A problem on the union of Helson sets

E. Galanis

Institut Mittag-Leffler, Djursholm, Sweden

Let G be a locally compact abelian group and let $\hat{G}$ be its dual group.

Definition 1. A compact subset $E \subset G$ is called *Kronecker* if for every continuous function f on E of modulus identically one ($|f(x)| \equiv 1, \ \forall x \in E$) and for every $\varepsilon > 0$ there exists $\chi \in \hat{G}$ such that

$$\sup_{x \in E} |f(x) - \chi(x)| \leq \varepsilon \qquad \text{(cf. [1] ch. 5, § 1)}.$$

We shall denote by $M(G)$ the set of all bounded complex valued Radon measures on G and by $M(E)$ the elements of $M(G)$ with support in a compact subset E of G.

We shall denote by $C(E)$ the set of all continuous complex valued functions on E.

Definition 2. A compact subset E of G is called a *Helson α-set* (*H_α-set*) if there exists a constant $\alpha > 0$ such that

$$\|\hat{\mu}\|_\infty = \sup_{\chi \in \hat{G}} |\hat{\mu}(\chi)| \geqq \alpha \, \|\mu\|$$

for every $\mu \in M(E)$, (observe that then $0 < \alpha \leq 1$).

If K is a compact subset of G we shall write $\mathrm{Gp}\,(K)$ for the group generated by K in G.

In this paper we shall prove the following theorem.

Theorem. *Let K be a totally disconnected Kronecker subset of G and D a countable compact H_α-subset of G such that*

$$\mathrm{Gp}\,(K) \cap \mathrm{Gp}\,(D) = \{0\}.$$

Then $K \cup D$ is an H_α-set.

Remarks. Varopoulos [3] has proved that if K is any totally disconnected compact H_1-subset of G and D is any H_α-subset of G then $K \cup D$ is $H_{\beta(\alpha)}$ with $\beta(\alpha) > 0$.

The interest of our theorem lies in the fact that in this special case we have $\beta(\alpha) = \alpha$.

However we must point out that the conclusion of our theorem fails if we replace the condition »Kronecker» by »H_1» for the set K. To see this observe that the set $\{(x, 0), (-x, 0), (0, -y), (0, y)\} \subset \mathbf{R}^2$ $(x, y \in \mathbf{R})$ is not an H_1-set of $\mathbf{R}^2$ despite the fact that it is the union of two H_1-sets

$$E_1 = \{(x, 0), (-x, 0)\}, \quad E_2 = \{(0, -y), (0, y)\}$$

that satisfy $\mathrm{Gp}(E_1) \cap \mathrm{Gp}(E_2) = \{0\}$.

We shall prove first:

LEMMA. *Let K and D be as in the theorem with $D = \{x_1, x_2, \ldots, x_r\}$ a finite set. Then for every $\chi \in \hat{G}$ there exists $\{\chi_n\}_{n=1}^{\infty}$, $\chi_n \in \hat{G}$ such that*

$$\chi_n \underset{n\to\infty}{\longrightarrow} 1 \quad \text{uniformly on } K \text{ and } \quad \chi_n(x_j) \underset{n\to\infty}{\longrightarrow} \chi(x_j)$$

for all $j = 1, 2, \ldots, r$.

Proof. Let L be the set of points $t \in \mathbf{T}^r$ for which there exists a sequence $\{\chi_n \in \hat{G}\}_{n=1}^{\infty}$ such that $\chi_n|_K \underset{n\to\infty}{\longrightarrow} 1$ uniformly, and $(\chi_n(x_1), \chi_n(x_2), \ldots, \chi_n(x_r)) \underset{n\to\infty}{\longrightarrow} t$ in $\mathbf{T}^r$.

Let us also denote by $H = \overline{\{\chi(x_1), \chi(x_2), \ldots, \chi(x_r)\}}$. We observe that both L and H are closed subgroups of $\mathbf{T}^r$ and that $L \subset H$. We shall prove that $L = H$.

Towards that let us suppose by contradiction that $L \neq H$. There exists then a character $\Theta \in \hat{\mathbf{T}}^r$ of $\mathbf{T}^r$ such that

$$\Theta(L) = 1 \quad \text{and} \quad \Theta(H) \neq 1. \tag{1}$$

Since $\Theta \in \hat{\mathbf{T}}^r$ and $\Theta(H) \neq 1$, there exist $n_1, n_2, \ldots, n_r \in \mathbf{Z}$ such that $\Theta(\chi(x_1), \chi(x_2), \ldots, \chi(x_r)) = \chi(n_1x_1 + n_2x_2 + \ldots + n_rx_r) = \chi(y)\ \forall\ \chi \in \hat{G}$ where $y = n_1x_1 + n_2x_2 + \ldots + n_rx_r \neq 0$.

Let now $\{x_n\}_{n=1}^{\infty}$ be a sequence of characters such that the sequence of points $\{(\chi_n(x_1), \ldots, \chi_n(x_r)) \in \mathbf{T}^r\}_{n=1}^{\infty}$ converges towards a point of L, (1) implies then that we have:

$$\Theta(\lim_n (\chi_n(x_1), \ldots, \chi_n(x_r))) = \lim_n \Theta(\chi_n(x_1), \ldots, \chi_n(x_r)) = \lim_n \chi_n(y) = 1.$$

But this implies that if $\{\psi_n \in \hat{G}\}_{n=1}^{\infty}$ is a sequence of characters such that

$$\psi_n\,|_K \underset{n\to\infty}{\longrightarrow} 1 \quad \text{uniformly on} \quad K \quad \text{then} \quad \psi_n(y) \to 1$$

and from that using Corollary 1 of [2] we deduce the required result that $y \in \mathrm{Gp}\,(K)$.

Proof of the theorem.
Case 1. D is finite.

Let $\mu \in M(K \cup D)$; we can write $\mu = \mu_1 + \mu_2$ where $\mu_1 \in M(K)$ and $\mu_2 \in M(D)$. Then

$$\|\hat{\mu}_1\|_\infty = \|\mu_1\| \quad \text{and} \quad \|\hat{\mu}_2\|_\infty \geqq \alpha\|\mu_2\| .$$

We first observe that there exists $\varphi \in \mathbf{R}$ and $\chi_1, \chi_2 \in \hat{G}$ such that

$$e^{i\varphi}\hat{\mu}_2(\chi_2) \geqq 0 , \quad e^{i\varphi}\hat{\mu}_2(\chi_2) \geqq \alpha\|\mu_2\| - \varepsilon \tag{2}$$

$$\mathrm{Re}\, e^{i\varphi}\hat{\mu}_1(\chi_1) \geq \|\mu_1\| - \varepsilon . \tag{3}$$

Indeed choose $\varphi \in \mathbf{R}$ and $\chi_2 \in \hat{G}$ such that (2) holds. Let us show that we can find $\chi_1 \in \hat{G}$ satisfying (3).

We can choose $\psi \in C(K)$, $|\psi| = 1$ such that

$$e^{i\varphi}\int \psi\, d\mu_1 \geq \|\mu_1\| - \varepsilon/2$$

and then since K is Kronecker we can approximate ψ uniformly by $\chi_1 \in \hat{G}$ as close as we like. This shows that χ_1 can be chosen to satisfy (3).

Now from the lemma we can find a sequence $\psi_n \in \hat{G}$ such that $\psi_n \to 1$ uniformly on K and

$$\psi_n\,|_D \to \chi_1^{-1}\chi_2\,|_D .$$

We have then

$$\mathrm{Re}\, e^{i\varphi}\hat{\mu}_1(\chi_1\psi_n) \to \mathrm{Re}\, e^{i\varphi}\hat{\mu}_1(\chi_1)$$

and

$$\mathrm{Re}\, e^{i\varphi}\,\hat{\mu}_2(\chi_1\psi_n) \to \mathrm{Re}\, e^{i\varphi}\,\hat{\mu}_2(\chi_2) .$$

Therefore

$$\|\hat{\mu}\|_\infty \geq \sup_n |\hat{\mu}(\chi_1\psi_n)| \geq \sup_n \mathrm{Re}\,\{e^{i\varphi}\hat{\mu}(\chi_1\psi_n)\} \geqq \|\mu_1\| + \alpha\|\mu_2\| - 2\varepsilon$$
$$\geq \alpha(\|\mu_1\| + \|\mu_2\|) - 2\varepsilon \geqq \alpha\|\mu\| - 2\varepsilon .$$

Since ε is arbitrary the conclusion of the theorem follows.
Case 2. D is countable.
Let $\mu \in M(K \cup D)$ we can write

$$\mu = \mu_1 + \mu_2 + \mu_3 , \quad \mu_1 \in M(K) , \quad \mu_2 , \mu_3 \in M(D)$$

with the support of μ_2 lying in a finite subset of D and $\|\mu_3\| \leq \varepsilon$ $(\varepsilon > 0)$.

Now by case 1

$$\varepsilon + \|\hat{\mu}\|_\infty \geqq \|\widehat{\mu_1 + \mu_2}\|_\infty \geqq \alpha\|\mu_1 + \mu_2\| \geqq \alpha(\|\mu\| - \varepsilon) \geq \alpha\|\mu\| - \varepsilon .$$

Therefore

$$\|\hat{\mu}\|_\infty \geqq \alpha\|\mu\| - 2\varepsilon .$$

And since ε is arbitrary this completes the proof of the theorem.

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E. Galanis
Department of Pure Mathematics
16, Mill Lane
Cambridge
England