Cohomology of operator algebras II. Extended cobounding and the hyperfinite case

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1. Introduction

We continue the programme begun in [1] of studying the (topological) cohomology of operator algebras. In that article, we proved that cohomology of a type I von Neumann algebra with coefficients in the algebra vanishes [1: Theorem 4.4]. Employing that theorem and the various preparatory results on centre adjustment of cocycles, we prove (Theorem 2.4), in this paper, that each cocycle on a (general) von Neumann algebra with coefficients in the algebra cobounds a cochain with coefficients in the algebra of all bounded operators on the Hilbert space on which the von Neumann algebra acts. This theorem is, then, used to prove (Theorem 3.1) that cohomology with coefficients in the algebra vanishes for hyperfinite von Neumann algebras.

The argument proving Theorem 2.4 is structurally the same as that appearing in [4; Theorem 4]. It is made more difficult by the fact that higher-dimensional (norm-continuous) cocycles do not satisfy automatic weak continuity conditions (as do derivations [4; Lemma 3]). This same difficulty rules out certain direct approaches to dealing with the hyperfinite case.

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2. Cobounding in $\mathcal{B}(\mathcal{H})$

In this section we show (Theorem 2.4) that each n-cocycle with coefficients in our von Neumann algebra $\mathcal{R}$ is the coboundary of a cochain with coefficients in $\mathcal{B}(\mathcal{H})$. This is precisely analogous to the step in the proof of the Derivation Theorem establishing that derivations are »spatial» (cobound an operator $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$ — see [4: Theorem 4]). The present proof is similar in structure to the proof for derivations. We extend our cocycle to the algebra generated by a maximal abelian subalgebra of the commutant and our von Neumann algebra, after establishing suitable boundedness conditions. Unlike the derivation case, the extension of the cocycle from this algebra to its weak-operator closure, a von Neumann algebra of type I, is not a result of automatic ultraweak continuity (as is the case with derivations [4; Lemma 3]). It is easy to convince oneself that (norm-continuous) higher-dimensional cocycles are not necessarily ultraweakly continuous by passing to the coboundary of a suitable (norm-continuous) cochain which is not ultraweakly continuous.

In fact that (Theorem 2.1), nonetheless, each cocycle on a (concretely represented) C^*-algebra with coefficients in its weak-operator closure can be adjusted by a coboundary so that the resulting cocycle has an extension to the weak-operator closure (which is, again, a cocycle) replaces the missing automatic ultraweak continuity. This fact is established with the aid of [1; Theorem 3.4] (the centre adjustment of cocycles) and the properties of the universal representation.

In the argument which follows, we will have occasion to extend a multilinear mapping from a C^*-algebra to its weak-operator closure. One would hope, of course, that when the mapping is ultraweakly continuous (separately) there is an ultraweakly continuous extension. This has been proved in another connection and will appear elsewhere. It involves other techniques; and, since we will be dealing, here, with the universal representation, we have chosen (and are able) to use the simpler (more awkward) procedure of successive extensions.

THEOREM 2.1. *If φ is a faithful representation of the C^*-algebra $\mathfrak{A}$ and $\varrho \in Z_c^n(\varphi(\mathfrak{A}), \varphi(\mathfrak{A})^-)$, there is a $\bar{\tau}$ in $Z_c^n(\varphi(\mathfrak{A})^-, \varphi(\mathfrak{A})^-)$ whose restriction τ to $\varphi(\mathfrak{A})$ is cohomologues to ϱ (in $Z_c^n(\varphi(\mathfrak{A}), \varphi(\mathfrak{A})^-)$).*

Proof. From the properties [3: pp. 181—182] of the universal representation ψ of $\mathfrak{A}$, there is a central projection P in $\psi(\mathfrak{A})^-$ and an isomorphism α of $\psi(\mathfrak{A})^-P$ onto $\varphi(\mathfrak{A})^-$ which extends the mapping $\psi(A)P \to \varphi(A)$. We denote by α_* the isomorphism induced by α of the cohomology of $\psi(\mathfrak{A})P$ and $\psi(\mathfrak{A})^-P$ having coefficients in $\psi(\mathfrak{A})^-P$ with that of $\varphi(\mathfrak{A})$ and $\varphi(\mathfrak{A})^-$, respectively, having coefficients in $\varphi(\mathfrak{A})^-$. Let ϱ_0 (in $Z_c^n(\psi(\mathfrak{A})P, \psi(\mathfrak{A})^-P)$) be $\alpha_*^{-1}(\varrho)$. If there exist ξ_0 (in $C_c^{n-1}(\psi(\mathfrak{A})P, \psi(\mathfrak{A})^-P)$) and $\bar{\tau}_0$ (in $Z_c^n(\psi(\mathfrak{A})^-P, \psi(\mathfrak{A})^-P)$) such that $\varrho_0 - \Delta\xi_0 = \bar{\tau}_0|\psi(\mathfrak{A})P$, let $\bar{\tau}$ (in $Z_c^n(\varphi(\mathfrak{A})^-, \varphi(\mathfrak{A})^-)$) and ξ (in $C_c^{n-c}(\varphi(\mathfrak{A}), \varphi(\mathfrak{A})^-)$) be $\alpha_*(\bar{\tau}_0)$ and $\alpha_*(\xi_0)$, respectively. Since, also, $\varrho = \alpha_*(\varrho_0)$, it follows that $\varrho - \Delta\xi = \bar{\tau}|\varphi(\mathfrak{A})$, proving the theorem. It is now sufficient to consider the case in which φ is the faithful representation $A \to \psi(A)P$ of $\mathfrak{A}$ (one could even assume that $\mathfrak{A}$ is given, acting on a Hilbert space $\mathcal{H}$, in its universal representation, and $\varphi(A) = AP$ for A in $\mathfrak{A}$).

With $\varphi(A) = \psi(A)P$ and ϱ replaced by ϱ_0, let ϱ_1 in $Z_c^n(\psi(\mathfrak{A}), \psi(\mathfrak{A})^- P)$ be $\beta_*^{-1}(\varrho_0)$, where β is the isomorphism $\psi(A) \to \psi(A)P$ of $\psi(\mathfrak{A})$ onto $\psi(\mathfrak{A})P$ (the identity isomorphism of $\psi(\mathfrak{A})^- P$ being used in defining β_*).

Since ϱ_1 is bounded, using the properties of the universal representation [3; pp. 181—182], $A \to \varrho(A, A_2, \ldots, A_n)$ has an ultraweakly continuous extension from $\psi(\mathfrak{A})$ to $\psi(\mathfrak{A})^-$. The resulting mapping $A_1, \ldots, A_n \to \varrho_{11}(A_1, \ldots, A_n)$ is multilinear from $\psi(\mathfrak{A})^- \times \psi(\mathfrak{A}) \times \ldots \times \psi(\mathfrak{A})$ to $\psi(\mathfrak{A})^- P$. Continuing, with successive extensions, we construct, for each k $(= 1, \ldots, n)$, a mapping ϱ_{1k} which is multilinear from $\psi(\mathfrak{A})^- \times \ldots \times \psi(\mathfrak{A})^- \times \psi(\mathfrak{A}) \times \ldots \times \psi(\mathfrak{A})$ (where the first k factors are $\psi(\mathfrak{A})^-$ and the last $n-k$ are $\psi(\mathfrak{A})$) into $\psi(\mathfrak{A})^- P$. Moreover, ϱ_{1k} is ultraweakly continuous in its kth argument and extends and has the same bound as ϱ_{1k-1}. For notational convenience, let ϱ_{10} be ϱ_1 and $\bar{\varrho}_1$ be ϱ_{1n} $(= \varrho_{1n+1})$. Then $\bar{\varrho}_1 \in C_c^n(\psi(\mathfrak{A})^-, \psi(\mathfrak{A})^- P)$.

By showing, inductively, that

$$(\Delta \varrho_{1k})(A_1, \ldots, A_{n+1}) = 0\,; \quad A_1, \ldots, A_k \in \psi(\mathfrak{A})^-, \quad A_{k+1}, \ldots, A_{n+1} \in \psi(\mathfrak{A}), \qquad (P_k)$$

we will establish (P_{n+1}) that $\bar{\varrho}_1 \in Z_c^n(\psi(\mathfrak{A})^-, \psi(\mathfrak{A})^- P)$. Since $\varrho_1 \in Z_c^n(\psi(\mathfrak{A}), \psi(\mathfrak{A})^- P)$, (P_0) follows. Suppose (P_{k-1}) is given. With $A_0, \ldots, A_k$ in $\psi(\mathfrak{A})^-$ and $A_{k+1}, \ldots, A_{n+1}$ in $\psi(\mathfrak{A})$, we have

$$\begin{aligned}(\Delta \varrho_{1k})(A_1, \ldots, A_{n+1}) &= A_1 \varrho_{1k}(A_2, \ldots, A_{n+1}) \\ &- \sum_{j=2}^{n+1} (-)^j \varrho_{1k}(A_1, \ldots, A_{j-2}, A_{j-1}A_j, A_{j+1}, \ldots, A_{n+1}) \\ &+ (-)^{n+1} \varrho_{1k}(A_1, \ldots, A_n) A_{n+1}\,.\end{aligned} \tag{1}$$

By construction of ϱ_{1k} and ϱ_{1k-1}, $A_k \to \varrho_{1k}(A_2, \ldots, A_k, \ldots, A_{n+1}) = \varrho_{1k-1}(A_2, \ldots, A_k, \ldots, A_{n+1})$ is ultraweakly continuous, as is each of the other terms on the right side of (1) in its argument A_k. Thus $A_k \to (\Delta \varrho_{1k})(A_1, \ldots, A_k, \ldots, A_{n+1})$ is ultraweakly continuous on $\psi(\mathfrak{A})^-$. Now, $(\Delta \varrho_{1k})(A_1, \ldots, A_{n+1})$ vanishes when $A_k \in \psi(\mathfrak{A})$, by inductive hypothesis, since (from (1)) it coincides with $(\Delta \varrho_{1k-1})(A_1, \ldots, A_{n+1})$, in this case. The ultraweak density of $\psi(\mathfrak{A})$ in $\psi(\mathfrak{A})^-$ combined with the foregoing, yields (P_k).

From [1; Theorem 3.4], there is a cochain $\bar{\xi}_1$ in $C_c^{n-1}(\psi(\mathfrak{A})^-, \psi(\mathfrak{A})^- P)$ such that $\bar{\varrho}_1 - \Delta \bar{\xi}_1$ $(= \bar{\tau}_1)$ vanishes when any of its arguments lies in the centre of $\psi(\mathfrak{A})^-$. Let ξ_0 (in $C_c^{n-1}(\psi(\mathfrak{A})P, \psi(\mathfrak{A})^- P)$) be β_* $(\bar{\xi}_1 | \psi(\mathfrak{A}))$. From [1; Lemma 3.2], $\bar{\tau}_1 \in NZ_c^n(\psi(\mathfrak{A})^-, \psi(\mathfrak{A})^- P)$; so that

$$\tau_1(A_1 P, \ldots, A_n P) = \bar{\tau}_1(A_1, \ldots, A_n) P = \bar{\tau}_1(A_1, \ldots, A_n)\,,$$

for $A_1, \ldots, A_n$ in $\psi(\mathfrak{A})$ (recalling that $\bar{\tau}_1(A_1, \ldots, A_n) \in \psi(\mathfrak{A})^- P$). Thus

$$\beta_*[\bar{\tau}_1|\psi(\mathfrak{A})] = \bar{\tau}_1|\psi(\mathfrak{A})P = \beta_*[(\bar{\varrho}_1 - \Delta\bar{\xi}_1)|\psi(\mathfrak{A})]$$
$$= \beta_*[\varrho_1 - \Delta(\bar{\xi}_1|\psi(\mathfrak{A}))] = \varrho_0 - \Delta\beta_*[\bar{\xi}_1|\psi(\mathfrak{A})] = \varrho_0 - \Delta\xi_0 .$$

It follows that $\varrho_0 - \Delta\xi_0 = \bar{\tau}_0|\psi(\mathfrak{A})P$, with $\bar{\tau}_0$ taken as $\bar{\tau}_1|\psi(\mathfrak{A})^-P$ $(\in Z_c^n(\psi(\mathfrak{A})^-P, \psi(\mathfrak{A})^-P))$.

The lemma which follows describes a canonical extension of a centre normalised n-cochain on a von Neumann algebra to a centre normalised n-cochain on a C^*-algebra with type I von Neumann algebra closure.

LEMMA 2.2. *If $\mathfrak{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{A}$ is a maximal abelian *-subalgebra of $\mathfrak{R}'$, and $\mathcal{S}$ is the C^*-algebra generated by $\mathfrak{R}$ and $\mathcal{A}$, then each ϱ in $NC_c^n(\mathfrak{R}, \mathfrak{R})$ extends uniquely to an element $\bar{\varrho}$ of $NC_c^n(\mathcal{S}, \mathcal{S})$. The mapping, $\varrho \to \bar{\varrho}$, is linear and isometric; and $\overline{\Delta\varrho} = \Delta\bar{\varrho}$.*

Proof. With $\mathcal{P}$ the lattice of projections in $\mathcal{A}$, and $\mathcal{S}_0$ the set of all operators of the form $E_1T_1 + \ldots + E_mT_m$, where $E_1, \ldots, E_m \in \mathcal{P}$ and $T_1, \ldots, T_m \in \mathfrak{R}$, $\mathcal{S}_0$ is a norm dense *-subalgebra of $\mathcal{S}$. Given $S_1, \ldots, S_n$ in $\mathcal{S}_0$, we may suppose that

$$S_j = E_{j,1}T_{j,1} + \ldots + E_{j,m(j)}T_{j,m(j)} \quad (j = 1, \ldots, n) \tag{2}$$

If $\sigma \in NC_c^n(\mathcal{S}, \mathcal{S})$ then, since each $E_{j,k}$ lies in the centre $\mathcal{A}$ of $\mathcal{S}$,

$$\sigma(S_1, \ldots, S_n) = \sum_{k(1)=1}^{m(1)} \cdots \sum_{k(n)=1}^{m(n)} \sigma(E_{1,k(1)}T_{1,k(1)}, \ldots, E_{n,k(n)}T_{n,k(n)})$$
$$= \sum_{k(1)=1}^{m(1)} \cdots \sum_{k(n)=1}^{m(n)} E_{1,k(1)} \ldots E_{n,k(n)}\sigma(T_{1,k(1)}, \ldots, T_{n,k(n)}) .$$

In particular, if $\bar{\varrho}$ satisfies the conclusions of the lemma, then

$$\bar{\varrho}(S_1, \ldots, S_n) = \sum_{k(1)=1}^{m(1)} \cdots \sum_{k(n)=1}^{m(n)} E_{1,k(1)} \ldots E_{n,k(n)}\varrho(T_{1,k(1)}, \ldots, T_{n,k(n)}) . \tag{3}$$

This equation determines $\bar{\varrho}(S_1, \ldots, S_n)$ uniquely whenever $S_1, \ldots, S_n \in \mathcal{S}_0$; and the uniqueness of $\bar{\varrho}$ (if it exists) now follows from its norm continuity, together with norm density of $\mathcal{S}_0$ in $\mathcal{S}$.

In order to prove the existence of $\bar{\varrho}$, we show first that, for $S_1, \ldots, S_n$ in $\mathcal{S}_0$, the right hand side of (3) depends only on $S_1, \ldots, S_n$ (but not on the particular way in which these operators are represented in the form (2)). For this, it is sufficient to show that the right hand side of (3) is zero if $E_{j,1}T_{j,1} + \ldots + E_{j,m(j)}T_{j,m(j)} = 0$ for some j, $1 \leq j \leq n$. By [2: Lemma 3.1.1], this last condition entails the existence of operators $C_{r,s}$ $(r, s = 1, \ldots, m(j))$ in the centre $\mathcal{C}$ of $\mathcal{R}$ such that

$$\sum_{r=1}^{m(j)} C_{r,s}T_{j,r} = 0 \qquad (s = 1, \ldots, m(j)) , \tag{4}$$

$$\sum_{s=1}^{m(j)} E_{j,s}C_{r,s} = E_{j,r} \quad (r = 1, \ldots, m(j)) . \tag{5}$$

Since $\varrho \in NC_c^n(\mathcal{R}, \mathcal{R})$, it follows from (4) and (5) that

$$\sum_{k(j)=1}^{m(j)} E_{j,k(j)}\varrho(T_{1,k(1)}, \ldots, T_{j,k(j)}, \ldots, T_{n,k(n)})$$
$$= \sum_{k(j)=1}^{m(j)} \sum_{s=1}^{m(j)} E_{j,s}C_{k(j),s}\varrho(T_{1,k(1)}, \ldots, T_{j,k(j)}, \ldots, T_{n,k(n)})$$
$$= \sum_{s=1}^{m(j)} E_{j,s} \sum_{k(j)=1}^{m(j)} \varrho(T_{1,k(1)}, \ldots, C_{k(j),s}T_{j,k(j)}, \ldots, T_{n,k(n)}) = 0 .$$

By operating on the left hand side of this chain of equations with $E_{1,k(1)} \ldots E_{j-1,k(j-1)}E_{j+1,k(j+1)} \ldots E_{n,k(n)}$, summing over each of the variables $k(1), \ldots, k(j-1), k(j+1), \ldots, k(n)$, and using the commutativity of the projections $E_{r,s}$, we deduce that

$$\sum_{k(1)=1}^{m(1)} \cdots \sum_{k(n)=1}^{m(n)} E_{1,k(1)} \ldots E_{n,k(n)}\varrho(T_{1,k(1)}, \ldots, T_{n,k(n)}) = 0 .$$

This shows that, if $S_1, \ldots, S_n$ in $\mathcal{S}_0$ are represented as in (2), then the equation

$$\varrho_0(S_1, \ldots, S_n) = \sum_{k(1)=1}^{m(1)} \cdots \sum_{k(n)=1}^{m(n)} E_{1,k(1)} \ldots E_{n,k(n)}\varrho(T_{1,k(1)}, \ldots, T_{n,k(n)}) \tag{6}$$

defines, unambiguously, an element $\varrho_0(S_1, \ldots, S_n)$ of $\mathcal{S}_0$. It is apparent that ϱ_0 is a multilinear mapping of $(\mathcal{S}_0)^n$ into $\mathcal{S}_0$; and ϱ_0 extends ϱ since $\varrho_0(R_1, \ldots, R_n) = \varrho(R_1 . I, \ldots, R_n . I) = \varrho(R_1, \ldots, R_n)$ when $R_1, \ldots, R_n \in \mathcal{R}$.

Our next objective is to prove that ϱ_0 is bounded. For this, suppose that $S_1, \ldots, S_n$ in $\mathcal{S}_0$ are represented as in (2). There is an orthogonal family $\{F_1, \ldots, F_m\}$ of projections in $\mathcal{P}$ such that each $E_{j,k}$ occurring in (2) is the sum of a collection of F_r's. Each S_j can be expressed in the form

$$S_j = F_1R_{j,1} + \ldots + F_mR_{j,m} , \tag{7}$$

with the $R_{j,k}$'s in $\mathcal{R}$; and, since $F_1, \ldots, F_m$ are pairwise orthogonal,

$$\varrho_0(S_1, \ldots, S_n) = \sum_{k=1}^{m} F_k\varrho(R_{1,k}, \ldots, R_{n,k}) \tag{8}$$

With Q_k the central carrier of F_k in $\mathcal{R}'$, we can replace F_k by F_kQ_k in (8). Since $\varrho \in NC_c^n(\mathcal{R}, \mathcal{R})$, we obtain

$$\varrho_0(S_1, \ldots, S_n) = \sum_{k=1}^{m} F_k\varrho(Q_kR_{1,k}, \ldots, Q_kR_{n,k}) . \tag{9}$$

Since $F_1, \ldots, F_k$ are orthogonal projections which commute with $\mathcal{R}$ (and thus with each value of ϱ), while the mapping $Q_kR \to F_kR$ is a *-isomorphism from $\mathcal{R}Q_k$ onto $\mathcal{R}F_k$ (and is therefore isometric), it follows from (9) and (7) that

$$\|\varrho_0(S_1, \ldots, S_n)\| = \max_{1 \le k \le m} \|F_k \varrho(Q_k R_{1,k}, \ldots, Q_k R_{n,k})\| \le \max_{1 \le k \le m} \|\varrho\| \, \|Q_k R_{1,k}\| \ldots \|Q_k R_{n,k}\|$$
$$= \max_{1 \le k \le m} \|\varrho\| \, \|F_k R_{1,k}\| \ldots \|F_k R_{n,k}\| = \max_{1 \le k \le m} \|\varrho\| \, \|F_k S_1\| \ldots \|F_k S_n\| \le \|\varrho\| \, \|S_1\| \ldots \|S_n\| .$$

Thus ϱ_0 is a bounded multilinear mapping of $(\mathcal{S}_0)^n$ into $\mathcal{S}_0$, and $\|\varrho_0\| \le \|\varrho\|$. The reverse inequality is apparent; so $\|\varrho_0\| = \|\varrho\|$. Since $\mathcal{S}_0$ is norm dense in $\mathcal{S}$, ϱ_0 extends by continuity to an element $\bar{\varrho}$ of $C_c^n(\mathcal{S}, \mathcal{S})$, and $\|\bar{\varrho}\| = \|\varrho_0\| = \|\varrho\|$; the linearity of the mapping $\varrho \to \bar{\varrho}$ is evident.

We show next that $\bar{\varrho} \in NC_c^n(\mathcal{S}, \mathcal{S})$; that is, $\bar{\varrho}(S_1, \ldots, S_{j-1}, ES_j, S_{j+1}, \ldots, S_n) = E\bar{\varrho}(S_1, \ldots, S_n)$ for all $S_1, \ldots, S_n$ in $\mathcal{S}$ and E in the centre $\mathcal{A}$ of $\mathcal{S}$. Since $\mathcal{A}$ is the norm closed linear span of $\mathcal{P}$ and $\mathcal{S}_0$ is norm dense in $\mathcal{S}$, it is sufficient, by the continuity and multilinearity of $\bar{\varrho}$, to establish this last equation for the case in which $S_1, \ldots, S_n \in \mathcal{S}_0$ and $E \in \mathcal{P}$ (whence, $\bar{\varrho}$ can be replaced by ϱ_0). We may suppose that $S_1, \ldots, S_n$ are represented as in (7); the corresponding expression for ES_j is then $EF_1R_{j,1} + \ldots + EF_mR_{j,m}$, and the appropriate equation of the form (6) yields

$$\varrho_0(S_1, \ldots, S_{j-1}, ES_j, S_{j+1}, \ldots, S_n) = \sum_{k=1}^{m} EF_k \varrho(R_{1,k}, \ldots, R_{n,k}) = E\varrho_0(S_1, \ldots, S_n) .$$

It remains to prove that $\Delta\bar{\varrho} = \overline{(\Delta\varrho)}$. With $S_0, \ldots, S_n$ in $\mathcal{S}_0$ represented as in (7), the corresponding expression for $S_{j-1}S_j$ is $F_1R_{j-1,1}R_{j,1} + \ldots + F_mR_{j-1,m}R_{j,m}$. Since $\bar{\varrho}$ extends ϱ_0, and $F_1, \ldots, F_n$ are pairwise orthogonal projections which commute with $\mathcal{R}$ (and hence with each value of ϱ), it follows from (8) that

$$\begin{aligned}
(\Delta\bar{\varrho})(S_0, \ldots, S_n) &= S_0\varrho_0(S_1, \ldots, S_n) + \sum_{j=1}^{n} (-1)^j \varrho_0(S_0, \ldots, S_{j-2}, S_{j-1}S_j, S_{j+1}, \ldots, S_n) \\
&\quad + (-1)^{n+1}\varrho_0(S_0, \ldots, S_{n-1})S_n \\
&= \sum_{k=1}^{m} F_k\{R_{0,k}\varrho(R_{1,k}, \ldots, R_{n,k}) \\
&\quad + \sum_{j=1}^{n} (-1)^j \varrho(R_{0,k}, \ldots, R_{j-2,k}, R_{k-1,k}R_{j,k}, R_{j+1,k}, \ldots, R_{n,k}) \\
&\quad + (-1)^{n+1}\varrho(R_{0,k}, \ldots, R_{n-1,k})R_{n,k}\} \\
&= \sum_{k=1}^{m} F_k(\Delta\varrho)(R_{0,k}, \ldots, R_{n,k}) \\
&= \overline{(\Delta\varrho)}(S_0, \ldots, S_n)
\end{aligned}$$

(where the last step results from the equation, corresponding to (8), for the element $\Delta\varrho$ of $NC_c^{n+1}(\mathcal{R}, \mathcal{R})$). The multilinear forms $\Delta\bar{\varrho}$ and $\overline{(\Delta\varrho)}$ on $\mathcal{S}^{n+1}$ are bounded, and take the same values on the dense subspace $(\mathcal{S}_0)^{n+1}$; so, by continuity, $\Delta\bar{\varrho} = \overline{(\Delta\varrho)}$.

COROLLARY 2.3. *With* $\mathcal{R}$, $\mathcal{A}$ *and* $\mathcal{S}$ *satisfying the conditions of Lemma 2.2, each* ϱ in $NZ_c^n(\mathcal{R}, \mathcal{R})$ *extends uniquely to an element* $\bar{\varrho}$ *of* $NZ_c^n(\mathcal{S}, \mathcal{S})$.

Proof. If $\varrho \in NZ_c^n(\mathcal{R}, \mathcal{R})$ then, by Lemma 2.2, ϱ extends uniquely to an element $\bar{\varrho}$ of $NC_c^n(\mathcal{S}, \mathcal{S})$, and $\Delta\bar{\varrho} = \overline{(\Delta\varrho)} = \bar{0} = 0$. Thus $\bar{\varrho} \in NZ_c^n(\mathcal{S}, \mathcal{S})$.

The possibility of extending a cocycle as in Lemma 2.2, »adjusting» it and then extending it to the (type I) weakoperator closure (of $\mathcal{S}$) combined with the fact [1; Theorem 4.4] that n-cocycles on type I von Neumann algebras cobound allows us to prove:

THEOREM 2.4. *If* $\mathcal{R}$ *is a von Neumann algebra acting on a Hilbert space* $\mathcal{H}$ *and* $\varrho \in Z_c^n(\mathcal{R}, \mathcal{R})$, *there is a* ξ *in* $C_c^n(\mathcal{R}, \mathcal{B}(\mathcal{H}))$ *such that* $\varrho = \Delta\xi$.

Proof. From [1; Theorem 3.4]; there is a ξ_1 in $C_c^{n-1}(\mathcal{R}, \mathcal{R})$ such that $\varrho - \Delta\xi_1$ $(= \varrho_1) \in NZ_c^n(\mathcal{R}, \mathcal{R})$. With $\mathcal{A}$ a maximal abelian *-subalgebra of $\mathcal{R}'$ and $\mathcal{S}$ the C^*-algebra generated by $\mathcal{R}$ and $\mathcal{A}$, there is, by Corollary 2.3, a (unique) ϱ_2 in $NZ_c^n(\mathcal{S}, \mathcal{S})$ extending ϱ_1. Theorem 2.1 provides a ξ_2 in $C_c^{n-1}(\mathcal{S}, \mathcal{S}^-)$ such that $\varrho_2 - \Delta\xi_2$ $(= \varrho_3)$ has an extension $\bar{\varrho}_3$ in $Z_c^n(\mathcal{S}^-, \mathcal{S}^-)$.

Since $\mathcal{S}$ contains both $\mathcal{R}$ and $\mathcal{A}$, and $\mathcal{A}$ is maximal abelian in $\mathcal{R}'$; $\mathcal{S}' \subseteq \mathcal{R}' \cap \mathcal{A}' = \mathcal{A}$. Thus $\mathcal{S}^-$, having an abelian commutant, is of type I. From [1; Theorem 4.4], there is a $\bar{\xi}_3$ in $C_c^{n-1}(\mathcal{S}^-, \mathcal{S}^-)$ such that $\bar{\varrho}_3 = \Delta\bar{\xi}_3$. Let η_3 be $\bar{\xi}_3|\mathcal{R}$; so that $\eta_3 \in C_c^{n-1}(\mathcal{R}, \mathcal{S}^-)$ and $\Delta\eta_3 = \bar{\varrho}_3|\mathcal{R}$. Let η_2 be $\xi_2|\mathcal{R}$; so that $\eta_2 \in C_c^{n-1}(\mathcal{R}, \mathcal{S}^-)$. With σ_2 taken as $\varrho_2|\mathcal{R}$, $\sigma_2 \in Z_c^n(\mathcal{R}, \mathcal{S}) \subseteq Z_c^n(\mathcal{R}, \mathcal{S}^-)$. Then $\sigma_2 - \Delta\eta_2 = \Delta\eta_3$. But $\varrho_1 = \varrho_2|\mathcal{R} = \sigma_2 = \Delta(\eta_2 + \eta_3) = \varrho - \Delta\xi_1$. Now $\xi_1 \in C_c^{n-1}(\mathcal{R}, \mathcal{R}) \subseteq C_c^{n-1}(\mathcal{R}, \mathcal{S}^-)$; so that $\varrho = \Delta(\eta_2 + \eta_3 + \xi_1)$, with $\eta_2 + \eta_3 + \xi_1$ in $C_c^{n-1}(\mathcal{R}, \mathcal{S}^-)$. Choosing $\eta_2 + \eta_3 + \xi_1$ as ξ completes the proof.

3. The hyperfinite case

In this section, we prove that the cohomology of a hyperfinite von Neumann algebra with coefficients in that algebra vanishes. The result that cohomology of u.h.f. C^*-algebras with coefficients in a dual module vanishes is established by meaning techniques.

THEOREM 3.1. *If* $\mathcal{R}$ *is a hyperfinite von Neumann algebra* $H_c^n(\mathcal{R}, \mathcal{R}) = 0$.

Proof. Suppose, first, that $\mathcal{R}'$ is hyperfinite. From [5; Lemma 5], there is a bounded projection π of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on the Hilbert space $\mathcal{H}$ on which $\mathcal{R}$ acts, onto $\mathcal{R}$, having (among others) the property that $\pi(ATB) = A\pi(T)B$, for A and B in $\mathcal{R}$. With ϱ in $Z_c^n(\mathcal{R}, \mathcal{R})$, Theorem 2.4 tells us that there is a ξ in $C_c^{n-1}(\mathcal{R}, \mathcal{B}(\mathcal{H}))$ such that $\varrho = \Delta\xi$. If η is $\pi \circ \xi$ then $\eta \in C_c^{n-1}(\mathcal{R}, \mathcal{R})$, and, with $A_1, \ldots, A_n$ in $\mathcal{R}$,

$$(\Delta\eta)(A_1,\ldots,A_n) = A_1\eta(A_2,\ldots,A_n) - \sum_{j=2}^{n}(-)^j\eta(A_1,\ldots,A_{j-2},A_jA_{j-1},A_{j+1},\ldots,A_n)$$
$$+ (-)^n\eta(A_1,\ldots,A_{n-1})A_n$$
$$= \pi[\Delta\xi(A_1,\ldots,A_n)] = \pi[\varrho(A_1,\ldots,A_n)] = \varrho(A_1,\ldots,A_n)\,.$$

Thus $H_c^n(\mathcal{R},\mathcal{R}) = 0$ when $\mathcal{R}'$ is hyperfinite.

From [6; Theorem 12.2], each von Neumann algebra can be represented in a »standard» form in which, in particular, it is * anti-isomorphic to its commutant. Since $H_c^n(\mathcal{R},\mathcal{R})$ is independent of the representation of $\mathcal{R}$ as a von Neumann algebra, we may assume that $\mathcal{R}$ is represented in this standard form. Then $\mathcal{R}'$ is hyperfinite; and, from the preceding, $H_c^n(\mathcal{R},\mathcal{R}) = 0$.

Remark 3.2. There are other routes we could take to a proof of the preceding theorem which would avoid [6]; but they refer to the »internal» properties of the hyperfinite $\mathcal{R}$ and have to be argued more closely. The use of π gives some information not noted in the statement of Theorem 3.1; viz. $H_c^n(\mathcal{R},\mathcal{R}) = 0$ if $\mathcal{R}'$ is hyperfinite. It is quite possible, though, that this occurs only when $\mathcal{R}$ is hyperfinite.

In the next result, the dual module might be, for example, the von Neumann algebra closure of the C^*-algebra in a given representation.

THEOREM 3.3. *If a C^*-algebra $\mathfrak{A}$ is the norm closed linear span of an amenable subgroup $\mathcal{V}$ of its unitary group, and $\mathcal{M}$ is a two-sided dual $\mathfrak{A}$-module, then $H_c^n(\mathfrak{A},\mathcal{M}) = 0$.*

Proof. From, [1; Lemma 3.3], there is a mean $\bar{\mu}$ from $l_\infty(\mathcal{V},\mathcal{M})$ into $\mathcal{M}$. With ϱ in $Z_c^n(\mathfrak{A},\mathcal{M})$, $A_1,\ldots,A_{n-1}$ in $\mathfrak{A}$ and V in $\mathcal{V}$, define $\xi(A_1,\ldots,A_{n-1})$ to be $\bar{\mu}(\varphi)$, where $\varphi(V) = V^*\varrho(V,A_1,\ldots,A_{n-1})$. Since φ depends multilinearly on the parameters $A_1,\ldots,A_{n-1}$ and $\bar{\mu}$ is linear, ξ is multilinear. Moreover

$$\|\xi(A_1,\ldots,A_{n-1})\| = \|\bar{\mu}(\varphi)\| \le \|\varphi\| \le \|\varrho\|\cdot\|A_1\|\ldots\|A_{n-1}\|\,;$$

so that $\|\xi\| \le \|\varrho\|$, and $\xi \in C_c^{n-1}(\mathfrak{A},\mathcal{M})$.

We prove that $\varrho = \Delta\xi$. With W in $\mathcal{V}$, since $(\Delta\varrho)(V,W,A_1,\ldots,A_{n-1}) = 0$ (in the notation of [1; Lemma 3.3]),

$$\varphi_W(V) = (VW)^*\varrho(VW,A_1,\ldots,A_{n-1})$$
$$= W^*V^*[V\varrho(W,A_1,\ldots,A_{n-1}) + \varrho(V,WA_1,A_2,\ldots,A_{n-1})$$
$$- \varrho(V,W,A_1A_2,A_3,\ldots,A_{n-1}) + \ldots$$
$$+ (-1)^n\varrho(V,W,A_1,\ldots,A_{n-3},A_{n-2}A_{n-1}) + (-1)^{n+1}\varrho(V,W,A_1,\ldots,A_{n-2})A_{n-1}]\,.$$

Thus, from the properties of $\bar{\mu}$ [1; Lemma 3.3],

$$\xi(A_1,\ldots,A_{n-1}) = \bar{\mu}(\varphi) = \bar{\mu}(\varphi_W) = W^*[\varrho(W,A_1,\ldots,A_{n-1}) + \xi(WA_1,A_2,\ldots,A_{n-1})$$
$$- \xi(W,A_1A_2,A_3,\ldots,A_{n-1}) + \ldots + (-1)^n\xi(W,A_1,\ldots,A_{n-3},A_{n-2}A_{n-1})$$
$$+ (-1)^{n+1}\xi(W,A_1,\ldots,A_{n-2})A_{n-1}]$$
$$= W^*[\varrho(W,A_1,\ldots,A_{n-1}) + W\xi(A_1,\ldots,A_{n-1}) - (\Delta\xi)(W,A_1,\ldots,A_{n-1})]\,;$$

and $$\varrho(W, A_1, \ldots, A_{n-1}) = (\Delta\xi)(W, A_1, \ldots, A_{n-1}), \tag{10}$$

for W in $\mathcal{V}$. As $\mathfrak{A}$ is the norm closed linear span of $\mathcal{V}$ and both ϱ and $\Delta\xi$ are multilinear and bounded; it follows from (10) that $\varrho = \Delta\xi$. Thus $H_c^n(\mathfrak{A}, \mathcal{M}) = 0$.

COROLLARY 3.4. *If* $\mathfrak{A}$ *is an abelian or a u.h.f. C*-algebra,* $H_c^n(\mathfrak{A}, \mathcal{M}) = 0$ *for each two-sided dual* $\mathfrak{A}$*-module* $\mathcal{M}$.

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