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Mukai–Sakai bound for equivariant principal bundles

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Abstract. Mukai and Sakai proved that given a vector bundle E of rank n on a connected smooth projective curve of genus g and any $r\in[1,n]$, there is subbundle S of rank r such that $\deg \operatorname{Hom}(S, E/S)\le r(n-r)g$. We prove a generalization of this bound for equivariant principal bundles. Our proof even simplifies the one given by Holla and Narasimhan for usual principal bundles.

1. Introduction

Let Y be a connected smooth projective curve of genus g_Y over an algebraically closed field k and E a vector bundle over Y of rank n. Fixing an integer $r\in[1,n]$, consider the space of all subbundles of E of rank r. It is easy to see that their degrees are bounded above. In [MS], Mukai and Sakai produced a lower bound for the maximum of these degrees. The main result of [MS] says that E has a subbundle S of rank r such that $\deg \operatorname{Hom}(S, E/S)\le r(n-r)g_Y$.

In [HN], Holla and Narasimhan extended this result to principal bundles. Let G be a connected reductive linear algebraic group over k and E_G a principal G-bundle over Y. Fix a reduced parabolic subgroup $P\subset G$ and consider the space of all reductions of E_G to P. There is a constant $c\in\mathbf{Z}$ such that for any reduction $E_P\subset E_G$, we have $\deg \operatorname{ad}(E_P)\le c$. In [HN], a lower bound for such a constant c is obtained. The main result of [HN] says that there is a reduction

$$\sigma\colon Y\longrightarrow E_G/P$$

such that $\deg\sigma^*T_{\mathrm{rel}}\le g_Y\dim G/P$, where T_{rel} is the relative tangent bundle for the projection of E_G/P to X. Since $\sigma^*T_{\mathrm{rel}}\cong\operatorname{ad}(E_G)/\operatorname{ad}(E_P)$, this implies that $c\ge -g_Y\dim G/P$.

We prove a generalization of the above bound on $\deg\sigma^*T_{\mathrm{rel}}$ for equivariant bundles (see Theorem 3.2). We use a certain Quot-scheme of $\operatorname{ad}(E_G)$, as opposed to the use of Hilbert schemes done in [HN]. This also yields a simpler proof of the bound obtained in [HN].

2. Equivariant reduction of a principal bundle

Let k be an algebraically closed field. Let Y be a connected smooth projective curve over k, and

$$\Gamma \subset \mathrm{Aut}(Y)$$

a finite reduced subgroup of the automorphism group of Y. So Γ acts on the left of Y.

Let G be a connected reductive linear algebraic group over k and E_G a principal G-bundle over Y. A *Γ-linearization* of E_G is a lift of the action of Γ on Y to the total space of E_G that commutes with the action of G. So a Γ-linearization of E_G is a left action of Γ on E_G such that for any $\gamma \in \Gamma$, the automorphism of the variety E_G defined by it is an isomorphism of the G-bundle over the automorphism γ of Y.

The Lie algebra of G will be denoted by $\mathfrak{g}$. Fix a reduced parabolic subgroup P of G. Let $\mathfrak{p} \subset \mathfrak{g}$ be the Lie algebra of P.

Let E_G be a Γ-linearized G-bundle. Its adjoint bundle $E_G \times^G \mathfrak{g}$ will be denoted by $\mathrm{ad}(E_G)$. Since the adjoint action of G preserves the Lie algebra structure of $\mathfrak{g}$, each fiber of $\mathrm{ad}(E_G)$ has the structure of a Lie algebra isomorphic to $\mathfrak{g}$.

Since G/P is complete and $\dim Y = 1$, the G-bundle E_G admits a reduction of structure group to P. If $E_P \subset E_G$ is a reduction of structure group of E_G to E_P, then $\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G)$, and the quotient $\mathrm{ad}(E_G)/\mathrm{ad}(E_P)$ is identified with the pullback of the relative tangent bundle on E_G/P by the section

$$\sigma : Y \longrightarrow E_G/P$$

(of the fiber bundle E_G/P over Y) corresponding to the reduction. Therefore, there is a constant $N(E_G)$ such that

$$\deg \sigma^* T_{\mathrm{rel}} \geq N(E_G),$$

where T_{rel} is the relative tangent bundle over E_G/P for the projection to Y.

A reduction $E_P \subset E_G$ is called *Γ-invariant* if the subvariety E_P is left invariant by the action of Γ on E_G. Let

$$c(E_G) \in \mathbf{Z} \tag{2.1}$$

be the minimum of $\deg \sigma^* T_{\mathrm{rel}}$ taken over all possible Γ-invariant reductions of E_G to P.

Let $\mathcal{Q}(\mathrm{ad}(E_G))$ be the Quot-scheme parametrizing all quotients of $\mathrm{ad}(E_G)$ of rank $\dim G/P$ and degree $c(E_G)$. (See [G] for properties of $\mathcal{Q}(\mathrm{ad}(E_G))$.)

Take a quotient $Q \in \mathcal{Q}(\mathrm{ad}(E_G))$. Let

$$0 \longrightarrow S \longrightarrow \mathrm{ad}(E_G) \longrightarrow Q \longrightarrow 0 \tag{2.2}$$

be the exact sequence defined by it. The subsheaf S is a subbundle of $\mathrm{ad}(E_G)$ over a nonempty open subset of Y.

The action of Γ on E_G defining the Γ-linearization induces an action of Γ on the vector bundle $\mathrm{ad}(E_G)$. Let

$$\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G)) \subset \mathcal{Q}(\mathrm{ad}(E_G))$$

be the closed subscheme consisting of all quotients Q such that the corresponding subsheaf S (as in (2.2)) is left invariant by the action of Γ. Note that if S is invariant under the action Γ, then there is an induced action of Γ on Q defined by the condition that the projection in (2.2) is Γ-equivariant.

The action of Γ on $\mathrm{ad}(E_G)$ induces an action of Γ on the scheme $\mathcal{Q}(\mathrm{ad}(E_G))$. The action of any $\gamma \in \Gamma$ sends a quotient $Q = \mathrm{ad}(E_G)/S$ to $\mathrm{ad}(E_G)/\gamma(S)$. Clearly, $\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$ coincides with $\mathcal{Q}(\mathrm{ad}(E_G))^{\Gamma}$.

The vector bundle $\mathrm{ad}(E_G)$ is associated with E_G for the adjoint action of G on $\mathfrak{g}$. So any closed point of the fiber $(E_G)_y$, $y \in Y$, gives a Lie algebra isomorphism of the fiber $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$. More precisely, the isomorphism defined by y sends any $w \in \mathfrak{g}$ to the equivalence class defined by (y, w) (recall that $\mathrm{ad}(E_G)_y$ is a quotient of $(E_G)_y \times \mathfrak{g}$). All such isomorphisms of $\mathrm{ad}(E_G)_y$, $y \in Y$, with $\mathfrak{g}$ (defined by $(E_G)_y$) will be called *distinguished isomorphisms.*

Therefore, any two distinguished isomorphisms of $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$ differ by an inner automorphism of $\mathfrak{g}$ (defined by some element in G).

Take any quotient $Q \in \mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$. Let $S \subset \mathrm{ad}(E_G)$ be the subsheaf defined as in (2.2). Let $U \subset Y$ be the nonempty open subset over which S is a subbundle of $\mathrm{ad}(E_G)$.

Lemma 2.1. *If there is a nonempty open subset $U' \subset U$ such that for any point $y \in U'$, there exists a distinguished isomorphism of $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$ that takes S_y isomorphically to $\mathfrak{p}$, then S is a subbundle of $\mathrm{ad}(E_G)$, that is, $U = Y$.*

Proof. Let $S' \subset \mathrm{ad}(E_G)$ be the (unique) subbundle of rank $\dim P$ that contains S. So S' is the inverse image of $\mathrm{Torsion}(Q)$ for the projection in (2.2). For any $y \in Y$, the fiber S'_y is a subalgebra of $\mathrm{ad}(E_G)$ identified with $\mathfrak{p}$ by (the restriction of) some distinguished isomorphism. Indeed, this follows from the fact that the subvariety of the Grassmannian $\mathrm{Gr}(\dim P, \mathfrak{g})$ (parametrizing all $\dim P$-dimensional subspaces of $\mathfrak{g}$), defined by all the conjugations of $\mathfrak{p}\subset\mathfrak{g}$, is the image of G/P in $\mathrm{Gr}(\dim P, \mathfrak{g})$. In particular, it is a complete subvariety. Therefore, if the fiber of a subbundle of $\mathrm{ad}(E_G)$ over the general point is identified with $\mathfrak{p}$ by some distinguished isomorphism, then the fiber over the subbundle over each point of Y has this property.

Since S is left invariant by the action of Γ on $\mathrm{ad}(E_G)$, it follows immediately that S' is invariant under the action of Γ.

Consequently, S' gives a (reduced) Γ-invariant sub-group-scheme

$$\widetilde{S}' \subset \mathrm{Ad}(E_G) := E_G \times^G G$$

of the gauge bundle defined by the condition that for any point $y\in Y$, the Lie algebra of $\widetilde{S}'_y$ coincides with S'. Now, $\widetilde{S}'$ defines a reduction of structure group

$$E_P \subset E_G$$

to the parabolic subgroup P. For any point $y\in Y$, the subvariety $(E_P)_y\subset(E_G)_y$ consists of all $z\in(E_G)_y$ such that the natural projection of $E_G\times G$ to $\mathrm{Ad}(E_G)$ sends $z\times P$ into $\widetilde{S}'_y$. That this defines a reduction of structure group of E_G to P is an immediate consequence of the fact that the normalizer of P in G coincides with P. This reduction E_P is Γ-invariant, since S' is left invariant under the action Γ.

From the definition of $c(E_G)$ in (2.1) it follows that

$$\deg(\mathrm{ad}(E_G)/S') = \deg \sigma^* T_{\mathrm{ref}} \geq c(E_G),$$

where σ is the section $Y\to E_G/P$ defining the reduction E_P. Therefore, as

$$\deg(\mathrm{ad}(E_G)/S') = \deg(\mathrm{ad}(E_G)/S) - \dim \mathrm{Torsion}(Q) = c(E_G) - \dim \mathrm{Torsion}(Q),$$

we have $\mathrm{Torsion}(Q)=0$. This implies that $S'=S$, and the proof of the lemma is complete. □

Using the above lemma we will construct a closed subscheme of $\mathcal{Q}^\Gamma(\mathrm{ad}(E_G))$.

3. The subscheme $\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$

Let $\mathrm{Gr}_Y(\dim P, \mathrm{ad}(E_G))$ be the Grassmann bundle over Y parametrizing all $\dim P$-dimensional subspaces in the fibers of $\mathrm{ad}(E_G)$. The space of all conjugates of the Lie subalgebra $\mathfrak{p}$ in $\mathrm{ad}(E_G)$ define a subbundle

$$\mathrm{Gr}_Y^{\mathfrak{p}} \subset \mathrm{Gr}_Y(\dim P, \mathrm{ad}(E_G)) \tag{3.1}$$

of the fiber bundle. In other words, for any $y \in Y$, a subspace $V \subset \mathrm{ad}(E_G)_y$ is in $\mathrm{Gr}_Y^{\mathfrak{p}}$ if and only if there is a distinguished isomorphism of $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$ that takes $\mathfrak{p}$ isomorphically to V. Therefore, any fiber of the fiber bundle $\mathrm{Gr}_Y^{\mathfrak{p}}$ is isomorphic to G/P. In fact, there is a canonical isomorphism

$$\mathrm{Gr}_Y^{\mathfrak{p}} \cong E_G/P. \tag{3.2}$$

Indeed, for any $y \in Y$ and any $z \in (E_G)_y$ (recall that $(E_G)_y$ parametrizes the distinguished isomorphisms of $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$), the image of $\mathfrak{p}$ in $\mathrm{ad}(E_G)$ by the distinguished isomorphism corresponding to z is a point in the fiber of $\mathrm{Gr}_Y^{\mathfrak{p}}$ over y. Since this image subalgebra does not change as y moves over a P-orbit (for the action of P on $(E_G)_y$), we get a natural isomorphism of the fiber bundle $\mathrm{Gr}_Y^{\mathfrak{p}}$ over X with E_G/P.

Note that the action of Γ on E_G induces an action of Γ on $\mathrm{Gr}_Y^{\mathfrak{p}}$.

Let

$$\mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G)) \subset \mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G)) \tag{3.3}$$

be the subscheme defined by the Γ-invariant sections of $\mathrm{Gr}_Y^{\mathfrak{p}}$ (defined in (3.1)). So a point of $\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$, representing a quotient Q of $\mathrm{ad}(E_G)$, lies in $\mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G))$ if and only if the corresponding subsheaf S (as in (2.2)) has the property that S is a subbundle of $\mathrm{ad}(E_G)$ and for each point $y \in Y$, there is a distinguished isomorphism of $\mathrm{ad}(E_G)_y$ with $\mathfrak{g}$ that takes the fiber S_y isomorphically to $\mathfrak{p}$.

Using Lemma 2.1 it can be shown that $\mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G))$ is in fact a closed subscheme of $\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$. Indeed, if we consider a morphism

$$f: C \setminus \{p\} \longrightarrow \mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G)),$$

where p is a point on a smooth curve C, then using the completeness of $\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$ it extends to a morphism $\bar{f}: C \to \mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$. Let $U \subset Y$ be the nonempty open subset over which the quotient $\bar{f}(c)$ of $\mathrm{ad}(E_G)$ is locally free. Let

$$\hat{f}: U \times C \longrightarrow \mathrm{Gr}_Y(\dim P, \mathrm{ad}(E_G))$$

be the map defined by $\bar{f}$. So, $\hat{f}(u,c)$ represents the subspace of $\mathrm{ad}(E_G)_u$ defined by $\bar{f}(c)$. Therefore, the map $\hat{f}$ has the property that $\hat{f}(U \times (C \setminus \{p\})) \subset \mathrm{Gr}_Y^{\mathfrak{p}}$. Since $\mathrm{Gr}_Y^{\mathfrak{p}}$ is a complete variety, we conclude that $\hat{f}(U \times C) \subset \mathrm{Gr}_Y^{\mathfrak{p}}$. In particular, we have $U \times \{p\} \subset \mathrm{Gr}_Y^{\mathfrak{p}}$. Now Lemma 2.1 implies that $\bar{f}(p) \in \mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G))$. Therefore, $\mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G))$ is closed in $\mathcal{Q}^{\Gamma}(\mathrm{ad}(E_G))$.

Take any quotient Q in $\mathcal{Q}_P^{\Gamma}(\mathrm{ad}(E_G))$. The action of Γ on Q induces an action on $H^i(Y,Q)$ for any $i \geq 0$. Let

$$H^i(Y,Q)^{\Gamma} \subset H^i(Y,Q)$$

be the invariant subspace on which Γ acts trivially.

Proposition 3.1. *For any* $Q\in\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$,

$$\dim G/P\geq\dim H^0(Y,Q)^\Gamma-\dim H^1(Y,Q)^\Gamma,$$

where $H^i(Y,Q)^\Gamma$ *is the invariant part.*

Proof. Let $\nu\colon Y\to\mathrm{Gr}_Y^{\mathfrak{p}}$ be a section fiber bundle defined in (3.1) (we saw in (3.2) that $\mathrm{Gr}_Y^{\mathfrak{p}}$ is naturally identified with E_G/P). As we saw in the proof of Lemma 2.1, such a section ν defines a reduction of structure group $E_P\subset E_G$ to P. Note that $\mathrm{ad}(E_G)/\mathrm{ad}(E_P)$ is identified with ν^*T_{rel}, where T_{rel} is the pullback of the relative tangent bundle for the projection of $\mathrm{Gr}_Y^{\mathfrak{p}}$ to Y. Therefore, from [K, p. 37, Theorem 2.17.1] it follows immediately that the dimension of the local moduli space, around ν, of sections of $\mathrm{Gr}_Y^{\mathfrak{p}}$ is at least

$$\dim H^0(Y,\mathrm{ad}(E_G)/\mathrm{ad}(E_P))-\dim H^1(Y,\mathrm{ad}(E_G)/\mathrm{ad}(E_P)).$$

(Set X and S in [K, p. 37, Theorem 2.17] to be the curve Y, with the identity map of Y as the projection of X to S; note that a morphism $Y/Y\to\mathrm{Gr}_Y^{\mathfrak{p}}/Y$ is a section of the fiber bundle $\mathrm{Gr}_Y^{\mathfrak{p}}$.) Similarly, if ν is Γ-invariant, then the local moduli space, around ν, of Γ-invariant sections of $\mathrm{Gr}_Y^{\mathfrak{p}}$ is of dimension not less than

$$\dim H^0(Y,\mathrm{ad}(E_G)/\mathrm{ad}(E_P))^\Gamma-\dim H^1(Y,\mathrm{ad}(E_G)/\mathrm{ad}(E_P))^\Gamma$$

(see [K, p. 37, Theorem 2.17.1] and [K, p. 35, Theorem 2.15]). To derive this from the previous assertion, note that a Γ-invariant section of $\mathrm{Gr}_Y^{\mathfrak{p}}$ is a section of $\mathrm{Gr}_Y^{\mathfrak{p}}/\Gamma$ over Y/Γ. The pullback of the relative tangent bundle by the section over Y/Γ defined by the above Γ-invariant section ν coincides with $\phi_*(\mathrm{ad}(E_G)/\mathrm{ad}(E_P))^\Gamma$, where ϕ is the projection of Y to Y/Γ. Since ϕ is a finite map, we have

$$H^i(Y/\Gamma,\phi_*(\mathrm{ad}(E_G)/\mathrm{ad}(E_P))^\Gamma)\cong H^i(Y,\mathrm{ad}(E_G)/\mathrm{ad}(E_P))^\Gamma.$$

This establishes the above lower bound for the dimension of the local moduli space, around ν, of Γ-invariant sections of $\mathrm{Gr}_Y^{\mathfrak{p}}$. Therefore, for any $Q\in\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$ we have

$$\dim T_Q\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))\geq\dim H^0(Y,Q)^\Gamma-\dim H^1(Y,Q)^\Gamma.$$

So, to prove the proposition it suffices to show that $\dim G/P\geq\dim\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$.

Fix a point $y\in Y$ and consider the map

$$f_y\colon\mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))\longrightarrow\mathrm{Gr}:=\mathrm{Gr}(\dim P,\mathrm{ad}(E_G)_y)$$

to the Grassmannian that sends a quotient Q to the quotient Q_y of $\mathrm{ad}(E_G)_y$. If $v \in \mathrm{Gr}(\dim P, \mathrm{ad}(E_G)_y)$ and C is an irreducible complete curve in $f_y^{-1}(v)$, consider the map

$$Y \times C \longrightarrow \mathrm{Gr}_Y(\dim P, \mathrm{ad}(E_G))$$

to the Grassmann bundle (parametrizing $\dim P$-dimensional subspaces in $\mathrm{ad}(E_G)$) that sends any (z, c) to the fiber at z of the subbundle corresponding to the quotient represented by c. This map is constant on $y \times C$, and hence using the rigidity lemma we conclude that this map factors through the projection of $Y \times C$ to Y (see [MS, pp. 254–255]). Therefore, all the fibers of f_y are of dimension zero.

The image of f_y is contained in the orbit of $\mathfrak{p}$ under the adjoint action of G on $\mathfrak{g}$ (recall the condition that any fiber of S in (2.2) is identified with $\mathfrak{p}$ by some distinguished isomorphism). This implies that $\dim \mathrm{Image}(f_y) \leq \dim G/P$. Consequently, we have $\dim G/P \geq \dim \mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$, and the proof of the proposition is complete. □

Set $X := Y/\Gamma$, and denote by ϕ the projection of Y. Let $R \subset Y$ be the collection of all points where the map ϕ is ramified, that is, all points with nontrivial isotropy. For any $y \in R$, the isotropy subgroup $\Gamma_y \subset \Gamma$ is a cyclic subgroup, which acts faithfully on $T_y Y$. Let $\tau_y \in \Gamma_y$ be the generator that acts as multiplication by $\exp(2\pi\sqrt{-1}/n_y)$, where $n_y = \#\Gamma_y$. Consider the action of τ_y on the fiber $\mathrm{ad}(E_G)_y$. The eigenvalues are of the form $\exp(2\pi\sqrt{-1}\, m/n_y)$, $m \in [0, n_y - 1]$. If $n_y > m_1^y \geq m_2^y \geq \ldots \geq m_{\dim \mathfrak{g}}^y \geq 0$ are such that $\exp(2\pi\sqrt{-1}\, m_i^y/n_y)$, $i \in [1, \dim \mathfrak{g}]$, are the eigenvalues, then set

$$N_y := \sum_{i=1}^{\dim G/P} m_i^y.$$

Theorem 3.2. *The bound $c(E_G)$ defined in* (2.1) *satisfies the inequality*

$$c(E_G) \leq g_X \#\Gamma \cdot \dim G/P + \sum_{y \in R} N_y,$$

where $g_X = \mathrm{genus}(X)$ and $\#\Gamma$ is the order of Γ.

Proof. This follows from Proposition 3.1 and the Riemann–Roch formula for the Euler characteristic $\dim H^0(Y, Q)^\Gamma - \dim H^1(Y, Q)^\Gamma$.

For any $Q \in \mathcal{Q}_P^\Gamma(\mathrm{ad}(E_G))$ we have $H^i(Y, Q)^\Gamma \cong H^i(X, (\phi_* Q)^\Gamma)$, where $(\phi_* Q)^\Gamma \subset \phi_* Q$ is the subsheaf on which Γ acts trivially.

For any point $y \in R$, consider the induced action of Γ_y on the fiber Q_y. Let $\exp(2\pi\sqrt{-1}\, l_i^y/n_y)$, $i \in [1, \dim G/P]$, be the eigenvalues, where n_y is defined above and $l_i^y \in [0, n_y - 1]$. We have

$$\deg Q = \#\Gamma \cdot \deg(\phi_* Q)^\Gamma + \sum_{y \in R} \sum_{i=1}^{\dim G/P} l_i^y$$

(see [B, p. 318, (3.10)]). Therefore,

$$\deg(\phi_* Q)^\Gamma = \frac{\deg Q - \sum_{y \in R} \sum_{i=1}^{\dim G/P} l_i^y}{\#\Gamma} \geq \frac{\deg Q - \sum_{y \in R} N_y}{\#\Gamma},$$

as $\{l_i^y\}_{i=1}^{\dim G/P}$ is a subcollection of $\{m_i^y\}_{i=1}^{\dim G}$ for each $y \in R$. Now, using the Riemann–Roch formula for $(\phi_* Q)^\Gamma$ we have

$$\dim H^0(X, (\phi_* Q)^\Gamma) - \dim H^1(X, (\phi_* Q)^\Gamma) \geq (1 - g_X) \dim G/P + \frac{\deg Q}{\#\Gamma} - \sum_{y \in R} \frac{N_y}{\#\Gamma}.$$

Combining this with Proposition 3.1 gives

$$\dim G/P \geq (1 - g_X) \dim G/P + \frac{\deg Q}{\#\Gamma} - \sum_{y \in R} \frac{N_y}{\#\Gamma}.$$

In other words, $\deg Q \leq g_X \#\Gamma \cdot \dim G/P + \sum_{y \in R} N_y$. Since $c(E_G) = \deg(Q)$, the proof of the theorem is complete. $\square$

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