \geqq 0$ for all $Y \in C^0$ by (14), Lemma 7, and $c_m \subseteqq (C \cap \mathbf{h})^*$. Thus $\langle H, Y \rangle \geqq 0$ for all $Y \in \overline{(C^0)} = C$. Q.e.d.

Corollary 6. $(c_M)^0 = \{H \in \mathbf{h} \colon T_H = -\mathrm{ad}\, Z\, \mathrm{ad}\, H \colon \mathbf{p} \to \mathbf{p}$ *is positive definite*$\}$.

Proof. Simply note that if $\alpha \in Q_+$, X_α and Y_α are eigenvectors for T_H with eigenvalue $i\alpha(H)$. Q.e.d.

Theorem 4. *An* $X \in \mathbf{h}$ *is contained in a unique maximal compact subalgebra if and only if no noncompact root vanishes on* X. *In particular, each* $X \in (c_M)^0$, *hence each* $X \in (C_{\mathrm{Max}})^0$ *(by Lemma 6) is contained in a unique maximal compact subalgebra.*

Proof. If $\alpha(X) = 0$ for some $\alpha \in Q_+$, then $[X, \mathrm{Ad}\,(e^{tX_\alpha})Z] = 0$, and $X \in \mathrm{Ad}\,(e^{tX_\alpha})\mathbf{k} \neq \mathbf{k}$.

Conversely, suppose that no $\alpha \in Q_+$ vanishes on X. If X is contained in the maximal compact $\{W \in \mathbf{g} \colon [W, \mathrm{Ad}\,(e^Y)Z] = 0\}$, determined uniquely by some $Y \in \mathbf{p}$, then necessarily $\mathrm{Ad}\,(e^Y)Z \in \mathbf{k}$, since $\mathrm{ad}\, X$ stabilizes $\mathbf{k}$ and $\mathbf{p}$, and vanishes on no nonzero vector in p by hypothesis.

Therefore $[Z, \mathrm{Ad}\,(e^Y)Z] = 0$, in which case we may assume that $Y \in \mathbf{a}$, and apply formula (1) to conclude that $Y = 0$. Q.e.d.

6. Self-dual cones

As noted in [12] and [15], $C_{\text{Min}}=C_{\text{Max}}$ if and only if $\mathbf{g}$ is isomorphic to some $\text{sp}(n, \mathbf{R})$ (using the notation of [4] for the classical algebras), and then clearly C_{Min} is self-dual: $(C_{\text{Min}})^*=C_{\text{Min}}$. In this case one finds that $C_{\text{Min}}\cap\mathbf{h}$ is orthogonally equivalent to the positive orthant

$$P_n = \{x_j\in\mathbf{R}^n\colon\ x_j \geqq 0\} \quad \text{in} \quad \mathbf{R}^n \approx \mathbf{h}.$$

In the other cases it is natural to look for intermediate cones $C\in\Omega_+{}^c$ such that $C_{\text{Min}}\subseteqq C\subseteqq C_{\text{Max}}$ and $C^*=C$. By Theorems 2 and 3 the problem is reduced to the consideration of W_K-invariant cones in $\mathbf{h}$. One finds that such self-dual cones exist and are unique for $\text{su}(2,1)$ and $\text{su}(2,2)$ [12]. The cases of $\text{so}^*(2n)$ also possess self-dual cones $C_n\in\Omega_+{}^c$, and again $C_n\cap\mathbf{h}\approx P_n$ (orthogonal equivalence).

However, at least in the cases $\text{so}^*(6)\approx\text{su}(3,1)$ and $\text{so}^*(8)$, such self-dual cones are not in general unique, as seen from the previous paragraph and the following existence result.

Theorem 5. *Let*

$$c_Z = \left\{X\in\mathbf{h}\colon\ \langle Z, X\rangle \geqq \left(\tfrac{1}{2}\right)^{\frac{1}{2}}\langle Z, Z\rangle^{\frac{1}{2}}\langle X, X\rangle^{\frac{1}{2}}\right\},$$

a self-dual cone in $\mathbf{h}$. *Then, among all classical simple Lie algebras* $\mathbf{g}$, $c_Z\in\omega_+{}^c$ *if and only if* $\mathbf{g}$ *is isomorphic to one of the following:*

$$\text{sp}(n,R) \ \textit{for}\ n=1,2;\quad \text{su}(n,1)\ \textit{for}\ n\geqq 1,\ \textit{and}\ \text{su}(2,2);$$
$$\text{so}^*(2n)\ \textit{for}\ n=3,4;\quad \text{so}(2,n)\ \textit{for}\ n=1\ \textit{and}\ n\geqq 3.$$

In these cases c_Z *is the intersection with* $\mathbf{h}$ *of a self-dual G-invariant convex cone in* $\mathbf{g}$.

Proof. One checks that all the h_α, $\alpha\in\varepsilon_0$ make angles $\leqq\frac{1}{4}\pi$ with Z only in the cases indicated. Q.e.d.

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