Holomorphic convexity and analytic structures in Banach algebras

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Introduction

In this paper every Banach algebra is commutative and semi-simple with a unit element. If B is a Banach algebra then M_B denotes its maximal ideal space and ∂_B its Shilov boundary. Since B is semi-simple the Gelfand transform identifies B with a subalgebra of $C(M_B)$. If B then becomes a uniformly closed subalgebra of $C(M_B)$ we say that B is a uniform algebra. Notice that B is a uniform algebra precisely when it is complete in its spectral radius norm.

If B is a uniform algebra and if A is a closed point-separating subalgebra of $C(M_B)$ such that $A \subset B$, then we simply say that A is a *subalgebra of* B. In Section 1 and 2 we analyze this situation. Here follow two results which are typical applications of this material.

THEOREM 1.1. *Let X be a reduced analytic space and let A be a point-separating subalgebra of $\mathcal{O}(X)$. Suppose that K is a compact $\mathcal{O}(X)$-convex set in X such that the set $\hat{K}_A = \{x \in X : |f(x)| \leq |f|_K$ for all f in $A\}$ is compact in X. Then $K = \hat{K}_A$ and $M_{A(K)} = K$, where $A(K)$ is the uniform algebra on K generated by the restriction algebra $A|K$.*

In Theorem 2.1. the following notations are used. D is the closed unit disc in $\mathbf{C}^1$ and $A(D)$ is the usual disc-algebra on D. An element $f \in A(D)$ is *smooth* if the restriction of f to T is continuously differentiable. Here T is the unit circle.

THEOREM 2.1. *Let A be a subalgebra of $A(D)$ such that A contains a dense subalgebra of smooth functions. Then $M_A = D$.*

Let us remark here that the results above also apply to Banach algebras. For if B is a Banach algebra and if B_c is the uniform closure of B in $C(M_B)$, then it is well known that $M_B = M_{B_c}$. If A is a closed subalgebra of B which separates points in M_B, then A_c is a subalgebra of B_c.

Using a famous principle due to Shilov (See [8, p. 214]), it follows that when M_B is identified with a subset of M_A, then $\partial_A \subset M_B$. This implies that M_A can be identified with M_{A_c}. In particular we have the basic principle below.

Shilov's Principle. *Let B be a Banach algebra and let A be a closed subalgebra of B which separates points in M_B. Then $M_A = M_B$ if and only if $M_{A_c} = M_B$.*

In Section 3 we prove a result about Banach algebras generated by analytic functions.

Theorem 3.1. *Let X be a compact subset of a reduced analytic space Y. Let B be a Banach algebra on X such that $X = M_B$. Suppose also that there exists an open neighborhood U of X in Y and a point separating subalgebra A of $\mathcal{O}(U)$ such that $A_X = \{f \in C(X) : f = \tilde{f}|X$ for some $\tilde{f} \in A\}$, is a dense subalgebra of B. If now J is a closed ideal of B such that $A_X \cap J$ is a dense subset of J, while* Hull $(J) = \{x \in X : j(x) = 0$ *for all j in $J\}$ is a compact subset of the interior of X, then J has a finite codimension in B.*

In Section 4 we study compact holomorphically convex sets in $\mathbf{C}^n$. If K is a compact set in $\mathbf{C}^n$ we let $\mathcal{O}(K)$ be the algebra of germs of analytic functions on K, and $H(K)$ is the uniform closure of $\mathcal{O}(K)$ in $C(K)$. We say that K is a *holomorphic set* if $K = M_{H(K)}$. It is well known that if K is an intersection of open domains of holomorphy, then K is a holomorphic set. We give an example which shows that the converse is false.

Finally Section 5 contains some material about integral extensions of Banach algebras, based on results from Section 1.

Very often we employ the Local Maximum Principle (abbreviated LMP) in the proofs. The LMP states that if B is a uniform algebra while V is a subset of $M_B \setminus \partial_B$, then $V \subset \mathrm{Hull}_B(bV)$, where bV is the topological boundary of V in M_B.

We refer to [5] for basic facts about uniform algebras.

1. Convexity and analytic structures in uniform algebras

Let X be a compact Hausdorff space and let B be a uniform algebra on X. When X is identified with a closed subset of M_B we have $\partial_B \subset X$. If K is a closed subset of M_B we put $\mathrm{Hull}_B(K) = \{y \in M_B : |f(y)| \leq |f|_K$ for all f in $B\}$. If S is a closed subset of M_B we let $B(S)$ be the uniform algebra on S which is generated by the restriction algebra $B|S$. Finally Σ_B denotes the Choquet boundary of B. Here Σ_B is dense in ∂_B.

If the set $M_B \setminus X$ is non empty, then we put $\Delta = M_B \setminus X$ and let $b\Delta$ be its topological boundary in M_B. Then the LMP shows that $\Delta \subset \mathrm{Hull}_B(b\Delta)$ and if $B_1 = B(\mathrm{Hull}_B(b\Delta))$, then $M_{B_1} = \mathrm{Hull}_B(b\Delta)$ while $\partial_{B_1} \subset b\Delta$. In the results which follow we show that ∂_{B_1} is contained in a smaller set than $b\Delta$ under certain assumptions.

Definition 1.1. Let B be a uniform algebra on X. We say that B has an *analytic structure* at a point $x \in X$ if there is an open neighborhood W of x in X and a homeomorphism Φ from W onto an analytic subset V of the open polydisc D^n in $\mathbf{C}^n$, such that the functions $f \circ \phi^{-1}$ are holomorphic on V for all f in B.

Notice that in Definition 1.1 the analytic set V may be 0-dimensional at $\phi(x)$. This happens precisely when x is an isolated point in X. In the non trivial case when $\dim(V) > 0$ at $\phi(x)$, the maximum principle for holomorphic functions on V implies that x cannot belong to ∂_B.

THEOREM 1.2. *Let B be a uniform algebra on X and let W be an open subset of X such that B has an analytic structure at each point in W. If now the set $\Delta = M_B \setminus \partial_B$ is non empty and if we put $B_1 = B(\mathrm{Hull}_B(b\Delta))$, then $\partial_{B_1} \subset (b\Delta \setminus W)$.*

Next we introduce some concepts from [10]. If W is a locally closed subset of M_B, then we put $H_B(W) = \{f \in C(W):$ to each $x \in W$ there is an open neighborhood U_f of x in M_B and a sequence (b_n) in B such that $\lim |f - b_n|_{W \cap U_f} = 0\}$.

Definition 1.2. Let W be a locally closed set in M_B and let V be a locally closed subset of W. Then V is called a *B_W-analytic variety* if to each point $x \in V$, there is an open neighborhood U of x in W and a family $\{f_i\}_{i \in I}$ in $C(U)$ such that the following holds. Firstly $U \cap V = \{x \in U : f_i(x) = 0$ for all $i\}$, and secondly each $f_i | U \setminus V$ belongs to $H_B(U \setminus V)$. The functions in $C(U)$ satisfying this condition are denoted by $H_B(U, U \cap V)$.

Suppose next that U is an open subset of M_B while V is a locally closed subset of bU. Then we say that V is a *U-analytic variety at bU* if V is a $B_{U \cup V}$-analytic variety. It is easily seen that a finite union of U-analytic varieties at bU is a U-analytic variety at bU.

THEOREM 1.3. *Let B be a uniform algebra on a compact space X and suppose that the set $M_B \setminus X = \Delta$ is non empty. Let V be a relatively open subset of $b\Delta$, such that V is a Δ-analytic variety. Then $\Delta \subset \mathrm{Hull}_B(b\Delta \setminus V)$.*

THEOREM 1.4. *Let B be a uniform algebra on X and let W be an open subset of X. Let K be a compact subset of W such that $\mathrm{Hull}_B(K) \cap W = K$. If now the set $M_B \setminus X = \Delta$ is non empty, while $b\Delta \cap W$ is a Δ-analytic variety, then $K = \mathrm{Hull}_B(K)$ and $K = M_{B(K)}$.*

Notice here that if Δ is empty in Theorem 1.4 then $K = \mathrm{Hull}_B(K)$ follows because each closed component of $\mathrm{Hull}_B(K)$ intersects K.

Suppose next that A is a uniform algebra and let B be a uniform algebra on M_A such that $A \subset B$. Then we say that A is *holomorphically dense* in B if the following holds. There exists a finite decreasing sequence of closed sets $M_A = U_0 \supset U_1 \supset \ldots \supset U_n \supset U_{n+1} = \emptyset$, such that U_{i+1} is a B_{U_i}-analytic variety for each $i = 0, \ldots, n-1$, and each restriction algebra $B|(U_i \setminus U_{i+1})$ is contained in $H_A(U_i \setminus U_{i+1})$.

THEOREM 1.5. *Let A be a uniform algebra and let B be a uniform algebra on M_A such that $A \subset B$. If A is holomorphically dense in B, then $M_A = M_B$ and $\partial_A = \partial_B$.*

Finally we show that Theorem 1.3 and 1.4 can be used to study analytic structures.

THEOREM 1.6. *Let B be a uniform algebra on X and let W be an open subset of X such that B has an analytic structure at each point in W. If now the set $\Delta = M_B \setminus X$ is non empty, then $b\Delta \cap W$ is a Δ-analytic variety.*

Notice here that Theorem 1.6 and 1.3 together imply Theorem 1.2. In a series of subsections we prove the results above.

1.a. *Proof of Theorem* 1.1.

We explain here why Theorem 1.1 is a consequence of Theorems 1.4 through 1.6. For consider the situation in Theorem 1.1 and choose a compact neighborhood K_1 of the set $\hat{K}_A$ in X. If $A_1 = A(K_1)$ it is obvious that A_1 has an analytic structure in the interior of K_1. Hence $\text{Hull}_{A_1}(\hat{K}_A) = \hat{K}_A$ and then an application of Theorems 1.6 and 1.4 implies that $\hat{K}_A = M_{A(\hat{K}_A)}$.

It remains to prove that $K = \hat{K}_A$. Let B be the uniform algebra on $\hat{K}_A$ generated by $\mathcal{O}(X)|\hat{K}_A$. Since K is $\mathcal{O}(X)$-convex we see that if the set $W = \hat{K}_A \setminus K$ is non empty, then $\partial_B \cap W \neq \emptyset$. On the other hand $\partial_{A(\hat{K}_A)} \subset K$, so if we can prove that $A(\hat{K}_A)$ is holomorphically dense in B, then Theorem 1.5 implies that $\partial_{A(\hat{K}_A)} = \partial_B$, which proves that W must be empty. This fact will be proved in Section 1.b.

We also remark that Theorem 1.1 essentially is contained in the work by H. Rossi in [13]. We will employ results from [13] in the proof of Theorem 3.1.

1.b. *Remarks about analytic spaces*

Let X be a reduced analytic space and let A be a point-separating subalgebra of $\mathcal{O}(X)$. The following assertions are well-known. See [6] or [9].

Let Reg (X) denote the open set of all regular points in X and let Sing (X) be the singular set. Then Sing (X) is a nowhere dense analytic subset of X. If $x \in \text{Reg}(X)$ there is a unique integer $s \geq 0$ such that there is some open neighborhood of x in X which is biholomorphic with the open polydisc in $\mathbf{C}^s$. The number s is denoted by dim (x), and the case dim $(x) = 0$ occurs when x is an isolated point in X.

If $x \in \text{Reg}(X)$ and if $s = \dim(x)$, then we say that $x \in \text{Reg}(A)$ if A gives local coordinates at x. The condition that $x \in \text{Reg}(A)$ is tested by looking at the Jacobians $J_F(x) = \det(\partial f_\nu / \partial u_i(x))$, where $u_1, \dots, u_s$ determine local coordinates at x, while $F = (f_1 \dots f_s)$ runs over s-tuples from A. Then $x \in \text{Reg}(A)$ if and only if $J_F(x) \neq 0$ for some F.

The definition of $\operatorname{Reg}(A)$ implies that all isolated points in X belong to $\operatorname{Reg}(A)$ while $\operatorname{Sing}(X) \cap \operatorname{Reg}(A)$ is empty. The discussion above shows that the set $\operatorname{Sing}(A) = X \setminus \operatorname{Reg}(A)$ is an analytic subset of X. Since A separates points in X it is an easy exercise to verify that $\operatorname{Sing}(A)$ is nowhere dense in X.

If we put $S_1 = \operatorname{Sing}(A)$, then S_1 is a reduced analytic space while $A|S_1$ is a subalgebra of $\mathcal{O}(S_1)$. Then we can define the set $\operatorname{Reg}(A|S_1)$ and put $S_2 = S_1 \setminus \operatorname{Reg}(A|S_1)$. Here S_2 is an analytic subset of X and S_2 is nowhere dense in S_1.

Inductively we get a sequence $S_1 \supset S_2 \supset \ldots$ such that $S_{i+1} = S_i \setminus \operatorname{Reg}(A|S_i)$. If $x \in X$ and if $n = \dim(x)$, then it follows that S_{n+1} does not contain x. More generally, if K is a compact subset of X, then there is an integer ν such that $S_\nu \setminus K$ is empty.

Clearly the assertions above imply that if K is a compact set in X such that $K = M_{A(K)}$, then $A(K)$ is holomorphically dense in the uniform algebra on K generated by $\mathcal{O}(X)|K$.

1.c. *Convexity and analytic phenomena*

Here we collect essentially well-known facts from the theory of functions locally approximable in a uniform algebra. The theory is developed in [10] and we also refer to [2].

The following result appears in [2, Theorem 5].

PROPOSITION 1.1.c. *Let A be a uniform algebra and let B be a uniform algebra on M_A such that $A \subset B$. Let F be a closed A-convex subset of M_A such that $B|(M_A \setminus F)$ contains a uniformly dense subalgebra contained in $H_A(M_A \setminus F)$. Then the (possibly empty) set $\Delta = M_B \setminus M_A$ is contained in* $\operatorname{Hull}_B(F)$. *Here* $\operatorname{Hull}_B(F) \cap M_A = F$ *and* $b\Delta \subset F$.

We explain here why Proposition 1.1.c follows from Theorem 5 in [2]. Firstly Theorem 5 is stated in the case when B is generated over A by a single function $f \in C(M_A)$ for which $f|(M_A \setminus F) \in H_A(M_A \setminus F)$. But the proof easily extends to the general case. The conclusion in Theorem 5 is that if $y \in \Delta$, then there is a point $\pi(y) \in F$ such that $a(y) = a(\pi(y))$ for all $a \in A$. Here π is the induced map from M_B into M_A given by the inclusion of A into B.

Since π is continuous while $\pi(\Delta) \subset F$ it follows that $b\Delta \subset F$. Now the LMP implies that $\Delta \subset \operatorname{Hull}_B(F)$. Since F is A-convex it follows trivially that $F = \operatorname{Hull}_B(F) \cap M_A$, and then $b\Delta \subset F$ follows.

PROPOSITION 1.2.c. *Let A be a uniform algebra and let U be an open subset of $M_A \setminus \partial_A$. Let $x \in bU$ and suppose there exists an open neighborhood W of x in bU such that W is an $A_{U \cup W}$-analytic variety. Then it follows that* $U \subset \operatorname{Hull}_A(bU \setminus W)$.

Proof. Put $A_1 = A(U \cup bU)$. Then the LMP shows that $\partial_{A_1} \subset bU$ and because ∂_{A_1} is the closure of Σ_{A_1} we conclude that if $\Sigma_{A_1} \cap W$ is empty, then $\partial_{A_1} \subset (bU \setminus W)$.

To prove that $\Sigma_{A_1} \cap W$ is empty we consider a point $x \in W$. Now we can choose a closed neighborhood Ω of x in M_A such that $\Omega \cap bU \subset W$ and where $\Omega \cap W = \{y \in \Omega \cap \bar{U} : f_i(y) = 0 \text{ for all } i \in I\}$. Here $F = \{f_i\}_{i \in I}$ is a family in $H_A(\Omega \cap \bar{U}, \Omega \cap W)$.

Next we consider the uniform algebra B on $\Omega \cap \bar{U}$ which is generated by F and $A|(\Omega \cap \bar{U})$. Using the LMP and we see that $\Omega \cap \bar{U} \subset \mathrm{Hull}_B(b\Omega \cap \bar{U} \cup (\Omega \cap W))$. So if $y \in \Omega^\circ \cap U$, then there is a so called Jensen measure m carried on $(b\Omega \cap \bar{U}) \cup (\Omega \cap W)$ satisfying $\log |b(y)| \leq \int \log |b| dm$ for all $b \in B$. Since this holds for all f_i we conclude that m carries no mass on the set $\Omega \cap W$. This implies that $y \in \mathrm{Hull}_B(b\Omega \cap \bar{U})$.

Since x belongs to the closure of $\Omega^\circ \cap U$ we conclude that $x \in \mathrm{Hull}_B(b\Omega \cap \bar{U}) \subset \mathrm{Hull}_{A_1}(b\Omega \cap \bar{U})$. Since $x \in \Sigma_{A_1}$ it follows that $x \in b\Omega \cap \bar{U}$, but this contradicts the fact that Ω is a neighborhood of x in M_A.

We have now proved that $\partial_{A_1} \subset (bU \setminus W)$ and then $U \subset \mathrm{Hull}_{A_1}(bU \setminus W) = \mathrm{Hull}_A(bU \setminus W)$ follows.

PROPOSITION 1.3.c. *Let A be a uniform algebra and let Z be a closed A-analytic variety in M_A. If F is a closed set in M_A, then* $\mathrm{Hull}_A(F \cup Z) = \mathrm{Hull}_A(F) \cup Z$.

Proof. Let us put $D = \mathrm{Hull}_A(F \cup Z) \setminus (\mathrm{Hull}_A(F) \cup Z)$, and suppose that D is non empty. If $A_0 = A(\mathrm{Hull}_A(F \cup Z))$, then D is an open subset of $M_{A_0} \setminus \partial_{A_0}$ while Z is an A_0-analytic variety.

The LMP shows that $D \subset \mathrm{Hull}_{A_0}(bD)$, where bD is the topological boundary of D in M_{A_0}. Clearly $W = (bD \setminus \mathrm{Hull}_A(F)) \subset Z$, and hence it is obvious that W is an $(A_0)_{W \cup D}$-analytic variety. It follows from Proposition 1.2.c that $D \subset \mathrm{Hull}_{A_0}(bU \setminus W) \subset \mathrm{Hull}_A(F)$, a contradiction.

PROPOSITION 1.4.c. *Let A be a uniform algebra and let B be a uniform algebra on M_A such that $A \subset B$. Suppose that V is a closed B_{M_A}-analytic variety in M_A and that $B|(M_A \setminus V)$ contains a uniformly dense subalgebra of functions from $H_A(M_A \setminus V)$. Then* $V = \mathrm{Hull}_A(V)$.

Proof. Consider a point $x \in V$. We can find a closed A-convex neighborhood Ω of x in M_A and a family $F = \{f_i\}$ in $H_B(\Omega, \Omega \cap V)$ for which $\Omega \cap V = \{y \in \Omega : f_i(y) = 0 \text{ for all } i\}$.

The hypothesis on B implies that every function in B is A-holomorphic of the second kind in $M_A \setminus V$. As a consequence the elements in F are A-holomorphic of the third kind in $\Omega \setminus V$. See [11] for a general concept of A-holomorphic functions.

Next we consider $A_1 = A(\mathrm{Hull}_A(V))$ and put $U = \mathrm{Hull}_A(V) \setminus V$. Then $bU \subset V$ holds and the discussion above shows that the hypothesis of Proposition 1.2.c is satisfied for each point in bU, except that we now deal with A-holomorphic functions of the third kind on $\Omega \setminus V$. But using Rickart's general theory in [11] it follows that Proposition 1.2.c remains true. Hence $U \subset \mathrm{Hull}_{A_1}(bU \setminus V) = \mathrm{Hull}_{A_1}(\emptyset)$, which implies that U is empty.

1.d. *Proof of Theorems* 1.3—1.6.

Proof of Theorem 1.3. Put $S = \Delta \cup b\Delta$ and introduce the uniform algebra $B(S)$. Then $\partial_{B(S)} \subset b\Delta$ and Δ is an open subset of $M_{B(S)} = \text{Hull}_B(b\Delta)$. Using Proposition 1.2.c it follows that $\Delta \subset \text{Hull}_{B(S)}(b\Delta \setminus V) = \text{Hull}_B(b\Delta \setminus V)$.

Proof of Theorem 1.4. Introduce the uniform algebra $A = B(\text{Hull}_B(K))$. Then $M_A = \text{Hull}_B(K)$ while $\partial_A \subset K$. Since $\text{Hull}_B(K) \cap W = K$ we see that K is a closed A-analytic variety in M_A. Then Proposition 1.4.c applied to the case $B = A$, shows that $K = \text{Hull}_A(K)$. Since $\partial_A \subset K$ it follows that $K = M_A = \text{Hull}_B(K)$, and as a consequence $K = M_{B(K)}$.

Proof of Theorem 1.5. Firstly Proposition 1.4.c shows that U_1 is A-convex in M_A. Then Proposition 1.1.c shows that the set $\Delta = M_B \setminus M_A$ is contained in $\text{Hull}_B(U_1)$. This means that if $A_1 = A(U_1)$ and $B_1 = B(U_1)$, then $\Delta = M_{B_1} \setminus M_{A_1}$ while $U_1 = M_{A_1}$.

Next we notice that $B_1|(U_1 \setminus U_2)$ contains a dense subalgebra of functions in $H_{A_1}(U_1 \setminus U_2)$. A new application of Proposition 1.4.c proves that U_2 is A_1-convex in M_B and that $M_{B_1} \setminus M_{A_1} \subset \text{Hull}_{B_1}(U_2)$. This means that $M_B \setminus M_A \subset \text{Hull}_B(U_2)$, and by induction we see that $M_B \setminus M_A \subset \text{Hull}_B(U_\nu)$ for all $\nu \geq 1$. Finally $\nu = n + 1$ gives $M_B = M_A$.

It remains to prove that $\partial_B = \partial_A$. So let us put $F = \text{Hull}_B(\partial_A)$ and suppose that the set $M_B \setminus F$ is non empty. Since $B|(M_A \setminus U_1) \subset H_A(M_A \setminus U_1)$ it follows from Corollary 2.4. in [10], that ∂_B cannot intersect $M_A \setminus (F \cup U_1)$.

Hence $\partial_B \subset (F \cup U_1)$ and if $y \in M_A \setminus F$, then $y \in \text{Hull}_B(F \cup U_1)$ while y does not belong to $\text{Hull}_B(F)$. Since U_1 is a closed B-analytic variety in M_B it follows from Proposition 1.3.c that $y \in U_1$. Hence $M_A \setminus F \subset U_1$ holds.

Since $M_A \setminus F$ is an open subset of M_B which is contained in U_1, the condition that $B|(U_1 \setminus U_2) \subset H_A(U_1 \setminus U_2)$ again implies that ∂_B cannot intersect $M_A \setminus (F \cup U_2)$. Then the same argument as above proves that $M_A \setminus F \subset U_2$. Inductively we see that $M_A \setminus F \subset U_\nu$ for all $\nu \geq 1$, so when $\nu = n + 1$ we conclude that $F = M_A$. But this means that $\partial_B = \partial_A$.

Proof of Theorem 1.6. We may consider W as a reduced analytic space while $B|W$ is a point separating subalgebra of $\mathcal{O}(W)$. Suppose now that $x \in b\Delta \cap W$ is such that x also belongs to $\text{Reg}(B|W)$. Let $s = \dim(x)$.

If $s = 0$ then x is an isolated point in W and hence also in $b\Delta$. Then $\{x\}$ is a Δ-analytic variety. If $s > 0$ we choose $f_1, \ldots, f_s$ in B which map an open neighborhood U of x in W homeomorphically onto the open polydisc D^s in $\mathbf{C}^s$. If $g \in B$ we put $\tilde{g}(z_1, \ldots, z_s) = g(\delta(z))$, where $\delta(z)$ is the unique point in U for which $z_i = f_i(\delta(z))$ for $i = 1, \ldots, s$. Then $\tilde{g} \in \mathcal{O}(D^s)$ holds.

Let us choose $0 < r < 1$ and put $K = \{y \in U : |f_i(y)| \leq r \text{ for } i = 1, \ldots, s\}$. Next we let Ω be an open neighborhood of x in M_B such that $|f_i|_\Omega < r$ for all i. We claim that $\Omega \cap U$ is a B_Ω-analytic variety.

For let $y \in \Omega \setminus U$ be given. We can choose $z \in U$ such that $f_i(y) = f_i(z)$ for all i. Then we take some $g \in B$ such that $g(y) = 1$ and $g(z) = 0$. Since $\tilde{g} \in \mathcal{O}(D^s)$ we can find a sequence of polynomials $P_\nu \in \mathbf{C}[Z_1, \ldots, Z_s]$ such that $\lim |P_\nu - \tilde{g}|_{D(r)} = 0$, where $D(r)$ is the closed polydisc of radius r in $\mathbf{C}^s$.

It follows that $\lim P_\nu(f_1, \ldots, f_s)$ exists uniformly in Ω and the limit function h belongs to $H_B(\Omega)$. Here $h = g$ holds on $\Omega \cap U$ while $h(y) = \lim P_\nu(f_1(y), \ldots, f_s(y)) = \lim P_\nu(f_1(z), \ldots, f_s(z)) = g(z) = 0$. Hence $h_1 = g - h$ belongs to $H_B(\Omega)$ and $h_1 = 0$ on $\Omega \cap U$ while $h_1(y) = 1$. This proves that $\Omega \cap U$ is a B_Ω-analytic variety.

We have now proved that the set $\operatorname{Reg}(B|W) = S_0$ is a Δ-analytic variety at $b\Omega$. Now we consider the general case. Using the notation of Section 1.b we have a decreasing sequence $S_1 \supset S_2 \supset \ldots$, where $S_{i+1} = S_i \setminus \operatorname{Reg}(B|S_i)$. If $x \in W$ we choose a relatively compact open neighborhood V of x in W. Then $S_n \cap V$ is empty for some $n \geq 1$. Using the same argument as above we can prove that the sets $U_i = (S_i \setminus S_{i+1}) \cap V$ are Δ-analytic varieties at $b\Delta$. But then $V = (S_0 \cap V) \cup U_1 \cup \ldots \cup U_{n-1}$ also becomes a Δ-analytic variety at $b\Delta$. Since this holds for all points in W, we conclude that W is a Δ-analytic variety at $b\Delta$.

2. Proof of Theorem 2.1.

Firstly we need some facts from the so called »arc-theory» dealing with uniform algebras presented on smooth Jordan arcs in $\mathbf{C}^n$. We refer to [3, 15, 18] for details. The following result is contained in [15, Lemma 5—8].

LEMMA 2.1. *Let A be a uniform algebra on a compact space X. Let $f \in A$ be such that there is an open set W in X which f maps homeomorphically onto an open Jordan arc J in $\mathbf{C}^1$. Suppose also that $W = \{x \in X : f(x) \in J\}$ and that J is a relatively open subset of $f(X)$, which borders the outer component Δ_∞ of the set $\mathbf{C}^1 \setminus f(X)$. If we now identify X with a closed subset of M_A then the following condition holds for each point $z \in J$: There exists an open disc $\Delta(z)$ such that f maps the open set $U = \{y \in M_A : f(y) \in \Delta(z)\}$ homeomorphically onto a subset of $\Delta(z)$, and here $f(U) \cap \Delta_\infty$ is empty.*

Let us now consider the uniform algebra A in Theorem 2.1. Clearly $\partial_A \subset T$ holds. If $\partial_A \neq T$ we can choose a closed interval I of T such that $\partial_A \subset I$. Then $A|I = A(I)$ becomes a proper subalgebra of $C(I)$ which is generated by continuously differentiable functions. But this contradicts [15, An Application, p. 186]. We conclude that $\partial_A = T$ holds.

If f is a smooth function in A we say that a point $a \in T$ is a *regular peak point of f* if the following condition holds: Firstly $f'(a) \neq 0$, where $f' \in C(T)$ is the continuous derivative determined by $f|T$. Secondly $\{a\} = \{x \in T : f(x) = f(a)\}$ and $f(a)$ belongs to the boundary of the outer component of the set $\mathbf{C}^1 \setminus f(T)$.

With the notations above we have the following result.

LEMMA 2.2. *Let f be a smooth function in A and let $a \in T$ be a regular peak point of f. Then there exists an open disc $\Delta_1(a)$ such that the set $D \cap \Delta_1(a)$ is an open subset of M_A.*

Proof. Clearly there exists an open arc W in T such that $a \in W$ while f maps W homeomorphically onto an open Jordan arc J in $\mathbf{C}^1$. In addition J is a relatively open subset of $f(T)$, and $W = \{x \in T : f(x) \in J\}$, while J borders the outer component Δ_∞ of the set $\mathbf{C}^1 \setminus f(X)$.

Since f is analytic and non constant in $D \setminus T$, we see that $f(D \setminus T)$ is contained in the interior of $f(D)$. It follows that J is a relatively open subset of the topological boundary of $f(D)$. Since $f(D \setminus T) \cap \Delta_\infty$ is empty, we see that J also borders a bounded component V of the set $\mathbf{C}^1 \setminus f(T)$. Because $f'(a) \neq 0$ it follows easily that there is an open disc $\Delta(f(a))$ and an open disc $\Delta(a)$ such that $f(\Delta(a) \cap D)$ contains $\Delta(f(a)) \cap (V \cup J)$.

Now we apply Lemma 2.1 to the uniform algebra $A|T$ and the point $f(a) \in J$. For then we can choose an open disc $\Delta_0(f(a))$, where $\Delta_0(f(a)) \subset \Delta(f(a))$, such that f maps the set $U_0 = \{y \in M_A : f(y) \in \Delta_0(f(a))\}$ homeomorphically onto a subset of $\Delta_0(f(a))$. Since $f(U_0) \cap \Delta_\infty$ is empty, we see that $f(U_0) \subset f(\Delta(a) \cap D)$. This means that $U_0 \subset \Delta(a) \cap D$, and since $a \in U_0$ we easily get the desired disc $\Delta_1(a)$.

LEMMA 2.3. *Let A be as in Theorem 2.1. Then there exists a smooth function f in A which determines a regular peak point.*

Proof. Let f be a non constant smooth function in A. Let us put $E = \{a \in T : f(a)$ belongs to the outer component Δ_∞ of $\mathbf{C}^1 \setminus f(T)\}$. Since $f(D) \cap \Delta_\infty$ is empty, it follows that $f(E) \cap f(D \setminus T)$ is empty. Hence $f(D \setminus T)$ is contained in the polynomially convex hull of the set $f(E)$. This implies that $f(E)$ is not simply connected. If $f' = 0$ on E, then it is easily verified that $f(E)$ is totally disconnected and simply connected.

We conclude that $f'(a) \neq 0$ for some $a \in E$. Next we consider the set $F = \{x \in T : f(x) = f(a)\}$. Since $F \neq T$ we know already that $A(F) = C(F)$. Since a is an isolated point in F we can choose $g \in A$ such that $g(a) = 1$ while $|g| < 1/2$ on $F \setminus \{a\}$ and g is smooth.

Now it is easily verified that we can choose some $\varepsilon > 0$ and a suitable »direction» e^{it}, such that if we put $h = f + e^{it}\varepsilon g$, then a is a regular peak point of h.

Proof of Theorem 2.1. Trivially A has an analytic structure in $D \setminus T$. If $\Delta = M_A \setminus D$ is non empty, then Theorem 1.2 shows that $\Delta \subset \mathrm{Hull}_A (b\Delta \cap T)$ holds. Now Lemma 2.2 shows that $b\Delta \cap T$ does not contain any regular peak point determined by a smooth function in A. Hence Lemma 2.3 implies that $b\Delta \cap T \neq T$. But then we know that $A(b\Delta \cap T) = C(b\Delta \cap T)$, and then $b\Delta \cap T$ is A-convex in M_A, a contradiction.

3. Proof of Theorem 3.1.

If B_c is the uniform closure of B in $C(X)$, then it is obvious that $B_c = A(X)$. Since $X = M_B$ it follows that $X = M_{A(X)}$, and because A separates points in Y we conclude that X is A-convex in Y. Let us put $J_0 = \{f \in A : f|X \in J\}$. Since Hull (J) does not intersect bX we can find an open neighborhood W of X in Y such that Hull (J_0) does not intersect $(W \setminus X)$.

By a standard argument we can find an analytic polyhedron $\Gamma = \{z \in W : |f_i(z)| < 1,\ i = 1, \ldots, s\}$, where $f_1, \ldots, f_s \in A$ and $X \subset \Gamma$ while $\bar{\Gamma} \subset\subset W$. It follows that if $A_0 = A|\Gamma$, then each compact subset of Γ has a compact A_0-convex hull in Γ. Next follows an important step in the proof.

LEMMA 3.1. *Let $\tilde{A}$ be the closure of A in the Fréchet space $\mathcal{O}(\Gamma)$. Then $\tilde{A}|X$ is contained in B, and if T is the map which sends $f \in \tilde{A}$ into $f|X \in B$, then T is a continuous map from the Fréchet algebra $\tilde{A}$ into B.*

Proof. Using Theorem 6.8 in [13] applied to the analytic space Γ, it follows that there exists a homeomorphism Φ from Γ onto a Stein space M such that if $H = \{g \in C(M) : g = f \circ \Phi^{-1}$ for some $f \in \tilde{A}\}$, then $H = \mathcal{O}(M)$.

Now $\Phi(X)$ is a compact $\mathcal{O}(M)$-convex subset of M. Since $H_0 = \{g \in C(M) : g = f \circ \Phi^{-1}$ for some $f \in A\}$, is a dense subalgebra of $\mathcal{O}(M)$, we can find an Oka — Weil domain Δ in M such that $\Phi(X) \subset \Delta$, while Δ is defined by functions from H_0. See [6, p. 211]. More precisely there are elements $g_1, \ldots, g_N$ in H_0 such that the map $G : y \to (g_1(y), \ldots, g_N(y))$, is a biholomorphic map from Δ onto a closed analytic subset Ω of the open polydisc D^N in $\mathbf{C}^N$.

If $K = G(\Phi(X))$, then K is the joint spectrum determined by the elements $b_1, \ldots, b_N$, where $b_i = g_i \circ \Phi^{-1}|X$. Then an application of the Symbolic Calculus shows that if $g \in \mathcal{O}(M)$, then $g \circ \Phi^{-1}|X$ determines an element of B. It follows that $\tilde{A}|X \subset B$.

To prove the continuity of T we use a result in [14, p. 161], which asserts that if $\{g_n\}$ is a sequence in $\mathcal{O}(\Omega)$ for which $\lim g_n = 0$ holds uniformly on compact subsets of Ω, then there exists a fixed open neighborhood U of K in $\mathbf{C}^N$ and elements $G_n \in \mathcal{O}(U)$ such that $G_n|K = g_n|K$ while $\lim |G_n|_U = 0$. Then the continuity principle for the Symbolic Calculus implies that if $u_n = g_n \circ \Phi^{-1}|X = G_n \circ \Phi^{-1}$, we have $\lim \|u_n\| = 0$.

The next result follows from the continuity of T in Lemma 3.1.

LEMMA 3.2. *Let us put $\tilde{J} = \{f \in \tilde{A} : f|X \in J\}$. Then $\tilde{J}$ is a closed ideal in the Fréchet space $\tilde{A}$.*

Proof of Theorem 3.1. The set $S = \text{Hull}(J)$ is a compact analytic subset of Y, and hence it is finite. Since Hull $(J_0) \cap (\bar{\Gamma} \setminus X^\circ)$ is empty while $J_0|\Gamma \subset \tilde{J}$, we can easily find elements $g_1, \ldots, g_N$ in $\tilde{J}$ such that the set $Z = \{y \in \Gamma : g_1(y) = \cdots = g_N(y) = 0\}$, is the finite set S.

Next we consider the closed ideal I which $g_1, \ldots, g_N$ generate in $\tilde{A}$. Since $\tilde{A}$ has been identified with $\mathcal{O}(M)$, where M is the Stein space from Lemma 3.1, we can apply standard sheaf theory. For $g_1, \ldots, g_N$ determine a coherent sheaf, called $\tilde{I}$, of $\tilde{A}$-ideals. The sections of this sheaf are the elements of I. See [6, p. 244] and [15].

Since the cohomology groups vanish for all coherent sheaves of $\tilde{A}$-modules, we conclude that the factor space $\tilde{A}/I$ is isomorphic with the sections of the quotient sheaf $R = \tilde{A}/\tilde{I}$. Here R has a non trivial stalk in the finite set S only. At each point $s \in S$ the complex vector space R_s is finite dimensional, because $\{s\}$ is an isolated point in the analytic set determined by $\tilde{I}$.

It follows that I has a finite codimension in $\tilde{A}$, and a fortiori $\tilde{J}$ has a finite codimension in $\tilde{A}$. Then we can write $\tilde{A} = \tilde{J} + Cf_1 + \ldots + Cf_m$, where $f_1, \ldots, f_m \in \tilde{A}$. Let $b_i = f_i|X$ be the corresponding elements in B.

Since J is a closed ideal in B it follows that $J + Cb_1 + \ldots + Cb_m$ is a closed subspace of B. This subspace contains the dense subset A_X, and hence J has a finite codimension in B.

Finally we remark that the work in [16] contains material related with Theorem 3.1.

4. Holomorphic sets in $\mathbf{C}^n$

Let A be a uniform algebra. If W is an open subset of M_A we denote by $\mathcal{O}_A(W)$ the algebra of all functions on W which are locally approximable in W by functions from A. If K is a compact subset of M_A we put $\mathcal{O}_A(K) = \{f \in C(K) : \exists$ an open neighborhood W of K in M_A and some $g \in \mathcal{O}_A(W)$ such that $g|K = f\}$. Finally $H_A(K)$ is the uniform closure of $\mathcal{O}_A(K)$ in $C(K)$. We say that K is a *holomorphic set* if $K = M_{H_A(K)}$.

Since each element of A determines an element in $H_A(K)$ via its restriction to K, we get induced map π from $M_{H_A(K)}$ into M_A satisfying $f(y) = f(\pi(y))$ for all $y \in M_{H_A(K)}$ and all f in A.

The result below is essentially well-known and a proof when K is a compact set in $\mathbf{C}^n$ occurs in [17].

PROPOSITION 4.1. *Let K be a compact set in M_A and put $K_1 = \pi(M_{H_A(K)})$. If now $f \in H_A(K)$ is such that there is some $g \in H_A(K_1)$ satisfying $g|K = f$, then $f(y) = g(\pi(y))$ for all $y \in M_{H_A(K)}$. So in particular g is determined by f.*

Proof. Suppose firstly that $g \in \mathcal{O}_A(K_1)$ and define $\tilde{g}(y) = g(\pi(y))$ for all $y \in M_{H_A(K)}$. The continuity of π shows that $\tilde{g}$ is locally approximable by functions from A on $M_{H_A(K)}$. So if B is the uniform algebra on $M_{H_A(K)}$ generated by $H_A(K)$ and $\tilde{g}$, then $\partial_B = \partial_{H_A(K)} \subset K$ holds. See [10, Corollary 2.4].

When $g_1 = g|K$ is considered as an element of $H_A(K)$ we see that $\tilde{g} = g_1$ on K. Since $\partial_B \subset K$ it follows that $\tilde{g} = g_1$ on $M_{H_A(K)}$. This proves the result when $g \in \mathcal{O}_A(K_1)$.

The general case follows if we let (g_n) be a sequence in $\mathcal{O}_A(K_1)$ such that $\lim |g_n - g|_{K_1} = 0$. For then $f_n = g_n|K$ belong to $H_A(K)$ while $\lim |f_n - f|_K = 0$. This implies that $f(y) = \lim f_n(y) = \lim g_n(\pi(y)) = g(\pi(y))$ for all $y \in M_{H_A(K)}$.

We notice that Proposition 4.1 immediately implies that if $K_1 = K$, then K is a holomorphic set. This was already proved in [2, Theorem 10]. In general K_1 is contained in the unique smallest holomorphic set $B(K)$ which contains K. Here $B(K)$ is called the *barrier of* K. See [2].

Suppose next that K is a holomorphic set in M_A. If T is a closed subset of K we define the algebra $\mathcal{O}_K(T) = \{f \in C(T) : \exists$ some open neighborhood U of T in K and some $g \in \mathcal{O}_{H_A(K)}(U)$ satisfying $g|T = f\}$. Finally $H_K(T)$ is the uniform closure of $\mathcal{O}_K(T)$ in $C(T)$. With these notations we say that T is a *K-holomorphic set* if $M_{H_K(T)} = T$ holds.

The result below is a direct consequence of C. E. Rickart's theory.

PROPOSITION 4.2. *Let T be a compact subset of the holomorphic set K. If now T is a holomorphic set, then T is K-holomorphic.*

We do not know if the converse is true, i.e. if the condition that T is K-holomorphic implies that T is a holomorphic set. But the following result shows that the converse is sometimes true.

PROPOSITION 4.3. *Let K be a compact holomorphic set in $\mathbf{C}^n$ and let W be a relatively open subset of K. Let S be a closed subset of W which is $\mathcal{O}_K(W)$-convex. Then S is a K-holomorphic set and if $g \in H(S)$, then g can be uniformly approximated on S by functions from $\mathcal{O}_K(W)$. If finally $B(S)$ is a compact subset of W, then S is a holomorphic set.*

Proof. The assertion that S is a K-holomorphic set is a consequence of Theorem 12 in [2]. Next we let B be the uniform algebra on S generated by $\mathcal{O}_K(W)|S$. Because S is $\mathcal{O}_K(W)$-convex in W, while W is an open subset of $M_{H(K)} = K$, it follows from the work in [12] that $S = M_B$.

If $z_1, \ldots, z_n$ are the coordinate functions in $\mathbf{C}^n$, then $z_i|S = b_i$ are elements in B. Here S is the joint spectrum of $b_1, \ldots, b_n$. It follows that if $g \in \mathcal{O}(S)$, then $g|S \in B$. This implies that $H(S) \subset B$ which proves the second assertion.

Suppose next that $B(S)$ is a compact subset of W. If $g \in H(K)$ it is obvious that $g|S \in H(B(S))$. It follows that if $f \in \mathcal{O}_K(W)$, then $f|B(S)$ is locally approximable on $B(S)$ by functions from $H(B(S))$. Since $B(S) = M_{H(B(S))}$ it follows from [10, Corollary 2.4] that $|f|_{B(S)} \leq |f|_T$ for all f in $\mathcal{O}_K(W)$, where $T = \partial_{H(B(S))}$. But now Theorem 11 in [2] shows that $T \subset S$. Then the condition that S is $\mathcal{O}_K(W)$-convex implies that $S = B(S)$. Hence S is a holomorphic set.

An example

We give an example of a compact holomorphic set in $\mathbf{C}^n$ which is not a hull, i.e. it cannot be represented as the intersection of open domains of holomorphy. This example gives a negative answer to the question raised in [7, p. 515].

Firstly we discuss compact Reinhardt sets in $\mathbf{C}^2$. If K is a compact Reinhardt set in $\mathbf{C}^2$ it can be represented as a compact subset of $\mathbf{R}^2 \supset \{(|z|, |w|) : (z, w) \in \mathbf{C}^2\}$.

A set of the type $Z(a_1, a_2, b_1) = \{(z, w) \in \mathbf{C}^2 : a_1 \leq |z| \leq a_2$ and $|w| = b_1\}$ is called a Z-segment. In a similar way we get the W-segments $W(a_1, b_1, b_2)$ and finally the squares $Q(a_1, a_2, b_1, b_2) = \{(z, w) \in \mathbf{C}^2 : a_1 \leq |z| \leq a_2$ and $b_1 \leq |w| \leq b_2\}$.

If K is a compact set of the form $W(a_1, b_1, b_2) \cup Z(a_1, a_2, b_1) \cup Z(a_1, a_2, b_2)$, where $a_1 < a_2$ and $b_1 < b_2$, then it is easily verified that $\pi(K) = Q(a_1, a_2, b_2, b_2)$. Using this fact it follows that if $K_0 = W(a_1, 0, b_2) \cup Z(a_1, a_3, 0) \cup Z(a_1, a_2, b_2) \cup Z(a_2, a_3, b)$ for some $0 < b < b_2$, then $\pi(K_0) = Q(a_1, a_2, 0, b_2) \cup Z(a_2, a_3, b) \cup Z(a_2, a_3, 0)$. It follows that $\pi(K_0) = K_1$ is not a holomorphic set, while $\pi(K_1) = Q(a_1, a_2, 0, b_2) \cup Q(a_2, a_3, 0, b)$ is the barrier of K.

The example. Let $K = W(0, 0, 1) \cup Z(0, 1, 0) \cup T$, where $T = \bigcup T_n$ and $T_n = Z(2^{-n-1}, 2^{-n}, 1 - 1/n)$ for all $n \geq 1$. If now V is a domain of holomorphy containing K, then the preceeding discussion easily implies that V contains all points (z, w) for which $2^{-n-1} \leq |z| \leq 2^{-n}$ and $|w| \leq 1 - 1/n$. It follows that K is not a hull.

But using the fact that the closed sets T_n are holomorphic sets and closed components of K, it follows easily that $\pi(K) = K$, and hence K is a holomorphic set.

In the example K is disconnected, and this is necessary because the following assertion can easily be verified.

PROPOSITION 4.4. *Let K be a connected compact Reinhardt set in $\mathbf{C}^n$. Then K is a holomorphic set if and only if K is a hull.*

Finally we show how to construct a connected holomorphic set which is not a hull. Let (z, w, t) be the coordinates in $\mathbf{C}^3$. Put $K_1 = \{(z, w, t) : (z, w) \in K$ and $t = 1\}$, where K is the set from the example above. To each $n \geq 1$ we let J_n be a closed arc in $\mathbf{C}^3$ which starts from the point $p_n = (2^{-n-1}, 1 - 1/n, 1)$ and has $q_n = (2^{-n-1}, 1 - 1/(n+1), 1)$ as an endpoint. As we pass from p_n to q_n the t-variable winds around the unit circle, i.e. we may take $J_n = \{p(x) : p(x) = (2^{-n-1}, 1 + x/n(n+1), e^{2\pi i x})$ as $0 \geq x \leq 1\}$.

Now the set $S = K_1 \cup \bigcup (J_n : n \geq 1)$, is connected. Because K is not a hull we see that S is not a hull. But S is a holomorphic set. For if $n \geq 1$ is given, then we can determine branches of the function $(2\pi i)^{-1} \log(t)$ in such a way that we get an element $f \in \mathscr{O}(S)$ satisfying $f = 1$ on the set $S_n = \{(z, w, t) : (z, w) \in T_n$ and $t = 1\}$, while $f = 0$ on $K_1 \setminus S_n$. In addition $f = 0$ on J_m, when $m \neq n$, and f_n maps J_n onto the closed unit interval. Using the existence of the family $\{f_n\}$ in $H(S)$, it follows easily that $S = M_{H(S)}$.

Finally we refer to [19] for more results about holomorphic sets.

Remark. For a very interesting theory, partly based on holomorphic sets in $\mathbf{C}^n$, we refer to R. Harvey's paper »The theory of hyperfunctions on totally real subsets of a complex manifold with applications to extension problems» in Amer. J. Math. 91 (1969), 853—873.

5. Integral extensions of Banach algebras

Let B be a Banach algebra and let $g \in C(M_B)$. We say that g is integral over B if g satisfies an equation $g^n + b_1 g^{n-1} + \ldots + b_n = 0$ in M_B with $b_i \in B$.

Let g be integral over B and put $A = [B, g]$ = the subalgebra of $C(M_B)$ generated by B and g. It is then possible to introduce a norm on A which makes A into a Banach algebra. We refer to [20, 22] for details of this construction.

When B and A are as above we get the induced map π from M_A into M_B. In general $\partial_A \neq \pi^{-1}(\partial_B)$, and we do not know if $\partial_A \supset \pi^{-1}(\partial_B)$ is always true. The result below was also proved in [22] under a slight additional assumption.

THEOREM 5.1. *Let B and A be as above. If π is an open map, then $\partial_A = \pi^{-1}(\partial_B)$.*

Next we introduce some concepts motivated by integral extensions.

Definition 5.1. Let B and A be two Banach algebras such that B is a closed subalgebra of A. Let π be the induced map from M_A into M_B. We say that π admits a *finite string of local charts,* if there is a decreasing sequence of closed sets $M_A = Z_0 \supset Z_1 \supset \ldots \supset Z_n \supset Z_{n+1} = \emptyset$ such that the following holds: If $p \in W_i = Z_i \setminus Z_{i+1}$, then there is an open neighborhood U of p in M_A such that $\pi|(U \cap W_i)$ is injective. If finally $a \in A$, then B contains a sequence $\{b_n\}$ such that $\lim \tilde{b}_n = \tilde{a}$ holds uniformly on $U \cap W_i$, where $\tilde{a}$ and $\tilde{b}_n$ are the Gelfand transforms on M_A.

In the special case when $A = [B, g]$ for some $g \in C(M_B)$ satisfying $Q(g) = 0$, where $Q(T) = T^n + b_1 T^{n-1} + \ldots + b_n$, then the map π from M_A into M_B admits a finite string of local charts. For let us put $Q_k(T) = \partial^k Q/\partial T^k$ for each $k \geq 1$, and consider the elements $g_i = Q_i(g)$ in A. If we then put $Z_i = \{y \in M_A : \tilde{g}_1(y) = \ldots = \tilde{g}_i(y) = 0\}$, then it is not hard to verify that $M_A = Z_0 \supset Z_1 \supset \ldots \supset Z_n \supset \emptyset$, gives the required string.

Now we begin the study of Shilov boundaries. The following result follows from easy topological considerations.

LEMMA 5.1. *Let $B \subset A$ be a pair of Banach algebras such that π admits a finite string of local charts. Then the fibers $\pi^{-1}(x)$ are totally disconnected for all $x \in M_B$.*

LEMMA 5.2. *Let $B \subset A$ be a pair of Banach algebras. Let $x \in \Sigma_B$ be such that $\pi^{-1}(x)$ is totally disconnected. Then $\pi^{-1}(x) \subset \Sigma_A$.*

Proof. Since $x \in \Sigma_B$ we know that $\{x\}$ is an intersection of peak sets in M_A

determined by functions in the uniform closure B_c of B. It follows that $A_0 = A_c|\pi^{-1}(x)$ is a uniform algebra on $\pi^{-1}(x)$ and hence $M_{A_0} = \pi^{-1}(x)$ holds. Since $\pi^{-1}(x)$ is totally disconnected it follows from a well-known result that $A_0 = C(\pi^{-1}(x))$. This means that $\pi^{-1}(x)$ is a peak interpolation set in $M_{A_c} = M_A$. Then $\pi^{-1}(x) \subset \Sigma_{A_c} = \Sigma_A$ follows.

LEMMA 5.3. *Let $B \subset A$ be such that π admits a finite string of local charts. If π is an open map, then $\partial_A \supset \pi^{-1}(\partial_B)$.*

Proof. Using Lemma 3.1 and 3.2 it follows that $\pi^{-1}(\Sigma_B) \subset \partial_A$. Since ∂_B is the closure of Σ_B in M_B while π is an open and continuous map it follows that $\pi^{-1}(\partial_B) \subset \partial_A$.

Now we obtain a generalization of Theorem 5.1.

THEOREM 5.2. *Let $B \subset A$ be such that π admits a finite string of local charts $M_A = Z_0 \supset Z_1 \supset \ldots \supset Z_n \supset \emptyset$. Suppose also that π is an open map and that Z_{i+1} is an A_{Z_i}-analytic variety for each i. Then $\partial_A = \pi^{-1}(\partial_B)$.*

Proof. Using Lemma 5.3 it is sufficient to prove that $\partial_A \subset \mathrm{Hull}_A(\pi^{-1}(\partial_B)) = S$. So let us assume that $M_A \setminus S = D$ is non empty. Now we prove that $\partial_A \cap D \subset Z_1$ holds. Since D is open while ∂_A is the closure of Σ_A, it is sufficient to prove that $\Sigma_A \cap D \subset Z_1$.

So let $y \in \Sigma_A \cap D$ be given and put $\pi(y) = x$. Then we can choose a closed neighborhood W of y in M_A such that π maps W homeomorphically onto $\pi(W)$ while $\pi(W) \subset M_B \setminus \partial_B$. In addition we may assume that each element in A can be uniformly approximated by functions from B on W. Since π is open we see that $b(\pi(W)) \subset \pi(bW)$. An application of the LMP to B shows that $|b(y)| = |b(x)| \leq |b|_{b(\pi(W))} \leq |b|_{bW}$ for all b in B. Then the approximation condition shows that $y \in \mathrm{Hull}_A(bW)$, and hence y cannot belong to Σ_A.

We have proved that $\partial_A \cap D \subset Z_1$ and hence $D \subset \mathrm{Hull}_A(\partial_A) \subset \mathrm{Hull}_A(S \cup Z_1)$. Using Proposition 1.3.c we deduce that $D \subset Z_1$. Since D is an open subset of M_A we can now use the local charts in $Z_1 \setminus Z_2$, and the same arguments show that $D \subset Z_2$. Inductively we get $D \subset Z_\nu$ for all $\nu \geq 1$, so finally $D \subset Z_{n+1} = \emptyset$. Hence $S = M_A$ which implies that $\partial_A = \pi^{-1}(\partial_B)$.

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