The center and the commutator subgroup in hopfian groups

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1. Abstract

We continue our investigation of the direct product of hopfian groups. Throughout this paper A will designate a hopfian group and B will designate (unless we specify otherwise) a group with finitely many normal subgroups. For the most part we will investigate the role of $Z(A)$, the center of A (and to a lesser degree also the role of the commutator subgroup of A) in relation to the hopficity of $A \times B$. Sections 2.1 and 2.2 contain some general results independent of any restrictions on A. We show here

(a) If $A \times B$ is not hopfian for some B, there exists a finite abelian group F such that if k is any positive integer a homomorphism θ_k of $A \times F$ onto A can be found such that θ_k has more than k elements in its kernel.

(b) If A is fixed, a necessary and sufficient condition that $A \times B$ be hopfian for all B is that if θ is a surjective endomorphism of $A \times B$ then there exists a subgroup B_* of B such that $A\theta B = A\theta \times B_*\theta$.

In Section 3.1 we use (a) to establish our main result which is

(c) If all of the primary components of the torsion subgroup of $Z(A)$ obey the minimal condition for subgroups, then $A \times B$ is hopfian.

In Section 3.3 we obtain some results for some finite groups B. For example we show here

(d) If $|B| = p^e q_1^{e_1} \dots q_s^{e_s}$ where $p, q_1 \dots q_s$ are the distinct prime divisors of $|B|$ and if $0 \leq e \leq 3$, $0 \leq e_i \leq 2$ and $Z(A)$ has finitely many elements of order p^2 then $A \times B$ is hopfian.

Several results of the same nature as (d) are obtained here.

In Section 4 we obtain some results similar to (d) by placing some restrictions on the commutator subgroup of A. We also show here

(e) $A \times B$ is hopfian if B is a finite group whose Sylow p subgroups are cyclic.

(f) $A \times B$ is hopfian if B is a perfect group.

Our main avenue of attack on the problems to be considered may be outlined here very briefly. Namely if B has finitely many normal subgroups and $A\times B$ is not hopfian we choose a homomorphic image C of B with as few normal subgroups as possible such that $A\times C$ is not hopfian. Then as in Lemma 7 of [3], $Z(C)$, the center of C is non-trivial and there exists a surjective endomorphism α of $A\times C$ such that α is not an isomorphism on A and such that $C\alpha^r \cap C = 1$ for all integers r, $r\neq 0$. Furthermore C does not have an abelian direct factor. Our approach in this paper is to assume $A\times B$ is not hopfian and to gather information about C. With suitable restrictions we achieve a desired contradiction. Throughout this paper C and α will be as defined here.

The existence or non existence of a hopfian group A with the properties (a) is unresolved. We show in our remarks following Theorem 1 that if $Z(A)$ has a finite torsion group and A has properties (a) then $A = A_1 \cdot F_1$ for some finite central subgroup F_1 and some subgroup A_1 which is a non-hopfian homomorphic image of A. Conversely if A can be decomposed in the above manner then regardless of the nature of $Z(A)$, A has the properties in (a). For if $F \approx F_1$ one can easily obtain a homomorphism of $A\times F$ onto A with arbitrarily large kernel. Baumslag and Solitar have shown that there exists a finitely generated hopfian group with a non-hopfian group of finite index [1]. In view of this anomolous result, we do not think that it is unreasonable to suspect that a group A with properties (a) exists.

In any case our result (c) together with the results of [2] and [3] show that $A\times B$ is hopfian for a wide range of A. In general, extensions of hopfian groups by hopfian groups are studied in [2] and [3] and the latter contains a bibliography of some relevant papers on the subject.

2. Some general results

2.1. *Strong hopficity*

We conjecture that if B has finitely many normal subgroups $A\times B$ must be hopfian. If this conjecture is false A is in a certain sense close to being non-hopfian. For write $A\alpha\cdot C = A\alpha\cdot C_1\alpha = C\times A_*$ where $C_1 \subset Z(C)$, $A_* \subset A$. Note $C\alpha$ is in the centralizer of $A\alpha\cdot C_1\alpha$ so that there is a homomorphism γ of $C\times A_*$ onto $C\alpha\cdot A_* = L$ such that γ is the identity on A_* and such that γ agrees with α on C. Note $L\cdot C = A\times C$ so that $L/L\cap C \approx A$. Hence $\alpha\cdot\gamma$ maps $A\times C_1$ onto L which in turn can be mapped onto A homomorphicly. If we designate the resulting homomorphism of $A\times C_1$ onto A by α_* we see α_* is not an isomorphism on A and since $|A\cap \text{kernel}\,\alpha|$ may be made as large as we please by choosing a suitable α, so may $|A\cap\text{kernel}\,\alpha_*|$. Also we note α_* may be extended to a homomorphism of $A\times Z(C)$ onto A for in the above discussion we may replace C_1 by $Z(C)$ and A_* by A^* where $A_*\subset A^*\subset A$. In the sequel α_* will be as above. These considerations prompt the definition: Let F be an arbitrary finite abelian group. We call a group A strongly hopfian if every homomorphism of $A \times F$ onto A has kernel of bounded order $\leq N$ where N is dependent only on A and F. Clearly, a strongly hopfian group is hopfian.

We may summarize the above discussion as

THEOREM 1. *If A is strongly hopfian and if B has finitely many normal subgroups, then $A \times B$ is hopfian.*

As an example of some conditions which imply strong hopficity suppose that the torsion subgroup of $Z(A)$, E, is finite. Suppose further that normal subgroups of finite index in A which are homomorphic images of A are hopfian. Then A is strongly hopfian. For if θ is a homomorphism of $A \times F$ onto A, F a finite abelian group, we have $A\theta^{j+1} \subset A\theta^j$, $j \geq 0$ and

$$A = A\theta^j \cdot F\theta \cdot F\theta^2 \ldots F\theta^j, \quad j \geq 1 .$$

Hence $A = A\theta^j \cdot E$ so that $[A : A\theta^j] \leq |E|$. Hence ultimately the subgroups $A\theta^j$ are identical, say for $j \geq k$. But then since θ maps $M = A\theta^k$ onto itself, θ is an isomorphism on M. Since $A = M \cdot E$, we see that kernel θ contains at most $|E|$ elements of A. It easily follows that A is strongly hopfian.

THEOREM 2. *If $Z(A)$ is contained in any normal subgroup of finite index in A which is a homomorphic image of A, then if L has finitely many normal subgroups or if L is finitely generated abelian group then $A \times L$ is hopfian.*

Proof. The hypothesis implies that A is strongly hopfian, so that if L has finitely many normal subgroups, $A \times L$ is hopfian by Theorem 1. If L is a finitely generated abelian group, we may assume by Theorem 3 of [3] that L is an infinite cyclic group. But then if $A \times L$ is not hopfian, almost exactly as before we can obtain a homomorphism δ of $A \times L$ onto A which is not an isomorphism on A. But then $A = A\delta \cdot L\delta$. If $A\delta$ is of infinite index in A, then $A = A\delta \times L\delta$, and $L\delta$ is infinite cyclic. But then $A \times L$ is hopfian by Theorem 3 of [3]. Hence $A\delta$ is of finite index in A so $L\delta \subset A\delta$, that is $A\delta = A$. But then δ is an isomorphism on A contrary to assumption.

Theorem 1 naturally leads us to ask what we can say about homomorphisms of $A \times F$ onto A where F is a finite abelian group. In this direction we may state,

THEOREM 3. *If A does not have a direct factor of prime order and if F is a finite abelian group of square free exponent and if θ is an arbitrary homomorphism of $A \times F$ onto A with kernel K, then θ is an isomorphism on A, $K = F$, and K is a central subgroup of $A \times F$.*

Proof. Let $A\theta \cap F\theta = F_1\theta$, $F_1 \subset F$. Hence we may find F_2 such that $F = F_1 \times F_2$ and $K \subset A \times F_1$. However if θ_1 is the restriction of θ to $A \times F_2$, θ_1 maps $A \times F_2$ onto A, so that if $A_1 = K \cap A$, then $A_1 =$ kernel θ_1 so that $(A/A_1) \times F_2 \approx A$. Hence $F_2 = 1$ so that $A\theta = A$. Hence θ is an isomorphism on A and $F\theta \subset A\theta$. If we write $f\theta = a_f\theta$, $f \in F$ $a_f \in A$, then one may show $K = \{f^{-1}a_f \mid f \in F\}$ and K is a central subgroup isomorphic to F.

2.2. *A necessary and sufficient condition that $A \times B$ be hopfian*

THEOREM 4. *A necessary and sufficient condition that $A \times B$ be hopfian for all B is that if θ is an arbitrary surjective endomorphism of $A \times B$ then there exists some group subgroup B_* of B such that $A\theta B = A\theta \times B_*\theta$.*

Proof. The necessity is obvious. For the sufficiency suppose that our hypothesis holds for all groups B but $A \times B$ is not hopfian for some fixed B. But then by hypothesis we may write

$$A\alpha C = A\alpha \times C_*\alpha\,, \quad C_* \subset C\,. \tag{1}$$

Now $A\alpha C = A_1 \times C$, $A_1 \subset A$. Note C_* is a central subgroup of C so that C_* is a finite abelian group. Now since $C_*\alpha \cap C = 1$, if we project $C_*\alpha$ into A_1, (by mapping C into 1 and A_1 onto itself via the identity map) and if say A_* is this projection of $C_*\alpha$ into A_1, then $A_* \approx C_*$. Furthermore we claim $A_* \cap A\alpha = 1$. To see this say C_* is the direct product of i cyclic groups E_1, $E_2 \ldots E_i$ generated by e_1, $e_2 \ldots e_i$ respectively, where each E_i is of order a power of a prime. Then

$$A\alpha \cdot C = (A \times E_1 \times \ldots \times E_{i-1})\alpha \times E_i\alpha\,.$$

Write $e_j = a_j e_j'$, $e_j' \in C$, $a_j \in A_1$. Let $A^k = A \times E_1 \times \ldots \times E_k$, $0 < k \leq i$, and let $A^\circ = A$. Let A_*^k be the subgroup generated by a_{k+1}, $a_{k+2}, \ldots a_i$, $k < i$, and let A_*^i be the identity group. Suppose

$$A\alpha C = A^k\alpha \times A_*^k\,, \quad k \leq i\,. \tag{2}$$

(2) is certainly true for $k = i$. But if $k > 0$, we may write from (2),

$$A\alpha \cdot C = F \times E_k\,, \quad F = A^{k-1}\alpha \times A_*^k\,.$$

Since e_k is of prime power order, say order $e_k = p^s$, either a_k or e_k' has order $p^s \bmod F$. If the order $e_k' \bmod F$ is p^s

$$A\alpha \cdot C = F \times \langle e_k' \rangle$$

which implies that C has a direct abelian factor which would contradict the "minimality" of C. Thus

$$A\alpha \cdot C = F \times \langle a_k \rangle = A^{k-1}\alpha \times A_*^{k-1}$$

so that (2) is true for $0 \leq k \leq i$.

Since $A_*^0 = A_*$, setting $k = 0$ in (2) gives us our assertion.

Now if γ is the projection of $A\alpha C$ onto $A\alpha$ which maps A_* into 1 and which is the identity on $A\alpha$, clearly $C\gamma \approx C$ and $C\gamma \cap A = 1$. Furthermore, $C\Delta A\alpha$ so certainly $C\gamma\Delta(A \times C)$. Hence, $A \times C = A \times C\gamma$. As in Lemma 4 of [3] this implies α is an isomorphism on A contrary to assumption.

We note that we have also established the following results in the proof of the theorem:

COROLLARY 1. *A sufficient condition that $A \times B$ be hopfian for fixed A and for fixed B is that for each homomorphic image E of B and for each surjective endomorphism γ of $A \times E$ we have $A\gamma E = A\gamma \times D\gamma$ for some $D \subset E$.*

COROLLARY 2. *If $A \times B$ is not hopfian, then it is impossible to find $C_* \subset C$ such that $A\alpha C = A\alpha \times C_*\alpha$.*

3. Restrictions on $Z(A)$

3.1. *$Z(A)$ with a torsion group with minimal condition for its primary subgroups*

The main results of this section depend mainly on the endomorphism α_* of the previous section and on the following result:

LEMMA 1. *Suppose $A \times B$ is not hopfian. If L is a Sylow p subgroup of $Z(C)$ there exists a basis $y_1, y_2, \dots y_e$ for L such that if θ is an arbitrary positive power of α then for any i, $1 \leq i \leq e$,*

$$y_i\theta \equiv y_1^{r_{1i}} y_2^{r_{2i}} \dots y_i^{r_{ii}} \dots y_e^{r_{ei}} \mod A$$

where the exponents $r_{1i}, r_{2i}, \dots r_{ii}$ are all divisible by p.

Proof. Let $Z(C) = M \times L$ where L is a Sylow p subgroup of $Z(C)$. Let

$$L = L_1 \times L_2 \times \dots \times L_s$$

where each L_j is a direct product of cyclic groups of the same order p^{n_j} where $n_{u+1} < n_u$, $u = 1, 2, \dots s-1$. Suppose $w \in L_k$. Let $w\theta \equiv w_1 w_2 w_3 \dots w_s \mod A$ where $w_i \in L_i$. We claim $w_1, w_2, \dots, w_k$ are p^{th} powers in $Z(C)$. Since w is of order p^{n_k} and each L_i for $i < k$ is a direct product of cyclic groups of order p^{n_k} and $n_i > n_k$ we can easily see that w_i is a p^{th} power in $Z(C)$ for $i < k$. It is not obvious however that w_k must be a p^{th} power. To see this, choose a basis $m_1, m_2, \ldots, m_j$ for L_k so that L_k is the direct product of the $\langle m_i \rangle$ and each m_i is of order p^{n_k}. Let $w_k = m_1^{t_1} m_2^{t_2} \ldots m_j^{t_j}$. To show w_k is a p^{th} power we show p is a divisor of each t_i. Suppose for example p is not a divisor of t_1. Let F be the subgroup generated by $m_2, m_3, \ldots, m_j$ and let E be the subgroup generated by the L_i, $i \neq k$. Let $A_1 = A \times M \times E \times F$. Hence, $A_1 C\theta = A_1 \times \langle m_1 \rangle$, $C\theta / A_1 \cap C\theta \sim \langle m_1 \rangle$. But the order of $w\theta \bmod A_1 \cap C\theta$ is p^{n_k}. Hence,

$$C\theta = \langle w\theta \rangle \times (A_1 \cap C\theta) .$$

Since θ is an isomorphism on C this implies that C has a cyclic direct factor of oder p^{n_k} which is impossible. Now if $y_1, y_2, \ldots, y_e$ is obtained by taking the union of basis' of each L_i and if the y's are indexed such that $r < t$ implies the y's in L_r precede the y's in L_t then the y's have the asserted property.

THEOREM 5. *Let B have finitely many normal subgroups. Suppose that for each prime p, the subgroup of elements in $Z(A)$ of order a power of p satisfies the minimal condition for normal subgroups. Then $A \times B$ is hopfian.*

Proof. Suppose the assertion is false. Let L_p be a Sylow p group of $Z(C)$ for the prime divisor p of $|Z(C)|$. Let P be the p^{th} powers of the elemests of order a power of p in $Z(A) \times Z(C)$. We will show that we can find subgroups $\overline{L_p} \subset Z(A) \times Z(C)$ and positive integers r_p such that

$$\overline{L_p} \approx L_p\,, \quad \overline{L_p} \cap A = 1\,, \quad \text{and} \quad \overline{L_p} \alpha^{r_p} \subset P\,. \tag{3}$$

To obtain the desired contradiction note that (3) implies that $A \times Z(C)$ is the direct product of the groups A and $\overline{L_p}$ for p a prime divisor of $|Z(C)|$. Hence if r is a positive common multiple of the r_p and $\gamma = \alpha^r$, then each element of $\overline{L_p}\gamma$ is a p^{th} power for all p and hence each element of $\overline{L_p}\gamma_*$ is a p^{th} power. But note that if H is an arbitrary group with a finite central p subgroup H_1 and if $H = H_1 H_2$ for some subgroup $H_2 \subset H$ and if δ is a homomorphism of H onto some group K such that every element in $H_1\delta$ is a p^{th} power then $K = H_2\delta$. Hence $A\gamma_* = A$, a contradiction of the hopficity of A.

We will give an inductive method for constructing the $\overline{L_p}$. Let p be a fixed prime divisor of $|Z(C)|$ and let $y_1, y_2, \ldots, y_e$ be a basis for L_p as in Lemma 1. We will show that there exists $u_1, u_2, \ldots, u_e$ in $Z(A) \times Z(C)$ such that for $1 \leq i \leq e$

$$u_i \equiv y_i \bmod A\,, \tag{4}$$

$$u_i \text{ and } y_i \text{ have the same order, and} \tag{5}$$

$$\text{some fixed power of } \alpha \text{ maps } u_i \text{ into } P \tag{6}$$

Once we do this we see that the subgroup generated by the u_i, $1 \leq i \leq e$ is isomorphic to $\overline{L_p}$ and may be taken as L_p. Our method first gives u_e, then u_{e-1}, then u_{e-2} and so forth.

Suppose that s is an integer, $1 < s \leq e$ and that we have already found $u_s, u_{s-1}, \ldots, u_e$ such that (4) and (5) hold for $s \leq i \leq e$ and that say some power θ of α maps $u_s, u_{s-1}, \ldots, u_e$ into P. We show that under this assumption we can find $u \in Z(A) \times Z(C)$ such that $u \equiv y_{s-1} \bmod A$ and u and y_{s-1} have the same order and some power of θ maps u into P. Then u may be taken as u_{s-1} and we may repeat the procedure until all the u's are constructed. (The inductive step of finding u_{s-1} also shows how to find u_e.)

Write $y = y_{s-1}$. Let K be the group generated by $u_s, u_{s+1}, \ldots, u_e$. Then we can write $y\theta \equiv a_1 y_1^{t_1} \ldots y_{s-1}^{t_{s-1}} \bmod K$, $a_1 \in A$ where each t_i above is divisible by p. Hence,

$$y\theta \equiv a_1 \bmod P \cdot K . \tag{7}$$

If $a_1\theta \equiv a_2 y_1^{q_1} \ldots y_{s-1}^{q_{s-1}} \bmod K$, $a_2 \in A$, then $y\theta^2 \equiv a_2 y_1^{q_1} \ldots y_{s-1}^{q_{s-1}} \bmod P \cdot K$ from which we deduce that each of the q_i are divisible by p. Hence $a_1\theta \equiv a_2 \bmod (K \cdot P)$. By considering $y\theta^3$ we see in a similar way that we may write $a_2\theta \equiv a_3 \bmod (KP)$, $a_3 \in A$ and that we can define $a_n \in A$ inductively so that

$$a_n\theta = a_{n+1} \bmod (KP) .$$

One may verify that $a_n \in Z(A)$ and that the order of a_n is a divisor of the order of y. Furthermore, since θ maps $K \cdot P$ into P we see $a_k\theta^m \equiv a_{k+m} \bmod (K \cdot P)$ and $y\theta^m \equiv a_m \bmod (K \cdot P)$. Now the elements of order a power of p in $Z(A \times C)$ form a direct product of a divisible group and a finite group. Hence not all the a_j can be distinct $\bmod P$. Hence we can find positive integers k and m such that $a_m \equiv a_{k+m} \bmod P$. Hence, $(ya_k^{-1})\theta^m \in K \cdot P$ and consequently, $(ya_k^{-1})\theta^{m+1} \in P$. Hence, if we define $u_{s-1} = ya_k^{-1}$ then $u_j\theta^{m+1} \in P$, $s - 1 \leq e$ so that the proof is complete.

COROLLARY 1. *If B is a finite group such that the subgroup of $Z(A)$ consisting of elements whose orders are divisors of $|B|$ obeys the minimal condition for subgroups then $A \times B$ is hopfian.*

Proof. Since C is a homomorphic image of B only prime divisors of $|B|$ come into play in the case where B is finite.

COROLLARY 2. *If B has finitely many normal subgroups and θ is a surjective endomorphism of $A \times B$ such that*

$$a\theta = a\theta^2 \quad \text{for} \quad a \in Z(A) \tag{8}$$

then θ is an automorphism. If B is finite and (8) *holds only for central elements of A whose orders are divisors of $|B|$ then θ is an automorphism.*

Proof. Suppose the assertion false. Then in passing from θ and $A \times B$ to α and $A \times C$ we note that (4) may be preserved; that is we may assume $a\alpha = a\alpha^2$

for $a \in Z(A)$ or for central elements of A whose orders divide $|B|$ in case B is finite. Now proceed exactly as in the theorem to construct the groups $\overline{L_p}$. Define $\theta, s, y, u_s, \ldots u_e$ as before. Now apply θ to (7) obtaining

$$y\theta^2 \equiv a_1\theta \equiv a_1\theta^2 \bmod P$$

so that u_{s-1} may be taken as ya_1^{-1}.

3.2. *Finite* B

We apply the results of section 2.2 in this section to finite groups with some special restricts on $|B|$. In contrast to Corollary 1 of Theorem 5 we show that in some cases we need not pay attention to all the elements in $Z(A)$ whose orders are divisors of $|B|$.

LEMMA 2. *If G is a group and if γ is an endomorphism of G and if $g \in G$ and the elements $g\gamma, g\gamma^2, g\gamma^3, \ldots$ are finite in number, we can find a positive integer r such that $g\gamma^r = g\gamma^{2r}$.*

Proof. Choose positive integers e and f such that $g\gamma^{2^e} = g\gamma^{2^{e+f}}$. Then for any $q > 0$, $g\gamma^{2^e} + q = g\gamma^{e+f} + q$. Choose q so that $2^{e+f} + q = 2(2^e + q)$ and choose $r = 2^e + q$.

LEMMA 3. *Suppose B is finite and $Z(A)$ has only finitely many elements of order p^2. If $A \times B$ is not hopfian, then $Z(C)$ is not of the form $L \times M$ where L is cyclic of order 1, p or p^2, p a prime and where M is of square free exponent prime to p.*

Proof. Suppose the assertion is false. Let

$$A\alpha C = A\alpha \cdot C_1\alpha\,, \quad A\alpha \cap C\alpha = C_2\alpha \quad \text{with} \quad C_2 \subset C_1 \subset Z(C)\,.$$

Then we claim C_1 is not of square free order or else $C_1 = C_2 \times C_3$ so that $A\alpha C = A\alpha \times C_3\alpha$ contrary to Corollary 2 of Theorem 4. Hence L is of order p^2 and $L \subset C_1$. Furthermore if $L = \langle w \rangle$,

$$w\alpha \notin A\alpha \tag{9}$$

or again we would obtain a contradiction of Corollary 2 of Theorem 4. Moreover, since $A(C\alpha) \equiv A \bmod Z(C)$ and $A\alpha(C) \equiv A\alpha \bmod Z(CT)$, one sees that $C\alpha/A \cap C\alpha$ and $C/A\alpha \cap C$ are isomorphic to subgroups of $Z(C)$. Hence

$$E = \langle w^p \rangle \times M \subset A\alpha^{-1} \cap A\alpha \tag{10}$$

or otherwise C would have a finite abelian direct factor which is impossible. Since $A\alpha \cap C\alpha \subset E\alpha$ we see $A\alpha \cap C\alpha \subset A$.

Now let $K = C\alpha \cap (C\times A \cap A\alpha)$. We claim $K = A\alpha \cap C\alpha$. We have already shown $A\alpha \cap C\alpha \subset K$. On the other hand suppose $k \in K$. Then

$$k = c\alpha = c_1 a\ ,\quad c \in Z(C)\ ,\quad C_1 \in Z(C)\ ,\quad a \in A \cap A\alpha\ . \tag{11}$$

From (10) we see that if $w\alpha \equiv w^q \bmod A$, then $(p, q) = p$. Hence (10) and (11) imply

$$c_1 \in A\alpha \tag{12}$$

so that $K = A\alpha \cap C\alpha$ as asserted. But then if we set $G = A\times C$ and $M = (A \cap A\alpha)C\alpha$ we see

$$G/M = [(A\alpha)(C\alpha)]/M \approx A\alpha/A \cap A\alpha \approx (A\cdot A\alpha)/A$$

so that $[G:M] \leq |C|$. But $M \cap C = 1$ so $[G:M] \geq |C|$. Hence, we conclude $A\times C = M\times C$.

Now in all of our above arguments we may replace α by α^i, $i \geq 1$, and A by any A_1 such that

$$A_1\times C = A\times C\ . \tag{13}$$

In particular as in (9)

$$w\alpha^i \notin A_1\alpha^i\ ,\quad i \geq 1 \tag{14}$$

for any A_1 in (13).

In any case our hypothesis concerning $Z(A)$ guarantees that the elements $w\alpha^i$, $i = 1, 2, 3 \ldots$ are finite in number. By Lemma 2 we may choose $r > 0$ such that $w\alpha^r = w\alpha^{2r}$. But then if we set $A_1 = A \cap A\alpha^r \cdot C\alpha^r$ and $a = w\alpha^r$ we see $a \in A_1$ and $w\alpha^r = a\alpha^r$, contrary to (14).

THEOREM 6. *If* $|B| = p^e q_1^{e_1} \ldots q_s^{e_s}$, *where* $p, q_1, \ldots, q_s$ *are the distinct prime divisors of* $|B|$, *and* $0 \leq e \leq 3$, $1 \leq e_i \leq 2$ *for* $1 \leq i \leq s$ *and if* $Z(A)$ *has finitely many elements of order* p^2, *then* $A\times B$ *is hopfian.*

Proof. Suppose the assertion is false. Then p^3 is not a divisor of $|Z(C)|$ or else C would have a direct abelian factor of order p^2 [4]. Similarly q_i^2, $i = 1, 2 \ldots s$, is not a divisor of $|Z(C)|$ and we arrive at a contradiction of Lemma 3.

In a similar way, the next two theorems follow easily with the aid of the previous theorem, Lemma 3 and Theorems 6 and 3 of [3].

THEOREM 7. *If* $|B| = p^4 q_1^{e_1} q_2^{e_2} \ldots q_s^{e_s}$ *where* $p, q_1, \ldots, q_s$ *are the distinct prime divisor of* $|B|$, $1 \leq e_i \leq 2$ *for* $1 \leq i \leq s$, *and if* $Z(A)$ *has only finitely many elemests of order* p^2, *and if a Sylow* p *group of* B *is non-abelian, then* $A\times B$ *is hopfian.*

THEOREM 8. *If $Z(A)$ has only finitely many elements of order p^2 and if $|B| = p^4$, p a prime, the $A \times B$ is hopfian.*

THEOREM 9. *Let $|B| = p_1^{e_1} p_2^{e_2} \ldots p_r^{e_r}$, $1 \leq e_i \leq 3$, where the p_i are the distinct prime divisors of $|B|$. Let L_i be a Sylow p_i group of B and suppose that at most one of the groups L_i are abelian. If $Z(A)$ has finitely many elements of order p_i^2, $i = 1, 2, \ldots, r$, then $A \times B$ is hopfian.*

Proof. Suppose the assertion false. Then $Z(C)$ is not divisible by p_i^3 for any p_i or C has an abelian direct factor. On the other hand, $|Z(C)|$ must (by Lemma 3) be divisible by $p_i^2 \cdot p_j^2$, $i \neq j$. Since C does not have any abelian direct factor we must have $e_i = e_j = 3$. But then the Sylow p_i and the Sylow p_j groups of C are abelian and isomorphic to the Sylow p_j and the Sylow p_i groups of B, contrary to assumption.

4. Restriction on the commutator subgroup

In investigating the hopficity of $A \times B$, we can obtain some further results by considering some of the following restrictions on A':

$$A' \subset Z(A)\,. \tag{15}$$

If B is finite, and p^e is a divisor of $|B|$, p a prime, then A' has only finitely many elements of order p^e. (16)

If K is an arbitrary normal subgroup of a homomorphic image D of B such that $Z(D) \neq 1$, then A' has only finitely many normal subgroups isomorphic to K. (17)

LEMMA 4. *If* (15), (16) *or* (17) *hold and $A \times B$ is not hopfian, then C' is a central subgroup of C. In any case, $C' \subset A\alpha$ and $C'\alpha \subset A$.*

Proof. As in the proof of Lemma 3, $C\alpha' A \cap C\alpha$ and $C' A\alpha \cap C$ are abelian so that the last two assertions are evident. Hence if (15) holds our assertion is evident. If (16) holds, let y be in a Sylow p group of C'. But then $y\alpha^i \subset A'$ for $i \geq 1$. By Lemma 2 we may choose a positive integer r such that

$$y\alpha^r = y\alpha^{2r}\,.$$

Hence $y\alpha^r \subset A\alpha^r \cap C\alpha^r$ so that $y\alpha^r$ is a central element. If (17) holds we note that since $C'\alpha^i \subset A'$, we may choose (exactly as in Lemma 2) $r > 0$ with $C'\alpha^r = C'\alpha^{2r}$. Hence $C'\alpha^r \subset A\alpha^r$ so $C'\alpha^r$ and hence C' is central.

Now recall that for any group G, $Z(G) \cap G' \subset \mathrm{Fr}\,(G)$ where $\mathrm{Fr}\,(G)$ is the Frattini subgroup of G, and for a finite group G, $G' \subset \mathrm{Fr}\,(G)$ implies that G is nilpotent. Hence we have the following:

LEMMA 5. *If* B *is finite and* $A \times B$ *is not hopfian and if* (15), (16) *or* (17) *holds, then* C *is nilpotent.*

THEOREM 10. *Suppose* B *is a finite group,* $|B| = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$, *where the* p_i *are the distinct primes dividing* $|B|$. *Suppose one of the conditions* (15), (16) *or* (17) *holds. If* $e_i \leq 3$ *for all* i, *then* $A \times B$ *is hopfian. If* $e_i \leq 4$ *for all* i *and* $Z(A)$ *has only finitely many elements of order* p_i^2, $i = 1, 2, \dots r$, *then* $A \times B$ *is hopfian.*

Proof. Suppose the assertion is false. By Lemma 5, C is nilpotent and hence is a direct product of p groups for various primes p, where p divides $|B|$. However, by Theorem 6 of [3], the direct product of a hopfian group with a group of order p^3 is hopfian. In any case with the aid of Theorem 8 and Theorem 3 of [3] we have our result.

THEOREM 11. *If* $B = B' \cdot L$ *where* L *is an abelian subgroup of* B *and if one of the conditions* (15), (16) *or* (17) *holds, then* $A \times B$ *is hopfian.*

Proof. Suppose the assertion is false. Note any homomorphic image of B satisfies the same hypothesis as B. Hence we may write, $C = C' \cdot M$ where M is abelian. By Lemma 4 C' is a central subgroup of C so that C is abelian and consequently finite contrary to Theorem 3 of [3].

COROLLARY. *If* B *is finite and if all the Sylow* p *groups of* B *are cyclic, then* $A \times B$ *is hopfian.*

Proof. B/B' is cyclic. ([6] Theorem 11).

THEOREM 12. *If* B *is a perfect group then* $A \times B$ *is hopfian.*

Proof. Suppose the assertion is false. Any homomorphic image of a perfect group is perfect. Hence C is perfect. By Lemma 4, $C \subset A\alpha$ which is contrary to Lemma 4 of [3].

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