On Besov, Hardy and Triebel spaces for $0<p\leqq 1$

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Introduction

The aim of the present paper is two-fold. Our first aim is to derive a Hardy—Littlewood type characterization for non-homogeneous Besov spaces defined by Peetre [13] in the case $0<p<1$ via traces of temperatures on the upper halfspace R^{n+1}_{+}, and thus we answer a question related to the one raised by Peetre [15; p. 258, Remark]. This characterization completes the work of Taibleson [16] and Flett [5] in the case $1\leqq p\leqq\infty$, and may be also considered as non-periodic version of another result of Flett [6]. The idea of the proof comes from the classical work of Gwilliam [9] as was done by Peetre [15] in a characterization of homogeneous spaces via harmonic functions (cf. also [6]); other tools are a sub-meanvalue property of temperatures proved in section 1, which is similar to a result of Hardy—Littlewood given in the paper of Fefferman—Stein [4], and results from interpolation theory. As a consequence of this characterization, we extend some results on translation invariant operators on Besov and Hardy spaces to the case $0<p\leqq 1$ (cf. [2], [12], [15]).

Our second (and main) aim concerns pseudo-differential operators. In [15], Peetre showed that if $\sigma\in C^{\infty}(R^n\times R^n)$, $1\leqq p\leqq\infty$ and

$$|D_x^{\alpha}D_{\xi}^{\beta}\sigma(x,\xi)|\leqq C_{\alpha,\beta}(1+|\xi|)^{-|\beta|}, \tag{1}$$

then the associated pseudo-differential operator $T=\sigma(D)$ is bounded on $B^s_{p,q}(-\infty<s<\infty,\ 0<q\leqq\infty)$. As for the case $0<p<1$, he required a "somewhat stronger" assumption on the symbol (cf. [15; pp. 285—287]); however, it is not difficult to prove that T is still bounded in this case under the same condition (1). On the other hand, Gibbons [7] has proven the boundedness of T on $B^s_{p,q}(0<s<1,\ 1\leqq p,\ q\leqq\infty)$ under the following assumption on the symbol:

$$\|D_{\xi}^{\beta}\sigma(\cdot,\xi)\|_{B^s_{\infty,q}}\leqq C_{\beta}(1+|\xi|)^{-|\beta|}. \tag{2}$$

We shall prove in this paper that if $0<p, q\leqq\infty$, $-\infty<s<\infty$ and

(3) $$\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B^\varrho_{\infty,\infty}} \leqq C_\beta(1+|\xi|)^{-|\beta|},$$

then $T=\sigma(D)$ is bounded on $B^s_{p,q}$ (resp. $F^s_{p,q}$, $p\neq\infty$), whenever $\varrho>\varrho_B$ (resp. ϱ_F) (see Theorem 3 for details). We also extend the above result of Gibbons to the case $0<p<1$ and study symbols satisfying conditions other than (2) and (3). Although our emphasis is put on the case $0<p<1$, some of our results seem new even in the case $1\leqq p\leqq\infty$. The proof of the boundedness of T is done by the well-known technique of decomposing σ into elementary symbols (cf. [3], [7]). However, in comparison to [7] where difference is used to deal with Besov spaces, our method relies heavily on the characterization of Besov (and Triebel) spaces via the spectral decomposition by Peetre and maximal function techniques ([4], [13], [14], [15], [17]). In fact, our point of view is influenced by both [3], and [15] and [17] where (ordinary) multipliers are studied.

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1. Notation and preliminaries

We use R^n and R^{n+1}_+ to denote the n-dimensional Euclidean space and the upper half-space, respectively; an element of R^{n+1}_+ will be generally denoted by (x, t), where $x\in R^n$ and $t>0$. The Fourier transform is defined by

$$\mathscr{F}f(\xi)=\hat{f}(\xi)=\int e^{-ix\cdot\xi}f(x)\,dx.$$

Here $x\cdot\xi=x_1\xi_1+\ldots+x_n\xi_n$, and the integral is extended over all of R^n unless otherwise indicated. For a multi-index $\alpha=(\alpha_1, \ldots, \alpha_n)$, let $D^\alpha=(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$. For convenience, we put $D_{n+1}=(\partial/\partial t)$. The *Gauss—Weierstrass kernel* for R^{n+1}_+ will be denoted by W, i.e.,

$$W_t(x)=W(x, t)=(4\pi t)^{-n/2}\exp(-|x|^2/4t),\quad (x, t)\in R^{n+1}_+.$$

For any $f\in\mathscr{S}'$, the space of tempered distributions (the dual of $\mathscr{S}$, the Schwartz class of rapidly decreasing functions), the function $u=W_t*f$ is well-defined and it is a temperature on R^{n+1}_+, i.e., a solution of the heat equation $\Delta_x u-D_{n+1}u=0$ on R^{n+1}_+; we shall usually call u the *Gauss—Weierstrass integral* of f. For a measurable function w on R^{n+1}_+, we put

$$M_p(w; t)=\|w(\cdot, t)\|_p,\quad 0<p\leqq\infty,$$

where $\|\cdot\|_p$ is the L_p-norm (it is only a quasi-norm in case $0<p<1$, but we still use the terminology "norm" for the sake of convenience; we shall use this abuse of terminology hereafter without explanation).

Next we recall the definitions of Besov, Hardy and Triebel spaces. Let ψ be a function in $\mathcal{S}$ such that $\operatorname{supp}\psi=\{1/2\leqq|\xi|\leqq 2\}$, and $\sum_{j=-\infty}^{\infty}\psi(2^{-j}\xi)=1$ for $\xi\neq 0$. Let Φ and ψ_j, $j=0, \pm 1, \pm 2, \dots$ be functions in $\mathcal{S}$ given by

$$\hat{\psi}_j(\xi)=\psi(2^{-j}\xi),\quad j=0,\ \pm 1,\ \pm 2, \dots,$$
$$\hat{\Phi}(\xi)=1-\sum_{j=1}^{\infty}\hat{\psi}_j(\xi).$$

Following Peetre ([13], [14], [15]) and Triebel ([17], [18]), we define

$$B_{p,q}^s=\{f\in\mathcal{S}';\ \|f\|_{B_{p,q}^s}=\|\Phi*f\|_p+(\sum_{j=1}^{\infty}(2^{js}\|\psi_j*f\|_p)^q)^{1/q}<\infty\},$$
$$\dot{B}_{p,q}^s=\{f\in\mathcal{S}';\ \|f\|_{\dot{B}_{p,q}^s}=(\sum_{j=-\infty}^{\infty}(2^{js}\|\psi_j*f\|_p)^q)^{1/q}<\infty\},$$
$$F_{p,q}^s=\Big\{f\in\mathcal{S}';\ \|f\|_{F_{p,q}^s}=\|\Phi*f\|_p+\Big(\int(\sum_{j=1}^{\infty}(2^{js}|\psi_j*f(x)|)^q)^{p/q}dx\Big)^{1/p}<\infty\Big\},$$
$$\dot{F}_{p,q}^s=\Big\{f\in\mathcal{S}';\ \|f\|_{\dot{F}_{p,q}^s}=\Big(\int(\sum_{j=-\infty}^{\infty}(2^{js}|\psi_j*f(x)|)^q)^{p/q}dx\Big)^{1/p}<\infty\Big\},$$

where $-\infty<s<\infty$ and $0<p,\ q\leqq\infty$. The following characterizations of Hardy and local Hardy spaces will be used as their definitions ([4], [8]).

$$H^p=\{f\in\mathcal{S}';\ \|f\|_{H^p}=\big\|\sup_{0<t<\infty}|\Phi_t*f(x)|\big\|_p<\infty\},$$
$$h^p=\{f\in\mathcal{S}';\ \|f\|_{h^p}=\big\|\sup_{0<t<1}|\Phi_t*f(x)|\big\|_p<\infty\},$$

where $0<p<\infty$ and $\Phi_t(y)=t^{-n}\Phi(y/t)$.

It was observed in [18; pp. 72—73] that the L_p-norm in the definition of $B_{p,q}^s$ $(p\neq\infty)$ can be replaced by the $F_{p,2}^0$-norm; the latter is equivalent to the h^p-norm by [1]. As for the space $\dot{B}_{p,q}^s$ $(p\neq\infty)$, Peetre [15] showed that the L_p-norm can be replaced by the H^p-norm. The following Fourier and ordinary multipliers' criteria for Besov and Triebel spaces are useful for our purpose. Hereafter, immaterial constants are denoted by $C, c, \dots$; they are not necessarily the same on any two consecutive occurrences.

Lemma 1. (Cf. [17], [18].) *Let* $0<p,\ q\leqq\infty$ *and* $-\infty<s<\infty$.

(i) *If* m *is sufficiently smooth and* $\hat{T}=m$, *then*

$$\|T*f\|_{B_{p,q}^s}\leqq C\{\sup_{|\alpha|\leqq N_B}\sup_{x\in R^n}(1+|x|)^{|\alpha|}|D^\alpha m(x)|\}\|f\|_{B_{p,q}^s},$$
$$\|T*f\|_{F_{p,q}^s}\leqq C\{\sup_{|\alpha|\leqq N_F}\sup_{x\in R^n}(1+|x|)^{|\alpha|}|D^\alpha m(x)|\}\|f\|_{F_{p,q}^s},\quad p\neq\infty,$$

where N_B *and* N_F *are positive integers that depend on* s, p, q *and* n.

(ii) *If* $\varrho_B=\max(s, n/p-s)$ *and* $\varrho_F=\max(s, n/\min(p, q)-s)$, *then*

$$\|af\|_{B^s_{p,q}} \leqq C\|a\|_{B^\varrho_{\infty,\infty}}\|f\|_{B^s_{q,q}}, \quad \varrho>\varrho_B,$$

$$\|af\|_{F^s_{p,q}} \leqq C\|a\|_{B^\varrho_{\infty,\infty}}\|f\|_{F^s_{p,q}}, \quad \varrho>\varrho_F,\ p\neq\infty.$$

Our preliminary result is a sub-mean-value property for temperatures.

Lemma 2. *Let* u *be a temperature on* R^{n+1}_+, $(x, t)\in R^{n+1}_+$ *and* $r>0$ *be such that* $r^2<t$. *Let* $R_r=\{(y, s);\ t-r^2\leqq s\leqq t,\ |x_j-y_j|\leqq r/2,\ j=1, \dots, n\}$ *and* $U_r=R_r\setminus R_{r/2}$. *Then*

$$|u(x, t|^p \leqq (C_p/r^{n+2})\iint_{U_r}|u(y, s)|^p\,dy\,ds, \quad 0<p<\infty.$$

Proof. It is easily seen that the result for the case $1\leqq p<\infty$ follows from that for the case $0<p<1$ by Hölder's inequality. (Note also that a proof in the former case can be obtained as in Hattemer [10; Lemma 5] where a proof for $p=2$ was given.) We may therefore assume that $0<p<1$ in the rest of the proof. We imitate the arguments given by Fefferman—Stein [4]. It is no loss of generality to assume that u can be extended to be continuous on $R^n\times[0, \infty[$, $(x, t)=(0, \dots, 0, 1)$ and $\iint_{U_1}|u(y, s)|^p\,dy\,ds=1$. For each $0<r<1$, let

$$K_r = \{(y, s);\ s = 1-r^2,\ |y_j| \leqq r/2,\ j = 1, \dots, n\}\cup\bigcup_{j=1}^{n}\{(y, s)\in R_r;$$
$$1-r^2\leqq s\leqq 1,\quad |y_j| = r/2\},$$

$$M_q(r) = \Big(\int_{K_r}|u(y, s)|^q\,dS_r(y, s)\Big)^{1/q}, \quad 0<q<\infty,$$

$$M_\infty(r) = \sup_{K_r}|u(y, s)|,$$

where dS_r is the (Lebesgue) surface measure on K_r properly normalized (cf. [10]). We may assume $M_\infty(r)\geqq 1$ for $0<r<1$, since the result would follow otherwise by maximum principle. Let $0<\varrho<r<1$. By known properties of temperatures (cf. [10]), there exists a kernel H_r such that for $(z, \delta)\in K_\varrho$,

$$u(z, \delta) = \int_{K_r} H_r((z, \delta), (y, s))\,u(y, s)\,dS'_r(y, s),$$

where dS'_r is the measure dS_r suitably oriented. Since

$$M_1(r) \leqq M_p(r)^p M_\infty(r)^{1-p},$$

and

$$\big|H_r((z, \delta), (y, s))\big| \leqq C(1-\varrho r^{-1})^{-2n}, \quad (y, s)\in K_r \text{ and } (z, \delta)\in K_\varrho$$

by the explicit formula for H_r given in [10], it follows that

$$M_\infty(\varrho) \leqq CM_p(r)^p M_\infty(r)^{1-p}(1-\varrho r^{-1})^{-2n}.$$

Taking $\varrho=r^\lambda$ ($\lambda>1$ to be chosen later), and using Jensen's inequality and the condition $\iint_{U_1}|u(y,s)|^p dy\,ds=1$, we see that

$$\int_{1/2}^{1}\log M_\infty(r)r^{-1}\,dr \leqq C_\lambda+(1-p)\int_{1/2}^{1}\log M_\infty(r)r^{-1}\,dr.$$

Thus, we obtain

$$(1/\lambda)\int_{(1/2)^\lambda}^{1}\log M_\infty(r)r^{-1}\,dr \leqq C_\lambda+(1-p)\int_{1/2}^{1}\log M_\infty(r)r^{-1}\,dr.$$

If λ is chosen so that $1/\lambda>1-p$, then we derive that

$$\int_{1/2}^{1}\log M_\infty(r)r^{-1}\,dr \leqq C'_p.$$

Hence, there exists r_0, $1/2<r_0<1$ such that

$$M_\infty(r_0)\leqq C_p,$$

which implies the lemma by maximum principle for temperatures.

2. Hardy—Littlewood type characterization of Besov spaces

In this section, we shall assume that $0<p\leqq 1$, $0<q\leqq\infty$ and $-\infty<s<\infty$.

Theorem 1. *Let f be a tempered distribution, u be its Gauss—Weierstrass integral and k be a non-negative integer greater than $s/2$.*

(i) *If $f\in B^s_{p,q}$, then*

$$\mathscr{B}^s_{p,q}(f)=\sup_{2\geqq t\geqq 1/2}\|u(\cdot,t)\|_{h^p}+\Big(\int_0^1[t^{k-s/2}\|D^k_{n+1}u(\cdot,t)\|_{h^p}]^q t^{-1}\,dt\Big)^{1/q}<\infty.$$

(ii) *Conversely, if*

$$B^s_{p,q}(f)=\sup_{2\geqq t\geqq 1/2}M_p(u;t)+\Big(\int_0^1[t^{k-s/2}M_p(D^k_{n+1}u;t)]^q t^{-1}\,dt\Big)^{1/q}<\infty,$$

then $f\in B^s_{p,q}$.

Furthermore, $\mathscr{B}^s_{p,q}(\cdot)$ and $B^s_{p,q}(\cdot)$ are norms equivalent to each other and to $\|\cdot\|_{B^s_{p,q}}$.

Proof. Let $\psi_0=\Phi$ and ψ_j ($j=1,2,\ldots$) be the functions in the definition of $B^s_{p,q}$ given in section 1. We observe the following easy consequence of Lemma 1:

For any non-negative integer m and real number λ, there exist constants C_m and C_λ such that for all $g\in h^p$ and $t\leqq 1$,

$$\|D^m_{n+1}W_t*g\|_{h^p}\leqq C_m t^{-m}\|g\|_{h^p}\leqq C_m t^{-m}\big(\textstyle\sum_{j=0}^\infty\|\psi_j*g\|^p_{h^p}\big)^{1/p}, \tag{4}$$

$$\|J^\lambda W_t*g\|_{h^p}\leqq C_\lambda(1+t^{\lambda/2})\|g\|_{h^p}.$$

(Here $(J^\lambda h)\hat{}=(1+|\xi|^2)^{-\lambda/2}\hat{h}$ for $h\in\mathscr{S}'$.)

This observation readily implies that

(5)

$$\left(\int_0^1 [t^{k-s/2}\|D^k_{n+1}u(\cdot,t)\|_{h^p}]^q t^{-1}\,dt\right)^{1/q} \sim \left(\sum_{j=0}^\infty (2^{-j(k-s/2)}\|D^k_{n+1}u(\cdot,2^{-j})\|_{h^p})^q\right)^{1/q},$$

where $F(h)\sim G(h)$ for h in some class H means that there exist C_1 and C_2 such that $C_1G(h)\leqq F(h)\leqq C_2G(h)$, $h\in H$.

We begin with the proof of (i). Assume first that $s<0$. Let s_0 and s_1 be negative numbers such that $s_0/2<s/2<s_1/2<k$. Let $f\in B^{s_i}_{p,q}$, $i=0, 1$. Then

$$\|D^k_{n+1}u(\cdot,2^{-j})\|_{h^p} = \|J^{s_i}W(\cdot,2^{-j-1})*D^k_{n+1}W(\cdot,2^{-j-1})*J^{-s_i}f\|_{h^p}$$
$$\leqq C2^{-j(s_i/2-k)}\|f\|_{B^{s_i}_{p,p}}$$

for $j=0, 1, 2, \dots$ by (4). Consider the map

$$T\colon f\mapsto \{D^k_{n+1}u(\cdot,2^{-j})\}_{j=0}^\infty.$$

Then the above shows that T maps $B^{s_i}_{p,p}$ into $l^{s_i/2-k}_\infty(h^p)$. (Here $l^\lambda_q(A)=\{\{a_j\}_{j=0}^\infty;\ \|\{a_j\}\|_{l^\lambda_q(A)}=\{\sum_{j=0}^\infty(2^{j\lambda}\|a_j\|_A)^q\}^{1/q}<\infty\}$, where $-\infty<\lambda<\infty$, $0<q\leqq\infty$ and A is a quasi-Banach space with norm $\|\cdot\|_A$.) Hence $T\colon B^s_{p,q}\to l^{s/2-k}_q(h^p)$ by interpolation (cf. [15; Chap. 5, Theorem 3 and Chap. 11, Theorem 10]). Since the proof of the fact that

$$\sup_{2\geqq t\geqq 1/2}\|u(\cdot,t)\|_{h^p}\sim\|u(\cdot,1/2)\|_{h^p}\leqq C\|f\|_{B^s_{p,q}}$$

is similar (in fact simpler), part (i) in case $s<0$ follows from (5). Next, let $s\geqq 0$ and $s_0/2<s/2<s_1/2<k$. Then the result for the case $s<0$ implies that

$$2^{-j(k-s_i/2)}\|D^k_{n+1}u(\cdot,2^{-j})\|_{h^p}\leqq C\|\Delta^k f\|_{B^{s_i-2k}_{p,q}}\leqq C'\|f\|_{B^{s_i}_{p,q}}.$$

Thus the desired result for $s\geqq 0$ follows by interpolation as above. The proof of (i) is hence complete.

Conversely, assume that the assumption in (ii) is satisfied. Keeping $j=1, 2, \dots$ fixed for a moment, let $t=2^{-2j}$. For simplicity, we write v for $D^k_{n+1}u$. Then

$$\psi_j*f = t^k\varphi_j*v(\cdot,t),$$

where $\hat\varphi_j(\xi)=\hat\psi_j(\xi)(-t|\xi|^2)^{-k}\exp(|\xi|^2t)$. Noting that $\varphi_j(x)=t^{-n/2}\Psi(x/\sqrt{t})$ with $\hat\Psi(\xi)=\psi(\xi)(-|\xi|^2)^{-k}\exp(|\xi|^2)$, we obtain

$$|\psi_j*f(x)|\leqq t^k\sum_{\mu\in Z^n}\Psi^*(\mu)v^*(x-\sqrt{t}\mu,t),$$

where

$$\Psi^*(\mu)=\sup_{y\in I_0}|\Psi(\mu+y)|,$$
$$v^*(z,t)=\sup_{y\in I_{0,\sqrt{t}}}|v(z+y,t)|,$$

and I_0 (resp. $I_{0,\sqrt{t}}$) denotes the cube with center at the origin and with length of side 1 (resp. $\sqrt{t}$). Hence, it follows that

$$\|\psi_j * f\|_p \leqq C_{p,\Psi} t^k M_p(v^*; t).$$

Assume first that $p\leqq q$. For any $x\in R^n$ and $y\in I_{0,\sqrt{t}}$, Lemma 2 gives

$$|v(x+y, t)|^p \leqq C2^{j(n+2)} \int_{t/2}^t \int_{|z-x|\leqq\sqrt{nt}} |v(z, \lambda)|^p dz\, d\lambda.$$

Thus

$$M_p(v^*; t) \leqq C\left(\int_{t/2}^t M_p(v; \lambda)^p \lambda^{-1} d\lambda\right)^{1/p}.$$

Jensen's inequality then implies that

$$S=\left(\sum_{j=1}^\infty (2^{js}\|\psi_j * f\|_p)^q\right)^{1/q} \leqq C\left(\int_0^1 (\lambda^{k-s/2} M_p(v; \lambda))^q \lambda^{-1} d\lambda\right)^{1/q}.$$

Next, let $q<p$. Then, again an application of Lemma 2 with q instead of p gives

$$M_p(v^*; t)^q \leqq C\int_{t/2}^t M_p(v; \lambda)^q \lambda^{-1} d\lambda.$$

Hence, it follows from Minkowski's inequality that

$$S \leqq C\left(\int_0^1 (\lambda^{k-s/2} M_p(v; \lambda))^q \lambda^{-1} d\lambda\right)^{1/q}.$$

Since the estimate for $\|\Phi * f\|_p$ is similar, we conclude that $f\in B^s_{p,q}$, and the proof of the theorem is thus complete.

Remark 1. (i) Results similar to those in Theorem 1 hold for homogeneous spaces. More explicitly, if $f\in \dot{B}^s_{p,q}$, and u and k are as above, then

$$\left(\int_0^\infty [t^{k-s/2}\|D^k_{n+1}u(\cdot, t)\|_{H^p}]^q t^{-1} dt\right)^{1/q} \sim \|f\|_{\dot{B}^s_{p,q}};$$

and conversely, if

$$\left(\int_0^\infty [t^{k-s/2} M_p(D^k_{n+1}u; t)]^q t^{-1} dt\right)^{1/q} < \infty,$$

then $f\in \dot{B}^s_{p,q}$ and the last quantity is equivalent to $\|f\|_{\dot{B}^s_{p,q}}$. Using harmonic functions instead of temperatures, Peetre [15] has given a similar characterization for $\dot{B}^s_{p,\infty}(s<1)$. Our proof of Theorem 1 (ii) is a modification of his.

(ii) A theory of Besov spaces in the case $0<p<1$ can probably be developed on the basis of the characterization given in (ii) of Theorem 1 as was done by Taibleson [16] and Flett [5] in the case $1\leqq p\leqq\infty$. In particular, if $\lambda>s$ and u is the Gauss—Weierstrass integral of an $f\in\mathcal{S}'$, then $f\in B^s_{p,q}$ if and only if

$$\sup_{2\geqq t\geqq 1/2} \|u(\cdot, t)\|_p + \left(\int_0^1 [t^{\lambda-s}\|J^{-\lambda}u(\cdot, t)\|_p]^q t^{-1} dt\right)^{1/q} < \infty;$$

furthermore, the above quantity is equivalent to $\|f\|_{B^s_{p,q}}$. The proof of these assertions is similar to that of Theorem 1.

3. Translation invariant operators

Using the characterization of Besov spaces given in Theorem 1 and the known technique from previous papers ([2], [12]), we extend many results on translation invariant operators to the case $0<p\leqq 1$. For quasi-Banach spaces X and Y such that $\mathcal{S}\subset X$, and X and Y are embedded in $\mathcal{S}'$, let $Cv(X, Y)$ denote the set of all bounded and translation invariant operators from X into Y, i.e.,

$$Cv(X, Y)=\{T\in\mathcal{S}';\ \|T*\varphi\|_Y\leqq C\|\varphi\|_X \text{ for all } \varphi\in\mathcal{S}\}.$$

Theorem 2. *If* $0<p\leqq q\leqq 1$, $0<r, s\leqq\infty$ *and* $-\infty<s_0, s_1<\infty$, *then*

(i) $Cv(B^{s_0}_{p,s}, B^{s_1}_{q,r})\subset B^{s_1-s_0+n(1/p-1)}_{q,\infty}$;

(ii) $Cv(B^{s_0}_{p,s}, B^{s_1}_{q,r})=B^{s_1-s_0+n(1/p-1)}_{q,\infty}$ *if* $s\leqq r$;

(iii) $Cv(h^p, B^{s_0}_{q,r})\subset B^{s_0+n(1/p-1)}_{q,\infty}$;

(iv) $Cv(h^p, B^{s_0}_{q,r})=B^{s_0+n(1/p-1)}_{q,\infty}$ *if either* $r\geqq 2$ *or* $p<q$ *and* $r\geqq p$;

(v) $Cv(h^p, h^q)\subset B^{n(1/p-1)}_{q,\infty}$;

(vi) $Cv(h^p, h^q)=B^{n(1/p-1)}_{q,\infty}$ *if* $p<q<1$;

(vii) $Cv(h^p, h^1)=Cv(h^p, L_1)=B^{n(1/p-1)}_{1,\infty}$, $p<1$.

The assertions (i) *and* (ii) *remain true in case* $0<p\leqq 1\leqq q\leqq\infty$.

Remark 2. (i) Theorem 2 is still valid for homogeneous spaces, that is, if we replace non-homogeneous Besov spaces and local Hardy spaces by homogeneous Besov spaces and Hardy spaces, respectively, then the corresponding statements of Theorem 2 hold.

(ii) Note that the assertion (ii) of Theorem 2 for $p=q$ and $r=s$ was obtained earlier by Peetre [15] (cf. also [18]). The author was informed by Per Nilsson that a sharpened version of Theorem 2 (ii) was also obtained by him independently.

Before proceeding on with the proof of Theorem 2, we need a lemma.

Lemma 3. (i) *If* k *is a non-negative integer and* $1/p+1/p'=1$, *then*

$$B^s_{p,q}(D^k_{n+1}W(\cdot,\lambda))\leqq C(1+\lambda^{-k-s/2-n/2p'}),\ 0<\lambda<\infty \text{ and } -\infty<s<\infty.$$

(ii) *If* $0<p<q\leqq\infty$ *and* $s_0-n/p=s_1-n/q$, *then*

$$B^{s_0}_{p,r}\subset B^{s_1}_{q,r},\quad 0<r\leqq\infty,$$

$$h^p=F^0_{p,2}\subset B^{n(1/q-1/p)}_{q,p},\quad q<\infty,$$

$$B^0_{p,p}\subset h^p\subset B^0_{p,2},\quad p\leqq 2$$

(cf. [1], [11], [15], [18]).

(iii) *If* $-\infty<s<\infty$, $0<p\leqq 1$ *and* $0<q\leqq\infty$, *then*

$$\|f*g\|_{B^s_{p,q}}\leqq C\|f\|_{B^s_{p,q}}\|g\|_{B^{n(1/p-1)}_{p,\infty}}$$

for all $f\in B^s_{p,q}$ *and* $g\in B^{n(1/p-1)}_{p,\infty}$ (*cf.* [15], [18]).

(iv) *If* k *is a non-negative integer and* $\lambda>0$, *then*

$$\|D_{n+1}^{k}W(\cdot,\lambda)\|_{h^p}\sim\begin{cases}\lambda^{-k-n/2p'} & \text{if } n/(n+2k)<p<\infty\\ (1+\lambda)^{-k-n/2p'} & \text{if } 0<p<n/(n+2k),\end{cases}$$

$$\|D_{n+1}^{k}W(\cdot,\lambda)\|_{h^p}\leqq C\big(1+\log((\lambda+1)/\lambda)\big)\quad\text{if}\quad p=n/(n+2k)$$

(*cf.* [2]).

Proof. The proofs of (ii), (iii) and (iv) are given in the referred papers. The assertion (i) follows from Theorem 1 and routine computations if one notes that for any $t>0$, $D_{n+1}^{k}W(x,t)=t^{-k}W(x,t)P(|x|^2/4t)$, where P is a polynomial of degree k.

Proof of Theorem 2. We begin with the proof of (i). Let $0<\lambda\leqq 1$, T be in $Cv(B_{p,s}^{s_0}, B_{q,r}^{s_1})$ and k be a non-negative integer such that $k+s_0/2+n/2p'>0$. Put $f=D_{n+1}^{k}W(\cdot,\lambda)$ and $u=W_t*T$. Then it follows from Lemmas 1 and 3 (i) that

$$\lambda^{-k-s_0/2-n/2p'}\geqq cB_{q,r}^{s_1}(T*f)\geqq c\left(\int_{\lambda/2}^{\lambda}\left(t^{m-s_1/2}\|D_{n+1}^{k+m}u(\cdot,\lambda+t)\|_{h^q}\right)^r t^{-1}dt\right)^{1/r}$$
$$\geqq C\lambda^{m-s_1/2}\|D_{n+1}^{k+m}u(\cdot,2\lambda)\|_{h^q},$$

where m is a non-negative integer greater than $s_1/2$. Thus

$$\sup_{0<\lambda\leqq 1}\lambda^{k+m-(s_1/2-s_0/2-n/2p')}\|D_{n+1}^{k+m}u(\cdot,\lambda)\|_{h^q}\leqq C.$$

Similarly, it can be seen that

$$\sup_{2\geqq t\geqq 1/2}\|u(\cdot,t)\|_{h^q}\leqq C.$$

The proof of (i) is hence complete on account of Theorem 1 since $k+m>(s_1-s_0+n(1/p-1))/2$.

The equality in (ii) follows from (i) and Lemma 3 (ii) and (iii).

The inclusion relation in (iii) is deduced from (i) by noting that $B_{p,p}^{0}\subset h^p$, whereas the equality in (iv) is derived by using the following two inclusion relations given by Lemma 3:

$$h^p\subset B_{p,2}^{0}\subset B_{p,r}^{0},\ r\geqq 2,$$
$$h^p\subset B_{q,p}^{n(1/q-1/p)}\subset B_{q,r}^{n(1/q-1/p)},\quad p<q\quad\text{and}\quad r\geqq p.$$

Next, let $T\in Cv(h^p,h^q)$ and k be a non-negative integer such that $n/(n+2k)<p$. Put $f=D_{n+1}^{k}W(\cdot,\lambda)$, $\lambda>0$, and $u=W_t*T$. Then it follows from Lemma 3 (iv) that

$$\|D_{n+1}^{k}u(\cdot,\lambda)\|_{h^q}\leqq C\|f\|_{h^p}\sim C\lambda^{-k-n(1-1/p)/2}.$$

Thus

$$\sup_{0<\lambda<\infty}\lambda^{k-n(1/p-1)/2}\|D_{n+1}^{k}u(\cdot,\lambda)\|_{h^q}\leqq C.$$

Similarly, we see that

$$\|u(\cdot, 1/2)\|_{h^q} \leqq C,$$

and hence the proof of (v) is complete.

The proofs of (vi) and (vii) can be done in the same spirit as that of (iv).

4. Pseudo-differential operators

In this section and the next one, we shall assume, unless otherwise indicated, that $-\infty < s < \infty$ and $0 < p,\ q \leqq \infty$. Let $\sigma = \sigma(x, \xi)$ be a bounded continuous function on $R^n \times R^n$, which is infinitely differentiable with respect to ξ. The *pseudo-differential operator* $T = \sigma(D)$ with *symbol* σ is defined by

$$Tf(x) = \sigma(D)f(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \sigma(x, \xi) \hat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}.$$

(All symbols hereafter are assumed to satisfy the above conditions.) We say that T is bounded on X if $\|Tf\|_X \leqq C \|f\|_X$ for all $f \in \mathcal{S}$, where X is a quasi-Banach space of (tempered) distributions containing $\mathcal{S}$; we also sometimes, by a slight abuse of terminology, say that σ is bounded on X. Our aim in this section is to prove the following theorem.

Theorem 3. (i) *If* $\varrho > \varrho_B = \max(s, n/p - s)$ *and* $\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B^\varrho_{\infty,\infty}} \leqq C_\beta (1 + |\xi|)^{-|\beta|}$, *then* $T = \sigma(D)$ *is bounded on* $B^s_{p,q}$.

(ii) *If* $\varrho > \varrho_F = \max(s, n/\min(p, q) - s)$ *and* $\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B^\varrho_{\infty,\infty}} \leqq C_\beta (1 + |\xi|)^{-|\beta|}$, *then* $T = \sigma(D)$ *is bounded on* $F^s_{p,q}$ $(p \neq \infty)$.

Since the proof of the theorem is rather long, we shall break it into steps.

I. The case where σ *is an elementary symbol*

A symbol σ is called an *elementary symbol* if it can be written in the form

$$\sigma(x, \xi) = \sum_{k=0}^\infty m_k(x) \hat{\varphi}_k(\xi),$$

where $\hat{\varphi}_k(\xi) = \varphi(2^{-k}\xi)$, $k = 1, 2, \ldots$ for a $\varphi \in \mathcal{S}$ such that $\operatorname{supp} \varphi \subset \{2^{-A} \leqq |\xi| \leqq 2^A\}$ for some $A \geqq 1$, $\varphi_0 \in \mathcal{S}$ and $\operatorname{supp} \hat{\varphi}_0 \subset \{|\xi| \leqq 2^A\}$, and $\sup_k \|m_k\|_{B^\varrho_{\infty,\infty}} < \infty$ (ϱ being as in Theorem 3).

Let σ be an elementary symbol with the above properties. Then the associated pseudo-differential operator is given by

$$Tf(x) = \sigma(D)f(x) = \sum_{h=0}^\infty m_h(x) \varphi_h * f(x).$$

We shall prove that T is bounded on $B^s_{p,q}$ and $F^s_{p,q}$. Since the estimate for $\psi_0 * Tf$ is simpler (recall that we put $\psi_0 = \Phi$ for simplicity), we only consider $\psi_k * Tf$ for

$k=1, 2, \ldots$. The above gives

$$\psi_k * Tf(x) = \sum_{j,h=0}^{\infty} \psi_k * [(\psi_j * m_h)(\varphi_h * f)](x)$$
$$= \int \mathscr{F}^{-1}\psi(y) \sum_{j,h=0}^{\infty} (\psi_j * m_h)(x-2^{-k}y)(\varphi_h * f)(x-2^{-k}y)\, dy.$$

A simple geometric consideration on the supports of $\hat{\psi}_k$ and $[(\psi_j * m_h)(\varphi_h * f)]^{\wedge}$ shows that the terms on the above sum vanish except for those where $|j-h|\leqq k+2A+2$. As in [17], we shall only consider three model cases; the other being similar.

$$(6)\qquad \begin{cases} S_k^1(x) = \sum_{j=0}^{k} \int \mathscr{F}^{-1}\psi(y)(\psi_j * m_k)(x-2^{-k}y)(\varphi_k * f)(x-2^{-k}y)\, dy, \\ S_k^2(x) = \sum_{h=0}^{k} \int \mathscr{F}^{-1}\psi(y)(\psi_k * m_h)(x-2^{-k}y)(\varphi_h * f)(x-2^{-k}y)\, dy, \\ S_k^3(x) = \sum_{h=k}^{\infty} \int \mathscr{F}^{-1}\psi(y)(\psi_h * m_h)(x-2^{-k}y)(\varphi_h * f)(x-2^{-k}y)\, dy. \end{cases}$$

Before proceeding on with the proof, we recall the following maximal function's inequality initially developed by Peetre [14] (cf. also [4]) in the study of the space $\dot{F}^s_{p,q}$. The present form is taken from [17] or [18].

Maximal function's inequality: *Let* Ψ *be a function in* $\mathscr{S}$ *such that* supp $\Psi\subset \{2^{-R}\leqq|\xi|\leqq 2^R\}$, R *being a positive number. Let* Ψ_j, $j=0, 1, 2, \ldots$, *be functions in* $\mathscr{S}$ *given by*

$$\hat{\Psi}_j(\xi) = \Psi(2^{-j}\xi), \quad j = 1, 2, \ldots,$$
$$\operatorname{supp} \hat{\Psi}_0 \subset \{|\xi| \leqq 2^R\}.$$

For each $j=0, 1, 2, \ldots$, *define*

$$(7)\qquad \Psi_j^* f(x) = \sup_{z\in R^n} \frac{|\Psi_j * f(x-z)|}{1+2^{j\lambda}|z|^\lambda}, \quad f\in\mathscr{S}', \ \lambda > 0.$$

Then

$$(8)\qquad \|\{2^{js}\psi_j^* f\}\|_{l_q(L_p)} \leqq CC_\Psi \|f\|_{B^s_{p,q}}, \quad \lambda > n/p,$$

$$(9)\qquad \|\{2^{js}\Psi_j^* f\}\|_{L_p(l_q)} \leqq CC_\Psi \|f\|_{F^s_{p,q}}, \quad p\neq\infty, \ \lambda > n/\min(p, q),$$

where

$$C_\Psi = \sup_{|\alpha|\leqq N} \sup_{x\in R^n} (1+|x|)^{|\alpha|}[|D^\alpha \Psi(x)| + |D^\alpha \hat{\Psi}(x)|]$$

and N *is a positive integer depending on* s, n, p *and* q.

Let λ_1 and λ_2 be positive numbers such that $\lambda_1 > n/p$ ($n/\min(p, q)$ when we are concerned with the F-space). Then, an easy argument implies that

$$|S_k^1(x)| \leqq C\varphi_k^* f(x) \sum_{j=0}^{k} \psi_j^* m_k(x),$$
$$|S_k^2(x)| \leqq C \sum_{h=0}^{k} \psi_k^* m_h(x)\varphi_h^* f(x),$$
$$|S_k^3(x)| \leqq C \sum_{h=k}^{\infty} 2^{(h-k)(\lambda_1+\lambda_2)} \psi_h^* m_h(x)\varphi_h^* f(x),$$

where $\varphi_k^* f, \ldots$ (resp. $\psi_j^* m_k, \ldots$) are maximal functions defined similarly to (7) with $\lambda = \lambda_1$ (resp. λ_2). Then, it follows from the assumption on $\{m_k\}$ that

$$|S_k^1 x)| \leqq C\varphi_k^* f(x) \sum_{j=0}^{\infty} \|\psi_j^* m_k\|_\infty \leqq C' \varphi_k^* f(x),$$

$$2^{ks}|S_k^2(x)| \leqq C 2^{k(|s|+\varepsilon-\varrho)} \left(\sum_{h=0}^{\infty} (2^{hs}\varphi_h^* f(x))^a\right)^{1/a}$$

and

$$2^{ks}|S_k^3(x)| \leqq C \sum_{h=k}^{\infty} 2^{-\varrho h + (h-k)(\lambda_1+\lambda_2-s)} 2^{hs} \varphi_h^* f(x)$$

$$\leqq C' 2^{-k(\lambda_1+\lambda_2+\delta(\varrho-\lambda_1-\lambda_2+s)-s)} \left(\sum_{h=0}^{\infty} (2^{hs}\varphi_h^* f(x))^a\right)^{1/a},$$

where $\varepsilon > 0$, $0 < \delta < 1$, λ_1 and λ_2 are chosen so that $|s| + \varepsilon - \varrho < 0$, $\varrho - \lambda_1 - \lambda_2 + s > 0$ and $\lambda_1 + \lambda_2 + \delta(\varrho - \lambda_1 - \lambda_2 + s) > s$, and $a = p$ (resp. q) if we are concerned with the B-space (resp. F-space). The boundedness of $T = \sigma(D)$ on $F_{p,q}^s$ then follows from these estimates and (9). On the other hand, choosing s_i $(i = 0, 1)$ so that $s_0 < s < s_1$ and $\varrho > \max(s_i, n/p - s_i)$, we derive from the estimates given above for S_k^1, S_k^2 and S_k^3 and (8) that T is bounded on $B_{p,p}^{s_i}$. Hence, the desired result for Besov spaces follows by interpolation.

Note that the proof gives

$$\|T\| \leqq C \sup_k \|m_k\|_{B_{\infty,\infty}^\varrho}.$$

Remark 3. Though there is a direct proof for Besov spaces (cf. section 5), we adopt here the above proof based on maximal function technique and interpolation theorems, since it gives a unified approach to both Besov and Triebel spaces. Our proof is to some extent modelled after the one given by Triebel for ordinary multipliers (cf. [17]).

II. The case σ has compact support

Lemma 4. *Let σ be a symbol such that*

$$\sigma(x, \xi) = 0 \quad \text{for} \quad |\xi| \geqq t,$$

$$\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B_{\infty,\infty}^\varrho} \leqq C_\beta t^{-|\beta|},$$

for some fixed $t > 0$ and let m be a positive integer. Then there exist a sequence of functions $\{a_k\}_{k \in Z^n}$ and a constant C such that

$$\sup_k \|a_k\|_{B_{\infty,\infty}^\varrho} \leqq C,$$

$$\sigma(x, \xi) = \sum_{k \in Z^n} (1 + |k|^2)^{-mn} a_k(x) e^{ik\cdot\xi/t} \varphi(\xi/t),$$

where $\varphi \in \mathscr{S}$, $\varphi = 1$ on $\{|\xi| \leqq 1\}$ and $\varphi = 0$ on the complement of $\{\xi;\ -\pi \leqq \xi_j \leqq \pi,$ $j = 1, \ldots, n\}$. Moreover, if $\sigma(x, \cdot)$ is supported in a ring $\{\gamma t \leqq |\xi| \leqq t\}$, $0 < \gamma < 1$, then φ can be chosen to be supported in a ring centered at the origin.

The proof of the lemma is similar to that of [3; Chap. II, Lemme 4 and Corollaire] by using Poisson summation formula.

Let σ be a symbol satisfying the properties in Lemma 4 with $t=1$ (for the sake of simplicity). We shall prove that the associated pseudo-differential operator $T=\sigma(D)$ is bounded on $B^s_{p,q}$ and $F^s_{p,q}$. By the decomposition given in Lemma 4, we see that

$$Tf(x) = \sum_{k\in Z^n}(1+|k|^2)^{-nm}a_k(x)(\varphi_k*f)(x),$$

where $\hat{\varphi}_k(\xi)=e^{ik\cdot\xi}\varphi(\xi)$. First, we note that

$$\sup_{\xi\in R^n}(1+|\xi|)^{|\alpha|}|D^\alpha\hat{\varphi}_k(\xi)| \leqq C_N(1+|k|)^N, \quad |\alpha| \leqq N \tag{10}$$

for any $k\in Z^n$ and positive integer N.

We begin with the Besov space case. Since the proof for the case $p\geqq 1$ is simpler, we assume that $0<p\leqq 1$. If $p\leqq q$, then the above representation for Tf, Minkowski's inequality, (10) and Lemma 1 imply that

$$\begin{aligned}(\textstyle\sum_{j=0}^\infty(2^{js}\|\psi_j*Tf\|_p)^q)^{p/q} &\leqq (\textstyle\sum_{j=0}^\infty 2^{jsq}(\sum_k(1+|k|^2)^{-nmp}\|\psi_j*(a_k(\varphi_k*f))\|_p^p)^{q/p})^{p/q}\\ &\leqq \textstyle\sum_k(1+|k|^2)^{-nmp}(\sum_j(2^{js}\|\psi_j*(a_k(\varphi_k*f))\|_p)^q)^{p/q}\\ &\leqq C(\textstyle\sum_k(1+|k|^2)^{(N_B-nm)p})\|f\|^p_{B^s_{p,q}} \leqq C'\|f\|^p_{B^s_{p,q}} \quad (m \text{ sufficiently large}).\end{aligned}$$

An inequality similar to the above can be also obtained in the case $p>q$. The proof for the B-space case is thus complete.

As for the F-space, we consider only the case $0<q\leqq 1$ and $0<p\leqq q$, since the other cases are similar. Observing that

$$(\textstyle\sum_{j=0}^\infty|\psi_j*Tf(x)|^q)^{1/q} \leqq (\sum_{k\in Z^n}(1+|k|^2)^{-nmq}(\sum_j|\psi_j*[a_k(\varphi_k*f)](x)|^q))^{1/q},$$

we obtain

$$\begin{aligned}\|\{2^{js}\psi_j*Tf\}\|_{L_p(l_q)} &\leqq C\big(\textstyle\sum_k(1+|k|^2)^{-nmp}\int(\sum_j 2^{jsq}|\psi_j*[a_k(\varphi_k*f)](x)|^q)^{p/q}dx\big)^{1/p}\\ &\leqq C'(\textstyle\sum_k(1+|k|^2)^{(N_F-nm)p})^{1/p}\|f\|_{F^s_{p,q}} \leqq C''\|f\|_{F^s_{p,q}}\\ &\qquad (m \text{ sufficiently large})\end{aligned}$$

by Lemma 1 and (10) again.

The proof of the boundedness for symbols with compact supports is thus complete.

III. Proof of Theorem 3

We write

$$\begin{aligned}\sigma(x,\xi) &= \sigma(x,\xi)\hat{\psi}_0(\xi)+\textstyle\sum_{j=1}^\infty\sigma(x,\xi)\hat{\psi}_j(\xi)\\ &= \tau_0(x,\xi)+\textstyle\sum_{j=1}^\infty\tau_j(x,\xi) = \tau_0+\tau.\end{aligned}$$

The boundedness of $\tau_0(D)$ is given by step II. Each τ_j, $j=1, 2, \ldots,$ satisfies the assumptions of Lemma 4 with $t=2^{j+1}$ and constants $\{C_\beta\}$ independent of j. Therefore, it follows that

$$\tau_j(x, \xi) = \sum_{k\in Z^n} (1+|k|^2)^{-nm} a_{j,k}(x) e^{i2^{-j-1}k\cdot\xi} \varphi(2^{-j-1}\xi),$$

$$\sup_{j,k} \|a_{j,k}\|_{B^{\varrho}_{\infty,\infty}} \leqq C.$$

Further, we may assume that φ is supported in a ring centered at the origin. Setting

$$\Phi_k(\xi) = e^{ik\cdot\xi/2}\varphi(\xi/2)(1+|k|^2)^{-m_0 n},$$

we see that

$$\sup_{\xi\in R^n} (1+|\xi|)^{|\alpha|} |D^\alpha \Phi_k(\xi)| \leqq C_\alpha \quad \text{for} \quad |\alpha| \leqq 2m_0 > N,$$

where N is the integer appearing in the maximal function's inequality given in step I, and $\{C_\alpha\}$ is independent of k. Put

$$\sigma_k(x, \xi) = \sum_{j=1}^{\infty} a_{j,k}(x) \Phi_k(2^{-j}\xi).$$

Then

$$\tau(x, \xi) = \sum_{k\in Z^n} (1+|k|^2)^{-n(m-m_0)} \sigma_k(x, \xi).$$

The boundedness of operators with elementary symbols and the proof in step II imply that $\tau(D)$ is bounded on $B^s_{p,q}$ and $F^s_{p,q}$.

Corollary. *If σ is a symbol in the class $S^0_{1,0}$, that is, σ satisfies the condition* (1), *then $\sigma(D)$ is bounded on $B^s_{p,q}$ and $F^s_{p,q}$ $(p\neq\infty)$.*

Remark 4. Per Nilsson kindly informed the author that he was able to obtain the boundedness of symbols in the class $S^0_{1,\delta}$, $0<\delta<1$, on h^p. (A symbol $\sigma\in C^\infty(R^n\times R^n)$ is in the class $S^0_{1,\delta}$ if $|D^\alpha_x D^\beta_\xi \sigma(x, \xi)| \leqq C_{\alpha,\beta}(1+|\xi|)^{-|\beta|+\delta|\alpha|}$.) Then, by using the relation $h^p=F^0_{p,2}$, he has established the boundedness of these symbols on Besov spaces. The proof of his result has been given in "Pseudo differential operators on Hardy spaces" (Technical report No. 12-1980, Lund).

5. More on pseudo-differential operators

Our aim in this section is to extend the result of Gibbons [7] to the case $0<p<1$ and study symbols satisfying conditions other than those in section 4. We shall be rather brief since proofs are similar to those in section 4 in the spirit. Our first result is

Theorme 4. *If* σ *is a symbol satisfying*

$$\|D_\xi^\beta \sigma(\cdot,\xi)\|_{B^s_{\infty,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (1\leqq q\leqq\infty),$$
$$\|D_\xi^\beta \sigma(\cdot,\xi)\|_{B^{s+\varepsilon}_{\infty,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (0<q<1)$$

for some $\varepsilon>0$, *then* $T=\sigma(D)$ *is bounded on* $B^s_{p,q}$ *for* $s>\max(0, n(1/p-1))$.

Proof. We give detailed proofs for only elementary symbols. Again, three model cases in (6) will be sufficient to illustrate the complete proof. First, we see that

$$\|S_k^1\|_p = \left\|\sum_{j=0}^k \psi_k*[(\psi_j*m_k)(\varphi_k*f)]\right\|_p$$
$$\leqq C\left(\sum_{j=0}^k \|\psi_j*m_k\|_\infty\right)\|\varphi_k*f\|_p \leqq C'\|\varphi_k*f\|_p.$$

(Here we use Lemma 8 in [15; Chap. 11] if $0<p<1$.) Hence it follows that

$$\|\{2^{ks}S_k^1\}\|_{l_q(L_p)} \leqq C\|f\|_{B^s_{p,q}}.$$

The rest of the proof is carried out only for the case $0<p\leqq 1$, since the case $p\geqq 1$ is simpler. As for S_k^2, we see that

$$\|S_k^2\|_p \leqq C\left(\sum_{h=0}^k \|\psi_k*m_h\|_\infty^p\,\|\varphi_h*f\|_p^p\right)^{1/p}.$$

If $0<q\leqq p$ so that $q/p\leqq 1$, then we see that

$$\|\{2^{ks}S_k^2\}\|_{l_q(L_p)} \leqq C\left(\sum_{h=0}^\infty 2^{-hsq}\sum_{k=0}^\infty (2^{ks}\|\psi_k*m_h\|_\infty)^q\right)^{1/q}\|f\|_{B^s_{p,q}} \leqq C'\|f\|_{B^s_{p,q}}.$$

If $p>q$, then Minkowski's inequality gives the same inequality.

Finally, we estimate S_k^3. Since

$$\|S_k^3\|_p \leqq C2^{kn(1-1/p)}\left(\sum_{h=k}^\infty 2^{hn(1/p-1)p}\|\psi_h*m_h\|_\infty^p\,\|\varphi_h*f\|_p^p\right)^{1/p},$$

we obtain

$$\|\{2^{ks}S_k^3\}\|_{l_q(L_p)} \leqq C\|f\|_{B^s_{p,q}}$$

as above. Here we must have $s>n(1/p-1)$.

Remark 5. The proof shows that for any elementary symbol σ, the condition $\sup_j\|m_j\|_{B^s_{\infty,q}}<\infty$ is sufficient to ensure the boundedness of $\sigma(D)$ for all q. This suggests that the number $\varepsilon>0$ appearing in the assumption in case $0<q<1$ could probably be dropped; we only need this stronger assumption to guarantee that $\sup_{k\in Z^n}\|a_k\|_{B^s_{\infty,q}}\leqq C$ in the decomposition of symbols with compact supports in Lemma 4, and thus the difficulty is of a technical nature.

Similar arguments give

Theorem 5. *Let* σ *be a symbol and* $1\leqq p\leqq\infty$. *Then* $\sigma(D)$ *is bounded on* $B^s_{p,q}$ *in the following two cases:*

(i) $s>n/p$ *and*

$$\|D_\xi^\beta \sigma(\cdot,\xi)\|_{B^s_{p,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (1\leqq q\leqq\infty),$$
$$\|D_\xi^\beta \sigma(\cdot,\xi)\|_{B^{s+\varepsilon}_{p,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (0<q<1)$$

for some $\varepsilon>0$.

(ii) $s=n/p>0$ *and*

$$\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B^s_{p,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (q=1),$$

$$\|D_\xi^\beta \sigma(\cdot, \xi)\|_{B^{s+\varepsilon}_{p,q}} \leqq C_\beta(1+|\xi|)^{-|\beta|} \quad (0<q<1)$$

for some $\varepsilon>0$.

As in Remark 5, the number $\varepsilon>0$ in the assumption in case $0<q<1$ could probably be dropped. Moreover, the boundedness of elementary symbols can be also given for $0<p<1$.

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