Estimates for the Fourier transform of the characteristic function of a convex set

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1. Introduction

Let C be a measurable set in R^{n+1} and set

$$\hat{u}_C(\xi) = \int_C u(x) e^{i\langle x, \xi\rangle}\, dx\,, \quad \xi \in R^{n+1}\,,\ u \in C_0^\infty(R^{n+1})\,.$$

The order of magnitude of $\hat{u}_C(\xi)$ when $\xi \to \infty$ is frequently of importance in harmonic analysis, for example in application to analytic number theory. However, even if one assumes that C is the closure of an open set with boundary $\partial C \in C^\infty$ the known results are far from complete. It is known then that

$$\hat{u}_C(\xi) = O(|\xi|^{-(n+2)/2})\,, \quad \xi \to \infty\,;\quad u \in C_0^\infty\,; \tag{1.1}$$

if and only if the Gaussian curvature of ∂C never vanishes (Herz [1], Hlawka [2], Littman [3]). Randol [4], [5] has also studied the case where C is convex and ∂C is analytic. His result is that the »maximal function»

$$\tilde{u}(\xi) = \sup_{r>0} r^{(n+2)/2} |\hat{u}_C(r\xi)|\,, \quad \xi \in S \tag{1.2}$$

is then in $L^p(S^n)$ for some $p > 2$ if ∂C is analytic. In fact, Randol proved that this is true for precisely those $p > 2$ such that

$$\int_{\partial C} K(x)^{(2-p)/2}\, dS(x) < \infty \tag{1.3}$$

where $K(x)$ is the Gaussian curvature at $x \in \partial C$. The necessity of (1.3) follows easily from the fact that

$$r^{(n+2)/2} |\hat{u}_C(r\xi)| \to c(|u(x_+)| K(x_+)^{-1/2} + |u(x_-)| K(x_-)^{-1/2})$$

when $r \to \infty$ provided that the Gaussian curvature of ∂C is $\neq 0$ at the points $x_\pm$ where the normal is $\pm \xi$ and that u vanishes at one of these points.

In this paper we shall prove that (1.3) implies that $\tilde{u} \in L^p(S^n)$ for all $u \in C_0^\infty$ provided that C is convex, $\partial C \in C^\infty$ and ∂C has no tangent of infinite order. This of course includes the result of Randol [5]. In fact, our methods allow us to treat also the case when ∂C has only a finite number of derivatives. Moreover, when $n = 1$, we shall give a very precise estimate for $\|\tilde{u}\|_{L^p(S^1)}$ valid for very general convex compact sets C. In that case the proof is a consequence of the Hardy — Littlewood maximal theorem.

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2. Variants of van der Corput's lemma

Let f be a convex increasing function on the interval $[0, 1]$ and let $u \in C_0^\infty(-\infty, 1)$. In this section we shall give some estimates for the integral

$$I(\lambda) = \int_0^1 e^{i\lambda f(r)}\, u(r) r^k dr \tag{2.1}$$

where $k > -1$. They are closely related to the van der Corput lemma (see [6], p. 197), and similar estimates also occur in Randol [5].

Let us split the integral in one from 0 to t and one from t to 1. The first part can be estimated by $\sup |u| t^{k+1}/(k+1)$. In the second we integrate by parts, assuming that $f'(t) > 0$

$$\int_t^1 e^{i\lambda f(r)}\, u(r) r^k dr = [(i\lambda)^{-1} e^{i\lambda f(r)} u(r) r^k / f'(r)]_t^1 - \int_t^1 (i\lambda)^{-1} e^{i\lambda f(r)} d(u(r) r^k / f'(r))\,.$$

We assume now that $k \leq 0$ so that $r^k/f'(r)$ is decreasing. Then the integral can be estimated by $M\lambda^{-1} t^k / f'(t)$ where

$$M = \sup_{[0,1]} |u| + \operatorname{var}_{[0,1]} u\,,$$

$\operatorname{var} u$ denoting the total variation of u. Hence

$$|I(\lambda)| \leq M(t^{k+1}/(k+1) + 3t^k/\lambda f'(t))\,.$$

Now we assume that

$$f'(r) \geq ar\,, \quad 0 < r < 1 \tag{2.2}$$

where $a > 0$. Then we have

$$|I(\lambda)| \leq M(t^{k+1}/(k+1) + 3t^{k-1}/a\lambda)\,.$$

With $t = 1/\sqrt{a\lambda}$ we obtain the bound $4(k+1)^{-1}M(a\lambda)^{-(k+1)/2}$ provided that $a\lambda \geq 1$. The same bound is also valid in the opposite case since $|I(\lambda)| \leq \max |u|$. In the proof we only used that $u \in C^1([0\,,1])$ so we have proved

LEMMA 2.1. *If* (2.2) *is valid* and $-1 < k \leq 0$, *then*

$$|I(\lambda)| \leq 4(k+1)^{-1}(\sup_{[0,1]} |u| + \operatorname*{var}_{[0,1]} u)(a\lambda)^{-(k+1)/2}\,, \quad u \in C^1([0\,,1])\,. \tag{2.3}$$

Remark. A change of variable shows that

$$\left|\int_0^d e^{i\lambda f(r)}\, u(r)dr\right| \leq 4(\sup_{[0,d]} |u| + \operatorname*{var}_{[0,d]} u)(a\lambda)^{-1/2} \tag{2.3$'$}$$

if $u \in C^1([0\,,d])$ and (2.2) is valid for $0 < r < d$. This will be useful in section 5.

We shall now give a similar estimate for larger values of k. To do so we have to integrate by parts several times in (2.1) and shall have to require additional bounds of the form

$$|r^i f^{(i+1)}(r)| \leq C_i f'(r)\,, \quad 0 < r < 1\,, \quad i = 1\,,2\,,\ldots,j\,. \tag{2.4}$$

This condition will be examined in section 3. We shall actually use a condition equivalent to (2.4) namely that if $g(r) = 1/f'(r)$, then

$$|r^i g^{(i)}(r)| \leq C_i' g(r)\,, \quad 0 < r < 1\,, \quad i = 1\,,2\,,\ldots,j\,. \tag{2.5}$$

The equivalence follows inductively if one differentiates the equation $g(r)f'(r) = 1$ using Leibniz' rule.

We shall now split the integral (2.1) as before in an integral from 0 to $t = 1/\sqrt{a\lambda}$ and one from t to 1. For the first part we clearly have the bound (2.3.) In the second part we shall integrate by parts j times if $k+1-2j < 0$. In doing so we note that

$$e^{i\lambda f(r)} = (i\lambda)^{-1} g(r) d(e^{i\lambda f(r)})/dr\,.$$

This gives, if D is the differential operator $d/dr\, g(r) = g(r)d/dr + g'(r)$:

$$\int_t^1 e^{i\lambda f(r)}\, u(r) r^k dr = (i/\lambda)^j \int_t^1 e^{i\lambda f(r)}\, D^j(u(r)r^k)dr - \sum_0^{j-1} (i/\lambda)^{\nu+1}\, e^{i\lambda f(t)}\, g(t) D^\nu(u(t)t^k)\,.$$

With

$$|u|_j = \sum_0^j \max |u^{(\nu)}|$$

it follows from (2.5) that

$$|(d/dr)^\mu D^\nu(u(r)r^k)| \le C|u|_j\, r^{k-\nu-\mu} g(r)^\nu\,, \quad \nu+\mu \le j\,. \tag{2.6}$$

In fact, this is obvious for $\nu = 0$, and if we know (2.6) for a certain value of $\nu < j$ it follows for ν replaced by $\nu + 1$ since

$$(d/dr)^{\mu-1} D^{\nu+1} = (d/dr)^\mu g(r) D^\nu = \sum \tbinom{\mu}{\sigma} g^{(\mu-\sigma)}(r)(d/dr)^\sigma D^\nu\,.$$

Using (2.6) with $\mu = 0$ and (2.2) we now obtain since $k - 2j < -1$

$$\left|\int_t^1 e^{i\lambda f(r)}\, u(r) r^k dr\right| \le C|u|_j \sum_0^{j-1} \lambda^{-\nu-1}\, t^{k-1-2\nu}\, a^{-\nu-1} = C'|u|_j (a\lambda)^{-(k+1)/2}$$

where we have introduced $t = 1/\sqrt{a\lambda}$. Thus we have proved

LEMMA 2.2. *Let f satisfy (2.2). Then we have if $k \ge 0$*

$$|I(\lambda)| \le C_k |u|_j (a\lambda)^{-(k+1)/2} \tag{2.7}$$

if (2.4) is valid for an integer $j > (k+1)/2$.

3. Remarks on the condition (2.4)

If we introduce the non-negative function $u = f''$, we can if $f'(0) = 0$ write (2.4) in the form

$$r^i |u^{(i)}(r)| \le C_{i+1} r^{-1} \int_0^r u(t)dt\,, \quad i = 0, 1, \ldots, j-1\,. \tag{3.1}$$

To study (3.1) we give a variant of the well known estimates between the maxima of the derivatives of a function.

LEMMA 3.1. *If I is an interval $\subset R$ with length $|I|$, then*

$$\max_I |u^{(i)}||I|^i \le C(|I|^{-1} \int_I |u(t)|dt + \max_I |u^{(k)}||I^k|)\,, \quad u \in C^k(I)\,, \tag{3.2}$$

provided that $0 \le i < k$

Proof. We may assume that $I = [0, 1]$. First assume that $i = 0$, $k = 1$. If $0 < \varepsilon < 1$ we have

$$|u(x)| \le \varepsilon \max_I |u'| + \min_{|x-y|<\varepsilon} |u(y)| \le \varepsilon \max_I |u'| + \varepsilon^{-1} \int_I |u(y)|dy\,.$$

Let now $i = 0$ but k be arbitrary. Then it is well known that

$$\max_I |u'| \le C(\max_I |u| + \max_I |u^{(k)}|) \tag{3.3}$$

and if we combine the two inequalities taking ε small enough we obtain (3.2). Having found an estimate for $\max |u|$ we obtain the general statement (3.2) by using the estimates of the form (3.3) which are valid for derivatives of order between 0 and k.

It follows from (3.2) that if (3.1) is valid for one value of i with $u^{(i)}(r)$ replaced by $\sup \{|u^{(i)}(t)|;\ 0 \leq t \leq r\}$ it is also fulfilled for any smaller value. Indeed, we need only apply Lemma 3.1 with $I = [0\,,r]$. So if $u^{(j-1)}$ is bounded, then a sufficient condition for (3.1) is of course that

$$\int_0^r u(t)dt \geq cr^j \tag{3.4}$$

for some $c > 0$.

If (3.4) is valid and u is in a bounded set in C^{j-1} we also obtain from (3.2) that

$$|u(0)| \leq Cr^{-1} \int_0^r u(t)dt \tag{3.5}$$

We have proved:

LEMMA 3.2. *Let M be a bounded set of convex functions in C^{j+1} such that $f'(0) = 0$ when $f \in M$ and for some constant $c > 0$*

$$\int_0^r f''(t)dt \geq cr^j\,, \quad 0 < r < 1\,, \quad f \in M\,. \tag{3.6}$$

Then we have (2.2) with $a = bf''(0)$, where b is independent of $f \in M$; in addition (2.4) is uniformly valid for $f \in M$.

To apply the preceding lemma we need the following one:

LEMMA 3.3. *Let $u_0 \in C^k(I)$ where I is a compact interval in R and assume that all derivatives of order $\leq k$ of u_0 never vanish simultaneously in I. Then there is a neighbourhood Ω of u_0 in $C^k(I)$ and an integer N such that for every $u \in \Omega$ and $\varepsilon > 0$ there exist at most N subintervals of length $\leq \varepsilon$ containing $\{x\,; x \in I, |u(x)| < \varepsilon^k\}$.*

Proof. There is nothing to prove when $k = 0$, so we assume that $k > 0$ and that the statement is proved for smaller values of k. The hypothesis implies that u_0 has only finitely many zeros. We can therefore find a finite decomposition $I = \bigcup I_\varepsilon$ in closed intervals such that in each I_ε either $u_0 \neq 0$ or else $\sum_1^k |u_0^{(\nu)}| \neq 0$. In the first case there is a fixed lower bound for $|u|$ in I_ε for all u in a neighbourhood of u_0, and in the second case the hypotheses of Lemma 3.3 with k replaced by $k-1$ are fulfilled in I_ε by u' for all u in a neighbourhood of u_0.

By the induction hypothesis we then have $|u'| > \varepsilon^{k-1}$ in I_ε outside N intervals of length $\leq \varepsilon$, which implies that $|u| > \varepsilon^k$ in I_ε outside these N intervals and $2N + 2$ additional ones of length at most ε. This completes the proof.

We note two important consequences: If M is a compact subset of $C^k(I)$ and if the hypotheses of Lemma 3.3 are fulfilled for all $u \in M$ we have for some positive constants c, C

$$\left| \int_x^y |u(t)|dt \right| \geq c|x-y|^{k+1} \text{ if } x, y \in I, \quad u \in M \tag{3.7}$$

$$\int_I |u(t)|^{-\delta} dt \leq C(1-\delta k)^{-1} \text{ if } 0 < \delta < 1/k, \quad u \in M. \tag{3.8}$$

In fact, by the Borel—Lebesgue lemma $M = \bigcup M_i$ where the union is finite and the conclusion of Lemma 3.3 is valid for each M_i and so for M. The estimate (3.7) follows if we choose ε in Lemma 3.3 so that $N\varepsilon = |x-y|/2$, for then the integral is at least $\varepsilon^k|x-y|/2$. The proof of (3.8) is obvious.

4. Estimates for the maximal function

We can now prove the extension of a result of Randol [5] referred to in the introduction. A surface is said to be flat of order at most j if the distance to the surface from a tangent has a zero of order $j + 2$.

THEOREM 4.1. *Let C be a convex set in R^{n+1} with boundary ∂C flat of order at most j where $j \geq \mu$ with μ the smallest integer $> (n+1)/2$. Then $\tilde{u} \in L^p(S^n)$ holds for all $u \in C^\mu(R^{n+1})$ if (1.3) holds and $\partial C \in C^{j1}$. These assumptions are fulfilled if $\partial C \in C^{h+2}$ and $2 < p < 2 + 2/h$ where $h = n(j-1)$.*

COROLLARY 4.2. *If $\partial C \in C^\infty$ and ∂C has no tangent of infinite order there is a j such that the hypotheses of Theorem 4.1 are valid and so $\tilde{u} \in L^p(S^n)$ for all $u \in C^\mu(R^{n+1})$.*

Proof. By the divergence theorem we have

$$\hat{u}(r\xi) = \int_C u(x) e^{ir<x,\xi>} dx =$$

$$i/r \int_{\partial C} u(x) <\xi, \nu(x)> e^{ir<x,\xi>} dS(x) + i/r \sum_{k=1}^n \int_C \xi_k \, \partial u(x)/\partial x_k \, e^{ir<x,\xi>} dx$$

Here ν is the interior normal and $|\xi| = 1$.

If we repeat this procedure μ times, we get

$$\hat{u}(r\xi) = \sum_{1}^{\mu} r^{-\nu} \int_{\partial C} w_\nu(x, \xi) e^{ir<x,\xi>} dS(x) + r^{-\mu} \int_{C} w_{\mu+1}(x, \xi) e^{ir<x,\xi>} dx$$

where $w_\nu(. , \xi)$ is in bounded set in $C^{\mu+1-\nu}(R^{n+1})$, $1 \leq \nu \leq \mu + 1$. We want to estimate $r^{(n+2)/2}|\hat{u}(r\xi)|$. Since $\mu > (n+1)/2$ the estimates of the last term in the sum and that with integral over C are obvious so it is sufficient to prove that for $1 \leq \nu \leq \mu - 1$

$$\sup_r r^{\frac{n+2}{2} - \nu} \left| \int_{\partial C} v(x, \xi) e^{ir<x,\xi>} dS(x) \right| \in L^p(S^n)$$

if $v(. , \xi)$ belongs to a bounded set in $C^{\mu+1-\nu}(R^{n+1})$.

Choose $\psi \in C_0^\infty(R)$ such that $\psi(t) = 1$, $|t| < \delta$ and $\psi(t) = 0$, $|t| > \delta$. Here δ will be chosen below. Denote by $X(\xi)$ the point on ∂C with interior normal ξ, and decompose v as a sum $v = \varphi_1 + \varphi_2 + \varphi_3$ where $\varphi_1(x, \xi) = v(x, \xi)\psi(\langle X(\xi) - x, \xi \rangle)$ and $\varphi_2(x, \xi) = v(x, \xi)\psi(\langle X(-\xi) - x, \xi \rangle)$. If (ϱ, ω) are polar coordinates in the tangent plane at $X(\xi)$, let $f(\varrho, \omega, \xi)$ describe the intersection of ∂C and the plane through ξ containing ω:

$$f(\varrho, \omega, \xi) = \inf\{t ; X(\xi) + \varrho\omega + t\xi \in C\}.$$

If δ_0 is small enough and $I = \{\varrho ; 0 \leq \varrho \leq \delta_0\}$ then $f(. , \omega, \xi) \in C^{j+1}(I)$ for all $\xi \in S^n$ and all tangent directions ω at $X(\xi)$.

Now we split the integral in three parts. If 2δ is smaller than the width of C, the integral involving φ_3 is $O(r^{-(\mu+1-\nu)})$ as $r \to \infty$, uniformly in ξ, for there is a lower bound independent of ξ, for the difference between ξ and a normal to ∂C in $\operatorname{supp} \varphi_3$, (cf [3]).

Now it is of course enough to examine

$$\left| \int_{\partial C} \varphi_1(x, \xi) e^{ir<x,\xi>} dS(x) \right|$$

In terms of the polar coordinate system in the tangent plane at $X(\xi)$ this integral becomes

$$\left| \int_{S^{n-1}} d\omega \int_0^\infty \varphi(\varrho, \omega, \xi) e^{irf(\varrho,\omega,\xi)} \varrho^{n-1} d\varrho \right| .$$

Here $\varphi(. , \omega, \xi)$ is in a bounded set in $C^{\mu+1-\nu}(I)$ and vanishes near the right hand end point.

Let us consider the map

$$(\omega, \xi) \to f(\cdot, \omega, \xi)$$

from the unit sphere bundle of the tangent space of S^n to $C^{j+1}(I)$. Since the domain is compact and the map is continuous the image set in $C^{j+1}(I)$ is compact. By hypothesis all derivatives of $f''_{\varrho\varrho}(\cdot\,,\omega\,,\xi)$ of order $\leq j-1$ do not vanish simultaneously so we can apply the lemmas in section 3. By (3.7) follows then

$$\int_0^r f''_{\varrho\varrho}(\varrho\,,\omega\,,\xi)d\varrho \geq cr^j$$

so by Lemma 3.2

$$f'_\varrho(\varrho\,,\omega\,,\xi) \geq bf''_{\varrho\varrho}(0\,,\omega\,,\xi)\cdot\varrho$$

and (2.4) is uniformly valid for $f(\cdot\,,\omega\,,\xi)$. We can now apply Lemma 2.2 and get

$$\left|\int_0^\infty \varrho^{n-1}\varphi(\varrho\,,\omega\,,\xi)e^{irf(\varrho,\omega,\xi)}d\varrho\right| \leq C_{n+1-2\nu}|\Phi|_{\mu+1-\nu}\,(rbf''_{\varrho\varrho}\,(0\,,\omega\,,\xi))^{-\frac{n+2}{2}+\nu}\,,$$

$$\Phi(\varrho) = \varphi(\varrho\,,\omega\,,\xi)\varrho^{2(\nu-1)}\,.$$

Next we prove that for $1\leq\nu\leq\mu-1$

$$\int_{S^{n-1}} f''_{\varrho\varrho}(0,\omega\,,\xi)^{-n/2+\nu-1}\,d\omega \leq CK(X(\xi))^{-1/2}$$

where $K(x)$, $x\in\partial C$, denotes the Gaussian curvature at x. Of course it is enough to take $\nu=1$ and then we shall prove equality with C equal to the volume of S^{n-1}.

Now

$$f''_{\varrho\varrho}(0\,,\omega\,,\xi)^{-n/2} = (A\omega\,,\omega)^{-n/2} = F(\omega)$$

where A is the curvature matrix of f at $\varrho=0$. The integral $\int_{S^{n-1}} F(\omega)d\omega$ is equal to the integral of the differential form

$$\sum_{i=1}^n(-1)^{i-1}F(\omega)\omega_i d\omega_1\wedge\ldots\wedge\widehat{d\omega_i}\wedge\ldots\wedge d\omega_n$$

over the unit sphere or any cycle in $R^n\setminus\{0\}$ homotopic to S^{n-1}, for the exterior derivative

$$[\sum_{i=1}^n(\omega_i\partial F(\omega)/\partial\omega_i+nF(\omega)]d\omega_1\wedge\ldots\wedge d\omega_n$$

is zero by Euler's theorem on homogeneous functions.

Thus we may integrate over an ellipsoid with axes $\omega^i f''_{\varrho\varrho}(0\,,\omega_i\,,\xi)^{-1/2}$ $i=1\,,2\,,\ldots,n$, where $\omega^1\,,\ldots,\omega^n$ are the directions of principal curvature at $X(\xi)$. The integral thus reduces to $C(K(X(\xi)))^{-1/2}$ where C is the volume of S^{n-1}.

Summing up, we have proved that

$$\tilde{u}(\xi) \leq C'(K(X(\xi))^{-1/2} + K(X(-\xi))^{-1/2} + 1)$$

The proof of the first part of the theorem is now complete since

$$\int K(X(\xi))^{-p/2}\, d\omega(\xi) = \int K(x)^{(2-p)/2}\, dS(x)\,.$$

To prove the second statement we want to estimate $\int K(x)^{-\delta} dS(x)$ over a neighbourhood of a point x_0 on ∂C.

As before we describe ∂C near $x_0 = X(\xi_0)$ by a set of functions $f \in M \subset C^{j+1}(I)$, where M is compact. We have $f(0, \omega, \xi_0) = f'(0, \omega, \xi_0) = 0$ and

$$f(\varrho, \omega, \xi_0) \geq C'' \varrho^{j+1} \quad \text{for some} \quad C'' > 0\,. \tag{4.1}$$

To prove (4.1) we note that Lemma 3.3 implies

$$m\{\varrho\,;\, \varrho \in I\,,\ f(\varrho, \omega, \xi_0) < t^{j+1}\} \leq Nt\,.$$

(4.1) follows if we take t so that $Nt = \varrho$, for f is an increasing function of ϱ.

We may assume that the coordinates are chosen so that $x_0 = 0$ and $\xi_0 = (1, 0, \ldots, 0)$. Write $x'' = (x_2, \ldots, x_{n+1})$. If $f(\varrho, \omega, \xi_0) \geq \varepsilon$ we have $\varrho \leq (\varepsilon/C'')^{1/(j+1)} = \gamma$ by (4.1) which implies that $(f, \varrho\omega) \in \Gamma$, where

$$\Gamma = \{x\,;\, x_1 \geq \varepsilon/\gamma |x''|\}\,.$$

If $\xi \in S^n$ and $X(\xi) \in \Gamma$ we have $\langle X(\xi), \xi\rangle < 0$ in view of the convexity of C so $\xi \notin \Gamma^*$ where

$$\Gamma^* = \{y\,;\, \langle x, y\rangle \geq 0\ \forall\, x \in \Gamma\} = \{y\,;\, |y''| \leq \varepsilon/\gamma y_1\}$$

Thus $\xi \in \Gamma^* \cap S^n$ implies $X(\xi) \notin \Gamma$ so $x_1(\xi) < \varepsilon$ and $|x''(\xi)| < \gamma$,

$$\int_{|x''|<\gamma} K(x) dS(x) \geq \int_{\Gamma^* \cap S^n} d\xi \geq C^{(3)} (\varepsilon/\gamma)^n = C^{(4)} \gamma^{nj} \tag{4.2}$$

From (4.2) it follows if $K \in C^h$, $h = n(j-1)$, that x_0 cannot be a zero of K of order $> h$. In this conclusion x_0 may of course be any point on ∂C.

Regarding K in a neighbourhood of x_0 in ∂C as a function of x'' in a neighbourhood of 0 in R^n we may assume that K, $\partial K/\partial x_2, \ldots, \partial^h K/\partial x_2^h$ do not vanish simultaneously. For a suitable $\sigma > 0$ it follows by (3.8) that

$$\int_{|x_2|<\sigma} K(x'')^{-\delta}\, dx_2 < C \quad \text{if} \quad \delta h < 1\,, \quad |x''| < \sigma\,.$$

This implies that $\int K(x)^{-\delta} dS(x)$ is finite over a neighbourhood of x_0. The proof of the theorem is complete.

To prove the corollary we only have to observe that if $f''_{\varrho\varrho}(\varrho, \omega, \xi)$ or some higher order derivative is different from zero at (ϱ, ω, ξ) then the same is true in a neighbourhood of (ϱ, ω, ξ). By the Borel—Lebesgue lemma this shows that the hypotheses of Theorem 4.1 are fulfilled for some j.

5. The case $n = 1$

Using Lemma 2.1 and the Hardy—Littlewood maximal theorem (see [6], p. 32) we shall give a very precise result in this case.

THEOREM 5.1. *Let C be any bounded strictly convex set such that the arc length s on the boundary is an absolutely continuous function of θ, where θ is the angle between the supporting line and some fixed direction.*

Then there is a constant M such that

$$\|\tilde{u}\|_{L^p(S^1)} \leq M(p/p-2)^{1/2} \left(\int_{\partial C} (ds/d\theta)^{p/2}\, d\theta \right)^{1/2} N(u) \tag{5.1}$$

where $N(u) = \sum_{|\alpha| \leq 2} l^{|\alpha|} \sup_{x \in C} |u^{(\alpha)}(x)|$ with l denoting the arc length of ∂C.

Proof. By the divergence theorem we have

$$\int u(x) e^{ir<x,\xi>}\, dx = i/r \int_{\partial C} \langle G(x, \xi), \nu(x) \rangle e^{ir<x,\xi>}\, ds(x)$$

if ν is the interior normal and

$$\begin{cases} \partial G_1(x, \xi)/\partial x_1 + \partial G_2(x, \xi)/\partial x_2 = 0 \\ \xi_1 G_1(x, \xi) + \xi_2 G_2(x, \xi) = u(x) . \end{cases}$$

We set $\langle G(x, \xi), \nu(x) \rangle = v(x, \xi)$ and study

$$\sqrt{r} \int_{\partial C} v(x, \xi) e^{ir<x,\xi>}\, ds(x) =$$

$$\sqrt{r} \int_{\gamma_1} v(x, \xi) e^{ir<x,\xi>}\, ds(x) + \sqrt{r} \int_{\gamma_2} v(x, \xi) e^{ir<x,\xi>}\, ds(x)$$

where γ_1 and γ_2 are the two arcs of ∂C separated by the points where a supporting line is parallel to ξ.

We study one of the integrals (the other is quite similar) and assume that $\xi = (0, 1)$. If we take the arc length s defined as 0 for $x_1 = 0$ we have by Lemma 2.1

$$\sqrt{r}\left|\int_{\gamma_1} v(x, \xi) e^{i<x,\xi>r} ds(x)\right| = \sqrt{r}\left|\int v(x(s), \xi) e^{ix_2(s)r} ds\right| \leq$$

$$8 \sup_s |s/x_2'(s)^{1/2}| (\operatorname{var}_{\partial C} v + \sup_{\partial C} |v|)$$

In fact if θ is the angle between the supporting line at the point with arc length s and the x_1-axis we have $dx_2/ds = \sin\theta$ which is an increasing function of s so x_2 is a convex function. Since $\theta/\sin\theta \leq \pi/2$ when $|\theta| \leq \pi/2$ we obtain

$$\sup_s |s/x_2'(s)|^{1/2} = \sup_\theta |s(\theta)/\sin\theta|^{1/2} \leq (\pi/2)^{1/2} \sup_\theta |s(\theta)/\theta|^{1/2} \leq (\pi/2)^{1/2} S(0)^{1/2}$$

where S denotes the Hardy—Littlewood maximal function of $ds/d\theta$.

We shall now estimate $\operatorname{var}_{\partial C} v + \sup_{\partial C} |v|$. We have

$$\sup_{\partial C} |v| \leq \sup_{\partial C} |G|$$

and

$$\operatorname{var}_{\partial C} v = \int |d\langle G, \nu\rangle| \leq \int |\langle dG, \nu\rangle| + \int |\langle G, d\nu\rangle| \leq$$

$$\leq \int |dG||\nu| + \int |G||d\nu| \leq \int |dG| + 2\pi \sup_{\partial C} |G| \leq$$

$$\leq \operatorname{var}_{\partial C} G_1 + \operatorname{var}_{\partial C} G_2 + 2\pi \sup_{\partial C} |G|.$$

Since $\xi = (0, 1)$ we can take

$$G_2(x, \xi) = u(x), \quad G_1(x, \xi) = -\int_0^{x_1} \partial u(t, x_2)/\partial x_2 dt,$$

and thus we have

$$\operatorname{var} G_j \leq l(l \sum_{|\alpha|=2} \sup_{x\in C} |u^\alpha| + \sum_{|\alpha|=1} \sup |u^\alpha|), \quad j = 1, 2.$$

Thus we have proved for $\theta = 0$

$$\tilde{u}(\theta) \leq (S(\theta)^{1/2} + S(-\theta)^{1/2}) N(u) M_1$$

if we have taken the angle θ as a parameter on S^1 so that $\theta = 0$ corresponds to $\xi = (0, 1)$. Since the estimate is invariant under a congruence transformation it is valid in general. By the Hardy—Littlewood maximal theorem we have if $q > 1$

$$\int_0^{2\pi} S(\theta)^q d\theta \leq 2(q/(q-1))^q \int_0^{2\pi} (ds(\theta)/d\theta)^q d\theta$$

so if $p > 2$ we obtain

$$\|u\|_{L^p(S^1)} \leq MN(u)(p/p-2)^{1/2} \left(\int_0^{2\pi} (ds(\theta)/d\theta)^{p/2}\, d\theta\right)^{1/2}$$

and (5.1) is proved.

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