On hypercontractivity for multipliers on orthogonal polynomials

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Nelson [10] proved that the Mehler transform is hypercontractive. More precisely, the kernel $(1-\omega^2)^{-1/2}\exp\left(\dfrac{\omega^2x^2-2\omega xy+\omega^2y^2}{2(1-\omega^2)}\right)d\mu$ yields a contraction of $L^p(\mu)$ into $L^q(\mu)$, where μ is the Gauss measure, for ω real, $\omega^2\leqq\dfrac{p-1}{q-1}$, $1<p<q<\infty$. This has been extended to imaginary ω by Beckner [3] (this enabled him to give a sharp version of the Hausdorff-Young inequality) and to complex ω by Weissler [16]. Several other proofs using widely different techniques have appeared [9], [6], [14], [11], [8], [1].

Weissler [17] proved that the Poisson integral $f\to P_\omega f$ on the unit circle also is hypercontractive (for the Haar measure) with the above bound on ω. It will be proved in this paper (Section 6) that the same is true for the Poisson integral on a sphere in R^3. It is an open problem whether this holds also in higher dimensions.

The eigenfunctions of the Mehler transform and the Poisson integral are the Hermite polynomials and the spherical harmonics respectively, which are the ortogonal polynomials for the respective measures.

In the present paper we use this as the definition of a family of operators for any probability measure on R^d (Section 1). The general and still unsolved problem is to decide when the operators are contractions and, in particular, for which measures this holds with Nelson's conditions on ω. It shown (Section 3) that this condition always is necessary, but no general sufficient condition is known. However, several theorems (Section 4) prove the hypercontractivity property for one measure, assuming it for others, and these theorems enable us to prove the result stated above for a sphere.

The final section treats multipliers on orthogonal polynomials of complex variables. In particular, it is proved that $\|f(\omega z)\|_q\leqq\|f(z)\|_p$ f analytic, $|\omega|^2\leqq\dfrac{p}{q}\leqq 1$, for certain measures on the complex plane.

Nelson's work was motivated by quantum field theory. In this connection several authors have extended the result to more general operator semigroups, see [7] and the references listed therein. Other extensions have been made by Borell [5] and Peetre [12].

1. Definitions

Let μ be a probability measure on R^d. We assume that the moments $\int x^\alpha d\mu$ exist. (α is used here and in the sequel to denote multiindices.) Hence, every polynomial belongs to $L^p(\mu)$, $p<\infty$. For simplicity, we will only consider L^p for $p\geqq 1$ and only measures μ such that the polynomials are dense in $L^p(\mu)$ for any finite p. This holds e.g. if the Fourier transform $\hat{\mu}$ is real analytic, or equivalently if $\int |x|^m d\mu\leqq C^m m^m, m=1, 2, \dots$.

Let P_k denote the space of polynomials in d variables of degree at most k, and let Q_k be the orthogonal complement of P_{k-1} in P_k (for the scalar product $\int f\bar{g}\, d\mu$). The spaces Q_k are orthogonal to each other and they span L^2, i.e. $L^2=\bigoplus_0^\infty Q_k$. The operator T_ω (alias $T_\omega(\mu)$) that we shall study is defined for a complex parameter ω by $T_\omega(\sum a_k q_k)=\sum a_k \omega^k q_k$, $q_k\in Q_k$. This defines T_ω on the space of polynomials, and it may be extended to L^p if it is continuous. T_1 is the identity and T_0 is the mapping $f\to\int f$. Since $T_\omega 1=1$, $\|T_\omega\|_{p,q}\geqq 1$. $\{T_\omega\}$ is a semigrogup; $T_\omega T_\lambda=T_{\omega\lambda}$, and thus $\|T_{\omega\lambda}\|_{p,r}\leqq\|T_\lambda\|_{p,q}\|T_\omega\|_{q,r}$.

When $d=1$, Q_k is (at most) one-dimensional and hence spanned by a polynomial φ_k. The sequence $\{\varphi_k\}$ is thus the sequence of orthogonal polynomials for the measure μ and the operator T_ω multiplies the k:th orthogonal polynomials by ω^k.

When $d>1$ we may select an orthonormal basis $\{\varphi_{k,l}\}_l$ in each Q_k. These yield together a sequence of orthogonal polynomials, although the sequence is in general not unique. For $|\omega|<1$ we define $K_\omega(x,y)=\sum_{k,l}\omega^k\varphi_{k,l}(x)\varphi_{k,l}(y)$ (the sum converges in $L^2(\mu\times\mu)$). K_ω is independent of the choice of $\{\varphi_{k,l}\}$, and $T_\omega f(x)=\int K_\omega(x,y)f(y)d\mu(y)$, i.e. T_ω is an integral operator with kernel K_ω. The dimension of Q_k is at most $\binom{k+d-1}{d}$. Equality holds for every k if and only if the monomials x^α are linearly independent in L^2, i.e. if and only if the support of μ does not lie in the zero-set of any polynomial. In that case we say that μ is *non-singular*. The opposite extreme case is when the support of μ is finite. Then L^2 has finite dimension, $Q_k=\{0\}$ for $k\geqq k_0$ and the sequence of orthogonal polynomials terminates.

Examples. 1. Gauss measure in R^1; $d\mu=(2\pi)^{-1/2}e^{-x^2/2}dx$. The orthogonal polynomials are the Hermite polynomials,

$$K_\omega(x,y)=(1-\omega)^{-1/2}\exp\left(\frac{\omega^2x^2-2\omega xy+\omega^2y^2}{2(1-\omega^2)}\right)$$

and T_ω is the Mehler transform. This is the case studied by Nelson [1] and others.

2. Symmetric two-point (Bernoulli) measure; $d\mu=\frac{1}{2}\delta_{-1}+\frac{1}{2}\delta_1$. Here $\varphi_0=1$, $\varphi_1=x$ and $T_\omega(a+bx)=a+b\omega x$. Thus $\|T_\omega\|_{p,q}\leqq 1$ if and only if

$$\left(\frac{|a+b\omega|^q+|a-b\omega|^q}{2}\right)^{1/q}\leqq\left(\frac{|a+b|^p+|a-b|^p}{2}\right)^{1/p},\quad a,b\in C.$$

This "two-point inequality" is an intermediate step in [3], [9] [16]. For real ω and $p\leqq q$, it holds if $|\omega|\leqq\sqrt{\frac{p-1}{q-1}}$ [4], for complex ω, see [16].

3. Uniform (Lebesgue) measure on the unit sphere in R^d. Q_k is the space of all spherical harmonics of degree k [15]. Hence, if f is an harmonic polynomial, $T_\omega f(x)=f(\omega x)$. Thus, if $0<\omega<1$, T_ω is the Poisson integral and K_ω the Poisson kernel. The case $d=1$ is Example 2, $d=2$ was studied by Weissler [1] and $d=3$ will be treated in Section 6.

An alternative approach is the following: Let $x_1 \dots x_d$ be random variables. Let P_k be the space of polynomials in them of degree at most k and orthogonalize as above for the scalar product $Ef\bar{g}$. The two approaches are clearly equivalent, letting μ be the joint distribution of the random variables. Both approaches may be extended to infinitely many variables.

2. Positivity

Theorem 1. $\|T_\omega\|_{p,q}=\|T_{\bar\omega}\|_{p,q}=\|T_\omega\|_{q',p'}$, *where p, p' and q, q' are conjugate exponents.*

Proof. The first equality follows from $\overline{T_\omega f}=T_{\bar\omega}\bar f$, and the second by duality since $T_\omega^*=T_{\bar\omega}$.

Theorem 2. *The following are equivalent*

i) $\|T_\omega\|_{p,p}=1$ *for every p,* $1<p<\infty$.
ii) $\|T_\omega\|_{1,1}=1$
iii) $\|T_\omega\|_{\infty,\infty}=1$
iv) T_ω *is a positive operator*
v) $K_\omega(x,y)\geqq 0$ a.e. (For $|\omega|<1$.)

Proof. i)$\Rightarrow$ii) by $p\to 1$, ii)$\Leftrightarrow$iii) by Theorem 1, ii) & iii)$\Rightarrow$i) by interpolation, and iv)$\Leftrightarrow$v) by definition.

ii)$\Rightarrow$iv). By the definition of T_ω, $\int T_\omega f\,d\mu=\int f\,d\mu$, $f\in L^1$. Hence, if $f\geqq 0$, $\int |T_\omega f|\leqq \|f\|_1=\int f=\int T_\omega f$ and thus $T_\omega f\geqq 0$ a.e. iv)$\Rightarrow$ii). $\int|T_\omega f|\leqq\int T_\omega|f|=\int |f|$.

It follows from this, and also from Theorem 3 below, that these properties are possible only if $-1\leqq\omega\leqq 1$. They obviously hold for $\omega=0,1$. $\omega=-1$ is treated in Theorem 5.

Problem. Is T_ω always a contraction in L^p for $0<\omega<1$?

For the examples 1—3 above, the answer is yes, since the kernel is positive.

3. A necessary condition

In the section we assume that μ is not a Dirac (one-point) measure.

Theorem 3. *If* $\|T_\omega\|_{p,q}=1$, *then*

$$(q-1)|\omega|^4-(p+q-2)(\mathrm{Re}\,\omega)^2-(pq-p-q+2)(\mathrm{Im}\,\omega)^2+p-1\geqq 0,\quad |\omega|^2\leqq p/q. \tag{1}$$

Note. For fixed p and q, the set of allowed ω is thus bounded by a quartic curve. $|\omega|^2\leqq p/q$ only serves to exclude the exterior of the outer branch of the curve. There are several equivalent forms of the condition, e.g.

$$|\omega a|^2+(q-2)(\mathrm{Re}\,\omega a)^2\leqq |a|^2+(p-2)(\mathrm{Re}\,a)^2\quad\text{for all}\quad a\in C. \tag{2}$$

Other forms are given in [16].

Proof. Let φ be a real linear function with $\int\varphi\,d\mu=0$ and $\int\varphi^2\,d\mu=1$. Thus $\varphi\in Q_1$. We use the binomial expansion to compute $\|1+\varepsilon\varphi\|_p$ for small (complex) ε.

$$\int|1+\varepsilon\varphi|^p d\mu=\int(1+2\,\mathrm{Re}\,\varepsilon\varphi+|\varepsilon|^2\varphi^2)^{p/2}d\mu$$

$$=\int_{|\varepsilon\varphi|<1/3}\left(1+\frac{p}{2}(2\,\mathrm{Re}\,\varepsilon\varphi+|\varepsilon|^2\varphi^2)+\frac{1}{2}\frac{p}{2}\left(\frac{p}{2}-1\right)(2\,\mathrm{Re}\,\varepsilon\varphi+|\varepsilon|^2\varphi^2)^2+O(|\varepsilon\varphi|^3)\right)d\mu$$

$$+O(|\varepsilon|^3)=\int\left(1+p\,\mathrm{Re}\,\varepsilon\varphi+\frac{p}{2}\left(|\varepsilon|^2+(p-2)(\mathrm{Re}\,\varepsilon)^2\right)\varphi^2\right)d\mu+O(|\varepsilon|^3)$$

$$=1+\frac{p}{2}\left(|\varepsilon|^2+(p-2)(\mathrm{Re}\,\varepsilon)^2\right)+O(|\varepsilon|^3),$$

whence by the binomial expansion again,

$$\|1+\varepsilon\varphi\|_p = 1+\tfrac{1}{2}\left(|\varepsilon|^2+(p-2)(\operatorname{Re}\varepsilon)^2\right)+O\left(|\varepsilon|^3\right).$$

This formula also yields

$$\|T_\omega(1+\varepsilon\varphi)\|_q = \|1+\varepsilon\omega\varphi\|_q = 1+\tfrac{1}{2}\left(|\omega\varepsilon|^2+(q-2)(\operatorname{Re}\omega\varepsilon)^2\right)+O\left(|\varepsilon|^3\right).$$

Comparing these two expressions, taking $\varepsilon=\delta a$ and letting $\delta\to 0$, we obtain (2). With $a=x+iy$, (2) expresses the positivity of a certain quadratic form. This is equivalent to the statement that the determinant and trace of the coefficient matrix are positive, which is (1). (We omit the elementary computations.)

Remarks. 1. For real ω, the condition is $|\omega|\leqq\sqrt{\dfrac{p-1}{q-1}}$ if $p\leqq q$ and $|\omega|\leqq 1$ if $p\geqq q$. For imaginary ω it is $|\omega|\leqq\min\left(p-1,\ \dfrac{1}{q-1}\right)$.

2. If $p>2>q$, (1) allows $|\omega|>1$. However, we may in this case improve the theorem, at least if the support of μ is infinite, by taking $\varphi\in Q_n$ and repeating the argument. This yields $|\omega^n|^2\leqq p/q$, and since n may be arbitrarily large, $|\omega|\leqq 1$. (This is obviously also sufficient for $\|T_\omega\|_{p,q}=1$ when $p\geqq 2\geqq q$.)

3. Theorem 3 holds also for $p<1$. It follows that if $q>p$, T_ω never is a contraction of L^p to L^q. If $q\leqq p$, $\|T_\omega\|=1$ is possible only if $\omega=\pm 1$, at least if the support of μ is infinite.

4. $p=q=1$ yields $-1\leqq\omega\leqq 1$ as stated earlier. If $p=1<q$, or dually if $p<q=\infty$, $\|T_\omega\|_{p,q}=1$ if and only if $\omega=0$.

The surprising fact is that the necessary condition (1), which comes from small perturbations of a constant function, in some cases also is sufficient. This is e.g. true for the Gauss and two-point measures for $p\leqq q$ unless possibly if $3/2<p\leqq q<2$ or $2<p\leqq q<3$ [16]. The condition is also sufficient for real ω when μ is the uniform measure on a circle [17] or a sphere in R^3 (Section 6 below).

To show that the condition not always is sufficient we use terms of higher order in the binomial expansions. In probabilistic terminology, the following theorem states that the variables $x_1\dots x_d$ and linear combinations of them have skewness zero and non-positive excess.

Theorem 4. *If* $\|T_\omega\|_{pq}=1$ *with* $\omega=\pm\sqrt{\dfrac{p-1}{q-1}}$ *for some p and q such that* $1<p$, $q<\infty$ *and* $p\neq q$, *then* $E\varphi^3=0$ *for* $\varphi\in Q_1$. *If this holds for all* $q>p$, *then* $E\varphi^4\leqq 3(E\varphi^2)^2$ *for* $\varphi\in Q_1$.

Proof. Let $\varphi\in Q_1$ and ε be real. We may assume that $\int\varphi^2\,d\mu=1$. Then

$$\int|1+\varepsilon\varphi|^p\,d\mu=\int\Big(1+p\varepsilon\varphi+\frac{1}{2}p(p-1)\varepsilon^2\varphi^2+\frac{1}{6}p(p-1)(p-2)\varepsilon^3\varphi^3$$

$$+\frac{1}{24}p(p-1)(p-2)(p-3)\,\varepsilon^4\varphi^4\Big)\,d\mu+O(|\varepsilon|^5)=1$$

$$+\frac{1}{2}p(p-1)\varepsilon^2+\frac{1}{6}p(p-1)(p-2)\varepsilon^3\int\varphi^3+\frac{1}{24}p(p-1)(p-2)(p-3)\varepsilon^4\int\varphi^4+O(|\varepsilon|^5).$$

$$\|1+\varepsilon\varphi\|_p$$

$$=1+\frac{1}{p}\Big(\frac{1}{2}p(p-1)\varepsilon^2+\frac{1}{6}p(p-1)(p-2)\varepsilon^3\int\varphi^3+\frac{1}{24}p(p-1)(p-2)(p-3)\varepsilon^4\int\varphi^4\Big)$$

$$+\frac{1}{2}\frac{1}{p}\Big(\frac{1}{p}-1\Big)\Big(\frac{1}{2}p(p-1)\,\varepsilon^2\Big)^2+O(|\varepsilon|^5)$$

$$=1+\frac{1}{2}(p-1)\varepsilon^2+\frac{1}{6}(p-1)(p-2)\varepsilon^3\int\varphi^3$$

$$+\frac{1}{24}(p-1)(p-2)(p-3)\varepsilon^4\int\varphi^4-\frac{(p-1)^3}{8}\varepsilon^4+O(|\varepsilon|^5).$$

Substituting this expression for both sides in $\|1+\varepsilon\omega\varphi\|_q=\|T(1+\varepsilon\varphi)\|_q\leqq\|1+\varepsilon\varphi\|_p$, we see that the zero and second order terms cancel. Thus the third order term on the left hand side is smaller than the one on the right hand side, and replacing ε by $-\varepsilon$ we see that they must be equal. Hence $\int\varphi^3=0$ or $(q-1)(q-2)\omega^3=(p-1)(p-2)$, but the latter equation implies $(q-2)(q-1)^{-1/2}=(p-2)(p-1)^{-1/2}$ which is impossible since $q\neq p$ and $(x-2)(x-1)^{-1/2}$ is increasing.

Since also the third order terms cancel, we obtain from the fourth order terms

$$\Big((q-1)(q-2)(q-3)\int\varphi^4-3(q-1)^3\Big)\Big(\frac{p-1}{q-1}\Big)^2\leqq(p-1)(p-2)(p-3)\int\varphi^4-3(p-1)^3$$

$$(q-p)(pq-p-q-1)\int\varphi^4\leqq3(q-p)(pq-p-q+1),$$

and we obtain $\int\varphi^4\leqq3$ as $q\to\infty$.

Theorem 5. $\|T_{-1}\|_{p,p}=1$ *if and only if* $p=2$ *or* μ *is symmetric about its center of mass.*

Proof. We may without loss of generality assume that the center of mass $\int x\,d\mu$ is the origin (cf. Theorem 7). Then μ is symmetric if and only if $\int f(x)d\mu=\int f(-x)d\mu$, $f\in L^1$. In that case $T_{-1}f(x)=f(-x)$ and thus $\|T_{-1}\|_{p,p}=1$.

Assume that $\|T_{-1}\|_{p,p}=1$ and that p is not an integer. Then T_{-1}, being its own inverse, is an isometry and hence $\|1+\varepsilon\varphi\|_p=\|1-\varepsilon\varphi\|_p$, where $\varphi\in Q_1$. Compute

$\int |1\pm\varepsilon\varphi|^p d\mu$ as above, carrying out the binomial expansion further on. All terms of even degree cancel, but the terms of odd degree have to be zero. From this follows that every odd order moment $\int x^\alpha d\mu$, $|\alpha|$ odd, is zero and hence $\int f(x)d\mu = \int f(-x)d\mu$ for every polynomial f, whence μ is symmetric.

If p is an integer $\neq 2$, we may by Theorem 1 replace p by the conjugate exponent, and, by interpolation, by e.g. 2.5.

For $|\omega|<1$, the Mehler kernel and the Poisson kernel are continuous. Since the circle is compact, the Poisson kernel is bounded whence the Poisson integral is bounded even on L^1 to L^∞. However, for the Mehler transform, (1) is necessary also for boundedness. This follows from explicit calculations with $f(x)=e^{ax}$ or e^{ax^2}. Conversely, if there is strict inequality in (1), the Mehler transform is bounded [6], [16], [12]. Hence, if $p\leqq q$, at least in many cases the transform is a contraction whenever it is bounded. If $p>q$ this fails unless p or q equal 2 [16]; if $p>2>q$ the Mehler transform is bounded for some ω with $|\omega|>1$.

Example 4. χ^2-distribution. (Laguerre polynomials.) $d\mu=(2\pi x)^{-1/2}e^{-x/2}dx$, $x>0$. We may regard x as the square of a normally distributed variable y. Thus the polynomials in x are the polynomials in y having only terms of even degree.

$T_{\omega^2}(\mu)$ is the restriction of $T_\omega(\nu)$ (ν the Gauss measure) to these polynomials. Hence $T_\omega(\mu)$ is bounded if $\sqrt{\omega}$ satisfies the sufficient condition for the Gauss measure. Conversely, if $T_\omega(\mu)$ is bounded it follows by computing norms for $e^{ax}=e^{ay^2}$ that (1) holds for $\sqrt{\omega}$. In particular, for real ω, $T_\omega(\mu)$ is a contraction (or bounded) if and only if $-\min\big((p-1)^2, (q-1)^{-2}, 1\big)\leqq\omega\leqq\min\big(1, (p-1)/(q-1)\big)$. The same is true for $\chi^2(n)$, $n=1, 2, \ldots$ and in particular for the exponential distribution.

4. Changing the measure

Theorem 6. *If* $\|T_\omega(\mu)\|_{p,q}=\|T_\omega(\nu)\|_{p,q}=1$ *and* $p\leqq q$ *then* $\|T_\omega(\mu\times\nu)\|_{p,q}=1$.

Proof. $Q_i(\mu)\otimes Q_j(\nu)$ and $Q_k(\mu)\otimes Q_m(\nu)$ are orthogonal in $L^2(\mu\times\nu)$ unless $i=k$ and $j=m$. Hence $Q_n(\mu\times\nu)=\bigoplus_{i+j=n} Q_i(\mu)\otimes Q_j(\nu)$ and thus $T_\omega(\mu\times\nu)=T_\omega(\mu)\otimes T_\omega(\nu)$. The theorem now follows by an application of Minkowski's inequality [4], [3].

Also if $p>q$ and $K_\omega(\mu)$ or $K_\omega(\nu)$ is positive, the conclusion holds [13].

Iterating and passing to the limit, we see that the theorem also holds for infinite products [10].

Theorem 7. *If* $\|T_\omega(\mu)\|_{p,q}=1$ and h is an affine mapping of R^d onto itself, then $\|T_\omega(h(\mu))\|_{p,q}=1$.

Proof. We regard $x_1, \ldots, x_d$ as d random variables. It is obvious that these generate the same spaces of polynomials, and hence the same operator as the transformed variables.

We may also do injections, increasing the dimension, but for projections severe restrictions are needed (at least for the following proof).

Theorem 8. *Let the projection* $(x, y)\to x$, $x\in R^{d_1}$, $y\in R^{d_2}$, *map* μ *to* ν. *If* $\|T_\omega(\mu)\|_{p,q}=1$ *and either*

a) *for any monomial* y^α, *the conditional expectation* $E(y^\alpha|x)$ *is a polynomial of degree at most* $|\alpha|$ *in* x, *or*

b) $p=2$ or $q=2$,

then $\|T_\omega(\nu)\|_{q,p}=1$.

Proof. $P_n(\nu)$, the polynomials in x of degree at most n, is a subspace of $P_n(\mu)$ and the embedding is an isometry for the L^p-norms.

Now assume a). If $\varphi(x)\in Q_n(\nu)$ and $x^\alpha y^\beta$ is a monomial with $|\alpha|+|\beta|<n$, $\int\varphi(x)x^\alpha y^\beta d\mu=\int\varphi(x)x^\alpha E(y^\beta|x)d\nu=0$, since $x^\alpha E(y^\beta|x)$ is a polynomial of degree less than n. Hence $\varphi(x)\perp P_{n-1}(\mu)$, i.e. $\varphi(x)\in Q_n(\mu)$. Thus $Q_n(\nu)\subset Q_n(\mu)$ and consequently $T_\omega(\nu)$ is the restriction of $T_\omega(\mu)$ to functions of x only. (Conversely, condition a) is necessary for this to be true.) Thus $\|T_\omega(\nu)\|\leqq\|T_\omega(\mu)\|$.

In general, we have only $Q_n(\nu)\subset P_n(\mu)$. Let $f\in Q_n(\nu)$. Then $f=\sum_0^n f_k$, $f_k\in Q_k(\mu)$. Hence $T_z^{-1}(\mu)T_z(\nu)f=T_z^{-1}(\mu)(z^n f)=\sum_0^n z^{n-k}f_k$ is a polynomial in the complex variable z. Thus $T_z^{-1}(\mu)T_z(\nu)$ is an analytic family of mappings. Since $T_z^{-1}(\mu)T_z(\nu)$ is an isometry of $L^2(\nu)$ into $L^2(\mu)$ when $|z|=1$, $\|T_\omega^{-1}(\mu)T_\omega(\nu)\|_{2,2}\leqq 1$ by the maximum principle. Hence, if $p=2$, $\|T_\omega(\nu)\|_{2,q}\leqq\|T_\omega(\mu)\|_{2,q}\|T_\omega^{-1}(\mu)T_\omega(\mu)\|_{2,2}\leqq 1$. The case $q=2$ follows similarly or by duality.

Given two measures on R^d, their product is mapped to their convolution by the mappings $(x, y)\to(x+y, y)\to x+y$ $(x, y\in R^d)$. Combining Theorems 6, 7 and 8 we obtain the following corollary.

Corollary 1. *Let* μ *and* ν *be probability measures on* R^d. *Assume that* $p\leqq q$, $\|T_\omega(\mu)\|_{p,q}=\|T_\omega(\nu)\|_{p,q}=1$ *and either*

a) $E(y^\alpha|x+y)$ *is a polynomial in* $x+y$ *of degree at most* $|\alpha|$ *for every multiindex* α, *or*

b) $p=2$ or $q=2$.

Then $\|T_\omega(\mu*\nu)\|_{q,p}=1$.

Theorem 9. *Let* $\mu, \mu_1, \mu_2, \dots$ *be probability measures on* R^d *such that* μ *is nonsingular,* $\mu_n\to\mu$ *weakly and* $\sup\int|x|^m d\mu_n<\infty$, $m=1, 2, \dots$. *If* $\|T_\omega(\mu_n)\|_{p,q}=1$ for $n=1, 2, \dots$, *then* $\|T_\omega(\mu)\|_{p,q}=1$.

Proof. It follows that $\int x^\alpha d\mu_n\to\int x^\alpha d\mu$. Let f be a polynomial of degree k. By assumption, the monomials x^α, $|\alpha|\leqq k$, are linearly independent in $L^2(\mu)$, i.e. the matrix $\left(\int x^\alpha x^\beta d\mu_n\right)_{\alpha\beta}$ is non-singular. By continuity, so is $\left(\int x^\alpha x^\beta d\mu_n\right)_{\alpha\beta}$ and thus the monomials are independent in $L^2(\mu_n)$ also, for every n except possibly finitely many. Order these monomials so that the degree $|\alpha|$ is non-decreasing and apply in $L^2(\mu)$ and $L^2(\mu_n)$ the Gram-Schmidt orthogonalization process to them to obtain orthogonal polynomials φ_α and $\varphi_{\alpha,n}$. The Gram-Schmidt process depends only on the scalar products and does so continuously. Hence $\varphi_{\alpha,n}\to\varphi_\alpha$ as $n\to\infty$ in the sense of convergence of the coefficients. Let $f=\sum a_\alpha\varphi_\alpha$ and define $f_n=\sum a_\alpha\varphi_{\alpha,n}$. Then

$$\int|f_n|^p d\mu_n-\int|f|^p d\mu=\int(|f_n|^p-|f|^p)d\mu_n+\int|f|^p d\mu_n-\int|f|^p d\mu\to 0,$$

as $n\to\infty$, and thus $\|f\|_{L^p(\mu)}=\lim\|f_n\|_{L^p(\mu)}$. Since $T_\omega(\mu_n)f_n=\sum a_\alpha\omega^{|\alpha|}\varphi_{\alpha,n}$, it also follows that

$$\|T_\omega(\mu)f\|_{L^q(\mu)}=\lim\|T_\omega(\mu_n)f_n\|_{L^q(\mu_n)}\leqq\liminf\|f_n\|_{L^p(\mu_n)}=\|f\|_{L^p(\mu)}.$$

Examples. 5. Assume that $p\leqq q$ and $\|T_\omega\|_{p,q}=1$ for a symmetric two-point measure (all symmetric two-point measures are equivalent by Theorem 7), cf. Example 2. By Corollary 1a and induction, $\|T_\omega\|_{p,q}=1$ also for the binomial distributions Bi $(n, 1/2)$. Normalizing these (by Theorem 7) and applying the Central Limit Theorem we conclude that the same is true for a normal distribution. This is implicitly done in the papers by Gross [9] and Beckner [3].

If $p=2$, we may conclude that $\|T_\omega\|_{p,q}=1$ for any finite or infinite convolution of symmetric two-point measures, cf. the proof of Theorem 10 below.

6. If $\|T_\omega\|_{p,q}=1$ for a normal distribution and $p\leqq q$, the same holds for all normal distributions in all dimensions.

7. Weissler's [17] result for the circle and Theorem 8a yield $\|T_\omega\|_{p,q}=1$ if ω real, $p\leqq q$ and $\omega^2\leqq\dfrac{p-1}{q-1}$ for the measure $\pi^{-1}(1-x^2)^{-1/2}dx$, $-1<x<1$. (Chebyshev polynomials.)

5. The infinitesimal inequality

The following technique to obtain hypercontractivity inequalities for real ω was invented by Gross [9]. We do not give the most general version.

$\left\|T_{\sqrt{\frac{p-1}{q-1}}}\right\|_{p,q}=1$ for all $p, q, 1<p\leqq q<\infty$ holds if and only if

$$\left\|\sum a_{k,l}\left(\frac{p-1}{q-1}\right)^{k/2}\varphi_{k,l}\right\|_q\leqq\left\|\sum a_{k,l}\varphi_{k,l}\right\|_p$$

for every polynomial $\sum a_{k,l}\varphi_{k,l}$. This may be written $\|\sum b_{k,l}(q-1)^{-k/2}\varphi_{k,l}\|_q\leqq\leqq\|\sum b_{k,l}(p-1)^{-k/2}\varphi_{k,l}\|_p$, $q\geqq p$, i.e. $\|\sum b_{k,l}(p-1)^{-k/2}\varphi_{k,l}\|_p$ is a decreasing function of p. This is checked by differentiation: (d/dr below is taken at $r=1$).

$$0 \geqq \frac{d}{dp} \|\Sigma b_{k,l}(p-1)^{-k/2}\varphi_{k,l}\|_p$$

$$= -\frac{1}{p}\|f\|_p \log\|f\|_p + \frac{1}{p}\|f\|_p^{1-p}\Big(\int |f|^p \log|f| - \frac{1}{2(p-1)}\frac{d}{dr}\int |T_r f|^p\Big),$$

for any polynomial $f=\sum b_{k,l}(p-1)^{-k/2}\varphi_{k,l}$. Thus the hypercontractive inequalities with $\omega=\sqrt{(p-1)/(q-1)}$ for all $p<q$ are equivalent to

$$\int |f|^p \log|f| \leqq \frac{1}{2(p-1)}\frac{d}{dr}\int |T_r f|^p + \|f\|_p^p \log\|f\|_p \quad \text{for all } p, \quad 1<p<\infty.$$

Introducing the operator H defined by $H\varphi_{k,l}=k\varphi_{k,l}$, so that $T_\omega=\omega^H$, $\frac{d}{dr}\int |T_r f|^p\, d\mu = p \operatorname{Re}\int |f|^{p-2}\bar{f} Hf\, d\mu$ (this is the version used by Gross).

6. The sphere S^2

We will now prove the promised theorem for the uniform measure on a sphere in R^3. We repeat that T is the Poisson integral so that the conclusion may be written $\|u(\omega x)\|_q \leqq \|u\|_p$, u harmonic in the interior.

Theorem 10. *Let μ be the uniform measure on the unit sphere S^2 in R^3. If $p\leqq q$ and* $-\sqrt{\frac{p-1}{q-1}} \leqq \omega \leqq \sqrt{\frac{p-1}{q-1}}$, *then* $\|T_\omega\|_{p,q}\leqq 1$.

Proof. The result is true for $p=q$. Hence we may assume that $p<q$ and $\omega=\sqrt{(p-1)/(q-1)}$. Let the projection $(x_1, x_2, x_3)\to x_1$ map μ to the one-dimensional measure ν. Condition a) of Theorem 8 is fulfilled, and thus $T_\omega(\nu)$ is the restriction of $T_\omega(\mu)$ to the set of functions of x_1. ν is the uniform measure on $[-1, 1]$ and the crucial fact is that this is an infinite convolution of two-point measures ν_n, viz. $\nu_n=(\delta_{2^{-n}}+\delta_{-2^{-n}})/2$, $n=1, 2, \ldots$. Since $\|T_\omega(\nu_n)\|_{p,q}\leqq 1$, Corollary 1b, induction and Theorem 9 yield $\|T_\omega(\nu)\|_{p,q}\leqq 1$ in the case $p=2$. Thus we have proved a special case: If f is a function of x_1 only and $p=2$, then $\|T_\omega f\|_q\leqq\|f\|_p$. This preliminary result yields by the preceding section

$$\int |f|^2 \log|f| \leqq \frac{1}{2}\frac{d}{dr}\int |T_r f|^2 + \|f\|_2^2 \log\|f\|_2,$$

if f is a function of x_1 only.

Next we remove the restriction $p=2$ by copying from the proof by Weissler [17] for the circle. Let $f\in C^\infty$ be positive and apply the last inequality to $f^{p/2}$.

$$\frac{p}{2}\int f^p \log f \leqq \frac{1}{2}\frac{d}{dr}\int |T_r f^{p/2}|^2 + \frac{p}{2}\|f\|_p^p \log \|f\|_p.$$

Let u and v be the harmonic extensions of f and $f^{p/2}$ respectively to the interior B of S^2. By Gauss' theorem

$$\frac{d}{dr}\int_{S^2} |T_r f^{p/2}|^2 = \int_{S^2}\frac{d}{dr}v^2 = \frac{1}{4\pi}\int_B \Delta(v^2) = \frac{1}{2\pi}\int_B |\nabla v|^2.$$

By Dirichlet's principle, this is less than

$$\frac{1}{2\pi}\int_B |\nabla(u^{p/2})|^2 = \frac{p}{8\pi(p-1)}\int_B \Delta(u^p) = \frac{p}{2(p-1)}\int_{S^2}\frac{d}{dr}u^p = \frac{p}{2(p-1)}\int \frac{d}{dr}|T_r f|^p.$$

This proves the infinitesimal inequality for an arbitrary p and a positive function f of x_1 only. Gross' machinery in the opposite direction yields $\|T_\omega f\|_q \leqq \|f\|_p$ if f a function of x_1 only.

Finally, let f and g be arbitrary functions in $L^p(\mu)$ and $L^{q'}(\mu)$ respectively, and let $\tilde{f}$ and $\tilde{g}$ be the symmetric non-decreasing rearrangements of $|f|$ and $|g|$. These are functions of x_1 only. By a rearrangement inequality by Baernstein and Taylor [2]

$$\left|\int gT_\omega f\right| \leqq \iint |g(x)||f(y)|K_\omega(x,y) \leqq \iint \tilde{g}(x)\tilde{f}(y)K_\omega(x,y)$$
$$= \int \tilde{g}T_\omega \tilde{f} \leqq \|\tilde{g}\|_{q'}\|T_\omega \tilde{f}\|_q \leqq \|\tilde{g}\|_{q'}\|\tilde{f}\|_p = \|g\|_{q'}\|f\|_p.$$

Since g is arbitrary $\|T_\omega f\|_q \leqq \|f\|_p$. Q.E.D.

Remarks. 1. It follows from the proof that $\|T_\omega(\nu)\|_{p,q}=1$ for

$$-\sqrt{(p-1)/(q-1)} \leqq \omega \leqq \sqrt{(p-1)/(q-1)},$$

where ν is the uniform measure on an interval. (Legendre polynomials.)

2. The only part of the proof that depends on the dimension is the decomposition of ν as a convolution of two-point measures. Thus, in higher dimensions also, hypercontractivity for the uniform measure on the sphere is equivalent to hypercontractivity for the projection $c_d(1-x^2)^{(d-3)/2}dx$, and the general result follows from the case $p=2$.

Conjecture. If μ is the uniform measure on a sphere in R^d, $\|T_\omega\|_{p,q}=1$ for $-\sqrt{(p-1)/(q-1)} \leqq \omega \leqq \sqrt{(p-1)/(q-1)}$.

We repeat that this is true for $d=1, 2, 3$. Remark 2 suggests the following generalization.

Conjecture. The same is true for the measures $c_\lambda(1-x^2)^\lambda dx$ on $(-1, 1)$, $\lambda>-1$. (Gegenbauer polynomials.)

7. Analytic functions

There is an analytic analogue of the theory developed in the preceding sections. We start with a probability measure μ on C^d and define the spaces $\tilde{P}_k$ and $\tilde{Q}_k$ as before but consisting of polynomials in $z_1, ..., z_d$. The corresponding operator is denoted $\tilde{T}_\omega$. Instead of the spaces $L^p(\mu)$, we study their subspaces $H^p(\mu)$ spanned by the analytic polynomials.

In fact, this version includes the earlier one. If the measure is supported on R^d, we evidently get the same result with both definitions. However, in general it is possible to obtain better bounds for the analytic version, i.e. Theorem 3 does not hold. We obtain, by the same method but averaging over all ε with the same modulus, the necessary condition $|\omega|\leqq\sqrt{p/q}$ and we will soon show that this can not be improved. Now, there is no reason to stop at $p=1$, the theory works for $p>0$, a common phenomenon when dealing with analytic functions.

The theorems in Section 4 hold with minor modifications: The mapping in Theorem 7 must be complex linear and in Theorem 8 (and similarly in Corollary 1) Condition a) may be replaced by the now weaker

$$\int \varphi(z)z^\alpha E(w^\beta|z)\,dv = \int \varphi(z)z^\alpha w^\beta\,d\mu = 0, \quad \varphi\in\tilde{Q}_n(v), \quad n > |\alpha|+|\beta|.$$

$\|\tilde{T}_{\sqrt{p/q}}\|_{p,q}=1$ for all $p<q$ is equivalent to the infinitesimal inequality

$$\int |f|^p\log|f| \leqq \frac{1}{2p}\frac{d}{dr}\int |\tilde{T}_r f|^p\,d\mu+\|f\|_p^p\log\|f\|_p = \frac{1}{2}\operatorname{Re}\int |f|^{p-2}\bar{f}Hf+\|f\|_p^p\log\|f\|_p$$

[9].

The theory is simplest when the measure is invariant for multiplication of the variables by a constant of modulus one. (Cf. Theorem 5 which expresses an analogue for the real theory.) In the sequel we assume that this is true. Then the monomials z^α are orthogonal and $\tilde{T}_\omega f(z)=f(\omega z)$. Hence, $\|\tilde{T}_\omega\|_{p,p}=1$ if and only if $|\omega|\leqq 1$, $p>0$, and the sets $\{\omega;\ \|\tilde{T}_\omega\|_{p,q}=1\}$ are discs. In fact, $z^\alpha\perp z^\beta\bar{z}^\gamma$ if $|\alpha|>|\beta|+|\gamma|$. Let us also regard μ as a measure on R^{2d} and define T_ω as before. Then $z^\alpha\in Q_{|\alpha|}$ and $\tilde{T}_\omega$ is the restriction of T_ω to the analytic polynomials. Thus $\|\tilde{T}_\omega\|_{p,q}\leqq\|T_\omega\|_{p,q}$. We will now show that it is possible to have better results for $\tilde{T}_\omega$ than for T_ω.

Theorem 11. *Let μ be either the uniform measure on the unit circle,* $(2\pi)^{-1} e^{-|z|^2/2}dx\,dy$ or $(2\pi)^{-1}(1-|z|^2)^{-1/2}dx\,dy$, $|z|<1$. Then $\tilde{T}\|_{\omega}\|_{p,q}=1$, i.e. $\|f(\omega z)\|_{H^q}\leqq\|f(z)\|_{H^p}$, for $|\omega|\leqq\sqrt{p/q}$, $0<y\leqq q<\infty$.

Proof. It suffices, by continuity, to take $0\leqq\omega<\sqrt{p/q}$. Choose n such that $\omega\leqq\sqrt{\frac{np-1}{nq-1}}$. By Example 3, Example 6 or Theorems 10 and 8 (the third allowed measure is the projection of the surface measure on the sphere in R^3), $\|T_\omega(|f|^{1/n})\|_{nq}\leqq\|\,|f|^{1/n}\|_{np}=\|f\|_p^{1/n}$. We will show that $T_\omega g(z)\geqq g(\omega z)$ for any subharmonic function g. This, applied to $|f|^{1/n}$, yields

$$\|f(\omega z)\|_q=\|\,|f(\omega z)|^{1/n}\|_{nq}^n\leqq\|T_\omega(|f|^{1/n})\|_{nq}^n\leqq\|f\|_p,\quad f\in H^p.$$

The pointwise inequality for subharmonic functions follows in the first case-since T_ω is the Poisson integral and in the second case since the kernel is $K_\omega(z,w)d\mu(w)=\big(2\pi(1-\omega^2)\big)^{-1}\exp\big(-|\omega z-w|^2/2(1-\omega^2)\big)dw_1dw_2$ which is positive and constant on circles around ωz. In the third case, let $g'(x_1,x_2,x_3)=g(x_1,x_2)$. Then $T_\omega g(x_1,x_2)=P_\omega g'\big(x_1,x_2,\sqrt{1-x_1^2-x_2^2}\big)$, where P_ω is the Poisson integral on S^2. Since g' is subharmonic, this is at least $g'\big(\omega x_1,\omega x_2,\omega\sqrt{1-x_1^2-x_2^2}\big)=g(\omega x_1,\omega x_2)$.

In the first case the spaces H^p are the classical Hardy spaces. (Special cases are given by Bonami [4].)

Problem. For which rotationally invariant measures in the complex plane is $\|f(\omega z)\|_q\leqq\|f\|_p$ for f analytic and $|\omega|\leqq\sqrt{p/q}$?

This class of measures is closed under convolution by the analytic version of Corollary 1a). This yields other examples from the thre measures of Theorem 11. A necessary condition, derived as Theorem 4, is $\int|z|^4d\mu\leqq2\big(\int|z|^2d\mu\big)^2$ which means that the real part of z has non-positive excess.

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