Exact bounds for the continuous spectrum of certain differential eigenvalue problems

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0. Introduction

Let Ω be an unbounded domain in the real n-dimensional cartesian spac. $\mathbf{R}^n$ and let a and k be real-valued and Lebesgue measurable functions on Ωe The function k is not required to have a constant sign. We shall consider a Hilbert space realization of the spectral problem

$$(-\sum_{j=1}^{n} \partial^2/\partial x_j^2 + a(x))u = \lambda k(x)u \quad \text{in} \quad \Omega, \tag{0.1}$$

$$u = 0 \quad \text{on the boundary,} \tag{0.2}$$

where λ is the eigenvalue parameter. Under certain conditions (Sections 1 and 2) we shall deduce exact bounds for the positive and for the negative continuous spectrum of this problem. The case when $k(x) = 1$ for all x in Ω was treated by Arne Persson in [7].

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1. Conditions, the spectral problem

Let $C_0^\infty(\Omega)$ be the set of all infinitely differentiable real-valued functions with compact support in Ω and write

$$(u, v) = \int_\Omega (\text{grad}\, u \,\text{grad}\, v + auv)dx \tag{1.1}$$

when u and v are in $C_0^\infty(\Omega)$. *It is assumed that* (u, u) *is positive definite on* $C_0^\infty(\Omega)$. *Furthermore there shall exist a constant* C *such that*

$$\int_{\Omega} |k| u^2 dx \leq C(u, u) \tag{1.2}$$

on $C_0^\infty(\Omega)$. Completion of $C_0^\infty(\Omega)$ with respect to $|.|$, $|u| = (u, u)^{1/2}$, gives a Hilbert space H on which (u, v) and, because of (1.2), also a form corresponding to

$$K(u, v) = \int_{\Omega} kuv\, dx \quad \text{on} \quad C_0^\infty(\Omega) \tag{1.3}$$

can be defined as limits on Cauchy sequences in $C_0^\infty(\Omega)$. The form $K(u, v)$ becomes symmetric and bounded. Hence

$$K(u, v) = (\mathbf{K}u, v) \tag{1.4}$$

defines a selfadjoint operator $\mathbf{K}$. Under suitable conditions the spectral problem for $\mathbf{K}$, i.e. for the equation $\mathbf{K}u = \mu u$, is equivalent to the eigenvalue problem (0.1), (0.2) if $\lambda = \mu^{-1}$, see [8], [9]. Our aim is not to discuss such conditions but to obtain exact bounds for the continuous spectrum of $\mathbf{K}$ as defined by (1.4).

The eigenspace corresponding to a real interval I is denoted by $H(I)$. For reference we state the following consequences of the spectral theorem. (1) *If* $u \neq 0$ *is an element of* $H(I)$, *then* $(\mathbf{K}u, u)/(u, u)$ *i.e.* $K(u, u)/(u, u)$ *belongs to* I. (2) *The spectrum is discrete in* I *if* $H(I_0)$ *is finite dimensional for every closed interval* I_0 *contained in the open kernel of* I.

2. Further conditions

Precompactness of a quadratic form Q on a linear space with scalar product $(.\,,.)$ can be characterized by a compactness inequality. To every $\varepsilon > 0$ there shall exist a finite number of $(.\,,.)$-bounded linear forms $L_1, L_2, \ldots, L_N$ such that

$$|Q(u, u)| \leq \varepsilon(u, u) + \sum_{j=1}^{N} |L_j(u)|^2. \tag{2.1}$$

Precompact is compact if the space is complete.

Let S_r be the intersection of Ω and the sphere $\{x : |x| < r\}$. Define a^- by $a^-(x) = \frac{1}{2}(|a(x)| - a(x))$. *We assume that for every finite* r *the integrals*

$$\int_{S_r} u^2 dx, \int_{S_r} a^-(x) u^2 dx, \int_{S_r} |k(x)| u^2 dx \tag{2.2}$$

are precompact on $C_0^\infty(\Omega)$ *with respect to the scalar product in* (1.1). Precompactness implies boundedness. Thus for every r and every u in $C_0^\infty(\Omega)$

$$\int_{S_r} u^2\, dx \leq C_1(r)(u, u), \tag{2.3}$$

where the constant $C_1(r)$ depends on r. Similar inequalities are valid for the other integrals in (2.2).

It is furthermore assumed that on $C_0^\infty(\Omega)$

$$\int_{S_r} \operatorname{grad}^2 u\, dx \leq C_2(r)(u, u), \tag{2.4}$$

where the constant $C_2(r)$ *depends upon* r.

The inequality (2.3) shows that every element u in H (Cauchy sequence) determines a function also denoted by u which is square integrable over S_r. Since r is arbitrary the definition of u can be extended to the whole of Ω. According to (2.4) the function u has generalized locally square integrable derivatives of the first order. For u in H, values can be attributed to the left hand sides of (2.3) and (2.4) either as limits on Cauchy sequences or as integrals in which the corresponding function and its generalized derivatives are introduced. The inequalities (2.3), (2.4) remain valid in H. On account of their boundedness the other two integrals in (2.2) can be similarly defined as limits or as integrals. The integrals (2.2) are compact on H.

From the inequality (1.2) it follows that for u and v in H the value of $K(u, v)$ equals the integral in (1.3) in which the corresponding functions in $L^2_{\text{loc}}(dx)$ are inserted. These functions belong to $L^2(|k|dx)$. The inequality (1.2) is valid also in H.

The possibility that a non-zero element of H may determine a function u which is 0 almost everywhere is not excluded. However, an eigenspace of $\mathbf{K}$ belonging to an interval I of positive distance from the origin is in one-to-one correspondence with its functions in $L^2_{\text{loc}}(dx)$. This follows from statement (1), end of Section 1.

3. Sufficient conditions

As in Courant & Hilbert [1], pp. 515—521, it can be proved that for every $\varepsilon > 0$ a Friedrich's inequality

$$\int_{S_r} u^2 dx \leq \varepsilon \int_{S_r} \operatorname{grad}^2 u\, dx + \sum_{j=1}^{N} \left(\int_{S_r} w_j u\, dx \right)^2 \tag{3.1}$$

holds on $C_0^\infty(\Omega)$. The functions $w_1, w_2, \ldots, w_N$ are bounded and integrable with compact supports. The proof is simplified since, when S_r is not contained in Ω, the functions in $C_0^\infty(\Omega)$ vanish identically in neighbourhoods of the common boundary of S_r and Ω.

If $a^-(x)$ *and* $k(x)$ *are bounded in every* S_r, *and if* (2.3) *and* (2.4) *hold, then the compactness of the integrals* (2.2) *is a consequence of* (3.1). For according to (2.3) the linear forms in (3.1) are $(.\,,.)$-bounded and by the use of (2.4) one can introduce (u, u) in the ε-term. Thus the first integral in (2.2) is precompact on $C_0^\infty(\Omega)$, compact on H. Because of the boundedness of $a^-(x)$ and $k(x)$ in S_r the last two integrals in (2.2) are majorized by the first one, hence compact.

Remark. Under the conditions of Sections 1 and 2 we shall obtain bounds for the continuous spectrum of $\mathbf{K}$. Any set of conditions which imply ours will evidently do for the same purpose. Such a set (not assuming boundedness of a^- in S_r) is due to Arne Persson when $k(x)$ is identically 1. His assumption that $a(x)$ be bounded from below for large values of $|x|$ is, however, not necessary. In Section 9 we shall indicate an example in which $a(x)$ is not bounded from below but all conditions of Sections 1 and 2 are fulfilled.

4. Result to be obtained

Let

$$L_r^+ = \sup\,(K(u, u)/(u, u)), \quad L_r^- = \inf\,(K(u, u)/(u, u))$$

when u varies in $C_0^\infty(\Omega - S_r)$, and consider the non-increasing and non-decreasing limits

$$L^+ = \lim_{r\to\infty} L_r^+, \quad L^- = \lim_{r\to\infty} L_r^-.$$

An easy consequence of the compactness of the third integral in (2.2) is that $L^+ \geq 0$ and $L^- \leq 0$. To see this take $R > r$. For $Q(u, u) = \int_{S_R} |k| u^2 dx$, the precompactness relation (2.1) holds with certain $L_1, L_2, \ldots, L_N$. The space $C_0^\infty(S_R - S_r)$ is of infinite dimension and thus contains functions $u \neq 0$ for which $L_j(u) = 0$, $j = 1, 2, \ldots, N$. For such a function the precompactness relation gives $\int_{S_R} |k| u^2 dx \leq \varepsilon(u, u)$ so that

$$|K(u, u)|/(u, u) \leq \varepsilon.$$

Thus $L_r^+ \geq -\varepsilon$ and $L_r^- \leq \varepsilon$ for any $\varepsilon > 0$ which shows that $L^+ \geq 0$ and $L^- \leq 0$. We shall prove the following

THEOREM. *The spectrum of* $\mathbf{K}$ *outside* $L^- \leq \mu \leq L^+$ *is discrete and this interval contains no smaller interval with this property.*

5. A basic inequality

For $0 < \varrho < r$ let φ be an infinitely differentiable function which is identically 0 for $|x| \leq \varrho$, identically 1 for $|x| \geq r$ and has all its values in the closed interval $[0, 1]$. With u, also φu belongs to $C_0^\infty(\Omega)$ and $(\varphi u, \varphi u) \geq 0$. Easy computations give

$$(\varphi u, \varphi u) = (u, u) - \int_{S_r} (1 - \varphi^2)\mathrm{grad}^2\, u\, dx - \int_{S_r} (1 - \varphi^2) a u^2 dx + $$
$$+ 2 \int_{S_r} u\varphi \,\mathrm{grad}\, u \,\mathrm{grad}\, \varphi\, dx + \int_{S_r} u^2 \,\mathrm{grad}^2\, \varphi\, dx.$$

According to the Cauchy—Schwarz inequality and by the help of (2.4) it follows that for all $u \neq 0$ in $C_0^\infty(\Omega)$

$$0 \leq (\varphi u, \varphi u)/(u, u) \leq 1 + \int_{S_r} a^- u^2 dx/(u, u) + C_1 \Big(\int_{S_r} u^2 dx/(u, u) \Big)^{\frac{1}{2}} + $$
$$+ C_2 \int_{S_r} u^2 dx/(u, u). \tag{5.1}$$

Here the constants C_1 and C_2 depend upon the choice of φ including the choice of ϱ and r.

A first consequence of (5.1) is that for any element $u = \{u_n\}_1^\infty$ in H, also $\varphi u = \{\varphi u_n\}_1^\infty$ belongs to H. For (5.1) shows, on account of the boundedness of the integrals (2.2), that with u, also $\{\varphi u_n\}_1^\infty$ is a Cauchy sequence (with elements in $C_0^\infty(\Omega - S_\varrho)$). A transition to the limit proves (5.1) for u in H.

6. Upper bound for the positive continuous spectrum

For u in $C_0^\infty(\Omega)$ it is clear that $K(u, u) = K(\varphi u, \varphi u) + \int_\Omega (1 - \varphi^2) k u^2 dx$, a relation which remains valid for u in H. With u in H this formula is divided by (u, u). If $\varphi u = 0$ the quotient $K(\varphi u, \varphi u)/(u, u)$ vanishes. If $\varphi u \neq 0$ the quotient equals $(K(\varphi u, \varphi u)/(\varphi u, \varphi u)) \cdot ((\varphi u, \varphi u)/(u, u))$. Here $K(\varphi u, \varphi u)/(\varphi u, \varphi u) \leq L_\varrho^+$. For u in H the quotient $(\varphi u, \varphi u)/(u, u)$ can be estimated according to (5.1). In both cases

$$K(u, u)/(u, u) \leq L_\varrho^+ \left\{1 + \int_{S_r} a^- u^2\, dx/(u, u) + C_1 \left(\int_{S_r} u^2\, dx/(u, u) \right)^{1/2} + \right.$$

$$\left. + C_2 \int_{S_r} u^2\, dx/(u, u) \right\} + \int_{S_r} |k| u^2\, dx/(u, u) \,, \tag{6.1}$$

where the constants depend upon the choice of φ including the choice of ϱ and r, $\varrho < r$.

Consider $H(L^+ + \delta \leq \mu < \infty)$ for $\delta > 0$. Because of (1.2) the entire spectrum of $\mathbf{K}$ lies in the closed interval $[-C, C]$ and $H(L^+ + \delta \leq \mu < \infty) = H(L^+ + \delta \leq \mu \leq C)$. The dimension of this space is claimed to be finite. Assume the contrary. First choose ε in $0 < \varepsilon < \delta/2$ and take ϱ so large that $L_\varrho^+ \leq L^+ + \varepsilon$. A choice of φ (and $r > \varrho$) gives certain values to C_1 and C_2 in (5.1). Take $\varepsilon_1 > 0$ so small that

$$(L^+ + \varepsilon)\{1 + \varepsilon_1 + C_1 \varepsilon_1^{1/2} + C_2 \varepsilon_1\} + \varepsilon_1 < L^+ + \delta \,.$$

The integrals (2.2) appearing in (6.1) are compact, thus can all be majorized by an expression

$$\varepsilon_1(u, u) + \sum_{j=1}^{N} |L_j(u)|^2 \,.$$

If $H(L^+ + \delta \leq \mu < \infty)$ is of infinite dimension it contains non-zero elements u for which the linear forms $L_j(u)$ vanish, $j = 1, 2, \ldots, N$. For such an element it follows that

$$K(u, u)/(u, u) \leq (L^+ + \varepsilon)\{1 + \varepsilon_1 + C_1 \varepsilon_1^{1/2} + C_2 \varepsilon_1\} + \varepsilon_1 < L^+ + \delta \,.$$

Since u belongs to $H(L^+ + \delta \leq \mu < \infty)$ the last inequality is violated by statement (1) at the end of Section 1. Thus, according to statement (2) in Section 1, the value of L^+ is an upper bound for the positive continuous spectrum.

7. Best upper bound for the positive continuous spectrum

This section depends only on (1.2) and the positive definiteness of (1.1); compare [10], p. 360.

Let $M > 0$ and assume that $H_1 = H(M \leq \mu < \infty)$ is finite-dimensional. Then H_1 is spanned by eigenfunctions $v_1, v_2, \ldots, v_N$ which can be taken orthonormalized with respect to $(.\,,.)$. The eigenspaces H_1 and $H_2 = H(-\infty < \mu < M)$ are orthogonal complements, also orthogonal with respect to $K(.\,,.)$. Put $u = u_1 + u_2$, where u_1 belongs to H_1 and u_2 belongs to H_2. Then $(u, u) = (u_1, u_1) + (u_2, u_2)$ and $K(u, u) = K(u_1, u_1) + K(u_2, u_2)$. According to (1.2),

$K(u_1, u_1) \leq C(u_1, u_1)$ and because of statement (1) in Section 1, $K(u_2, u_2) \leq M(u_2, u_2)$. Hence

$$K(u, u)/(u, u) \leq (C - M)(u_1, u_1)/(u, u) + M . \tag{7.1}$$

Write $u_1 = c_1 v_1 + c_2 v_2 + \ldots + c_N v_N$, where $c_j = (u_1, v_j) = (u, v_j)$. The eigenvalues μ_j corresponding to v_j, $j = 1, 2, \ldots, N$, are positive and $c_j = \mu_j^{-1} K(u, v_j)$.

Let u be a function in $C_0^\infty(\Omega - S_r)$. On account of (1.2)

$$|c_j| = \mu_j^{-1} \left| \int_{\Omega - S_r} k u v_j dx \right| \leq \mu_j^{-1} \left(C(u, u) \int_{\Omega - S_r} |k| v_j^2 dx \right)^{1/2}$$

so that $|c_j|^2 (u, u)^{-1}$ tends to 0 when r tends to infinity, $j = 1, 2, \ldots, N$. If $(u_1, u_1) = c_1^2 + c_2^2 + \ldots + c_N^2$ is introduced into (7.1) the limit property of the coefficients c_j shows that, according to the definition of L^+ in Section 4,

$$L^+ = \limsup \, (K(u, u)/(u, u)) \leq M.$$

Thus L^+ is the least upper bound of the positive continuous spectrum.

8. Bound for the negative continuous spectrum

A change of sign of $k(x)$ shows that L^- is the greatest lower bound for the negative continuous spectrum. Thus the spectrum of $\mathbf{K}$ outside $L^- \leq \mu \leq L^+$ is discrete and this interval is the smallest one with this property. For a differential problem of type (0.1), (0.2) this means that the spectrum in

$$1/L^- < \lambda < 1/L^+$$

is discrete and that no larger interval has this property.

9. Example

We shall show that our conditions can be fulfilled even with a function $a(x)$ which is not bounded from below for large values of $|x|$.

The domain Ω is taken as the entire space $\mathbf{R}^n$. Let p and P be functions on $\mathbf{R}^n$, p real-valued and P vector-valued with values in $\mathbf{R}^n$. If u belongs to $C_0^\infty(\mathbf{R}^n)$, partial integrations give

$$\int_{\mathbf{R}^n} (\operatorname{grad}^2 u + a u^2) dx = \int_{\mathbf{R}^n} (\operatorname{grad} u + uP)^2 dx + \int_{\mathbf{R}^n} (p - P^2) u^2 dx , \tag{9.1}$$

where $a = p - \operatorname{div} P$ and P^2 is the scalar product of P by itself. The integrated parts vanish since u has compact support. Thus the left hand side of (9.1), i.e. (u, u) is positive definite on $C_0^\infty(\mathbf{R}^n)$ provided $p > P^2$ for all x.

Take p = constant and $P(x) = \operatorname{grad} f(r)$ with $r = |x|$ and $f'(r) = A + \sin(r^2)$, where A is a constant. Then

$$a(x) = p - 2r\cos(r^2) - (n-1)r^{-1}(A + \sin(r^2))$$

which is not bounded from below. If $A < 0$ the function $a^-(x)$ is locally bounded. We have $P^2 = (A + \sin(r^2))^2$ and it is easily seen that we can take $p > (|A| + 1)^2$ such that (2.3) and (2.4) hold with the integrals extended over the entire space $\mathbf{R}^n$ (the constants of course independent of r). This is still true if a positive locally bounded function is added to $a(x)$. The negative part of the new function $a(x)$ is still locally bounded. This gives a simple way of constructing a couple of locally bounded functions $a(x)$ and $k(x)$ for which (1.2) and, on account of Section 3, also the other conditions of Sections 1 and 2 are satisfied. The new function $a(x)$ can be chosen unbounded from below.

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