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Polya's inequalities, global uniform integrability and the size of plurisubharmonic lemniscates

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Abstract. First we prove a new inequality comparing uniformly the relative volume of a Borel subset with respect to any given complex euclidean ball $\mathbf{B}\subset\mathbf{C}^n$ with its relative logarithmic capacity in $\mathbf{C}^n$ with respect to the same ball $\mathbf{B}$. An analogous comparison inequality for Borel subsets of euclidean balls of any generic real subspace of $\mathbf{C}^n$ is also proved.

Then we give several interesting applications of these inequalities. First we obtain sharp uniform estimates on the relative size of plurisubharmonic lemniscates associated to the Lelong class of plurisubharmonic functions of logarithmic singularities at infinity on $\mathbf{C}^n$ as well as the Cegrell class of plurisubharmonic functions of bounded Monge–Ampère mass on a hyperconvex domain $\Omega\Subset\mathbf{C}^n$.

Then we also deduce new results on the global behaviour of both the Lelong class and the Cegrell class of plurisubharmonic functions.

1. Introduction

Local uniform integrability and estimates on the size of sublevel sets of plurisubharmonic functions in terms of capacities or various measures have been studied earlier in several works (see [CDL], [Ki], [K2], [Z2], [Z3], [P] and [BJ]). Such estimates turn out to be useful in many areas of complex analysis such as pluripotential theory, Padé approximation and complex dynamics (see [Ki], [K1], [K2], [CDL] and [FG]).

Our aim here is to generalize the classical Polya's inequality to subsets of any generic subspace of $\mathbf{C}^n$ and to give several new applications to the study of the global behaviour of two important classes of plurisubharmonic functions.

More precisely, given a generic subspace $\mathbf{G}\subset\mathbf{C}^n$, we prove a new inequality estimating from above the relative volume in $\mathbf{G}$ of a Borel subset with respect to a euclidean ball $B\subset\mathbf{G}$ in terms of its relative logarithmic capacity in $\mathbf{C}^n$ with respect

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to the same ball B, up to a multiplicative numerical constant which depends only on the dimension of $\mathbf{G}$ but not on the "condenser" considered.

Formulated in this way, Polya's inequalities turn out to play an important role in applications, implying interesting results which improve significantly earlier results obtained by several authors (see [CDL], [K2], [Z1] and [Z2]).

Indeed, first we easily deduce new estimates on the relative volume with respect to balls in a generic subspace of $\mathbf{C}^n$ of the plurisubharmonic lemniscates associated to the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbf{C}^n$ as well as the Cegrell class of plurisubharmonic functions with bounded Monge–Ampère mass on a bounded hyperconvex domain of $\mathbf{C}^n$.

Then we give estimates on global uniform integrability of the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbf{C}^n$ with respect to the Lebesgue measure on any generic subspace. These estimates can be considered as precise quantitative versions for the Lelong class of the well-known John–Nirenberg inequalities for BMO-functions on $\mathbf{R}^n$ (see [St]).

In particular we prove that restrictions to any generic subspace $\mathbf{G}\subset\mathbf{C}^n$ of plurisubharmonic functions with logarithmic singularities at infinity on $\mathbf{C}^n$ are in BMO($\mathbf{G}$) with a uniform explicit bound on their BMO($\mathbf{G}$)-norms depending only on the dimension of $\mathbf{G}$.

Finally we give a general sufficient condition for uniform integrability of a given class of plurisubharmonic functions on some domain in terms of the behaviour of the relative Monge–Ampère capacity of their sublevel sets with respect to this domain. In particular, we deduce a new global uniform integrability result for the Cegrell class of plurisubharmonic functions of uniformly bounded Monge–Ampère masses on a bounded hyperconvex domain.

2. Preliminaries

Let us recall the classical Polya's inequality (see [R] and [T]). For convenience, let us first recall the definition of the logarithmic capacity $c(K)$ of a compact subset $K\subset\mathbf{C}$. Let $D_\infty\subset\mathbf{C}$ be the unbounded component of $\mathbf{C}\setminus K$ and let g_{D_∞} be the (subharmonic) Green function of the domain D_∞ with logarithmic pole at infinity. Then the *logarithmic capacity* of K is defined by the formula

$$-\log c(K) := \limsup_{z\to\infty}(g_{D_\infty}(z)-\log|z|). \tag{2.1}$$

It is well known that $K\subset\mathbf{C}$ is a polar compact subset if and only if $c(K)=0$; and also that if $c(K)=0$ then the area of K is 0. Moreover if $K\subset\mathbf{R}$ and $c(K)=0$ the length of K is 0. There are quantitative versions of such results known as Polya's inequalities which we state now.

For any compact subset $K \subset \mathbf{C}$,

$$\lambda_2(K) \leq \pi c(K)^2, \tag{2.2}$$

with equality for a disc, where λ_2 is the area measure on $\mathbf{C}=\mathbf{R}^2$ and $c(K)$ is the logarithmic capacity of K.

Apart from this inequality, there is a corresponding inequality for sets of the real line $\mathbf{R} \subset \mathbf{C}$. Namely, for any compact subset $K \subset \mathbf{R}$,

$$\lambda_1(K) \leq 4c(K), \tag{2.3}$$

with equality for an interval, where λ_1 is the length measure on $\mathbf{R}$.

Recall that the logarithmic capacity $c(K)$ of the compact subset K defined by (2.1) coincides with its Chebyshev constant (see [R] and [T]), so that

$$c(K) = \inf_{d \geq 1} \inf\{\|P\|_K^{1/d} ; P \in \dot{\mathcal{P}}_d\},$$

where $\dot{\mathcal{P}}_d$ is the set of monic polynomials of degree d and $\|P\|_K := \sup_{z \in K} |P(z)|$.

We want to introduce similar quantities in $\mathbf{C}^n$. In this case, it is more convenient to normalize polynomials by requiring that $\|P\|_B := \max_B |P| = 1$ for some fixed non-pluripolar compact subset $B \subset \mathbf{C}^n$. Then following classical notation (see [AT] and [Si2]), we introduce the following *Chebyshev constant* associated to a compact subset $K \subset \mathbf{C}^n$,

$$T_B(K) := \inf_{d \geq 1} \inf\{\|P\|_K^{1/d} ;\ P \in \mathbf{C}[z],\ \deg P = d \text{ and } \|P\|_B = 1\}. \tag{2.4}$$

For $n=1$, it is easy to prove that the two constants c and T_B are equivalent as we shall see below.

The constant defined by (2.4) is related to the pluricomplex Green function with logarithmic singularities at infinity on $\mathbf{C}^n$, which we will recall below. Its definition is based on the usual *Lelong class* of plurisubharmonic functions of logarithmic growth at infinity on $\mathbf{C}^n$ defined as

$$\mathcal{L}(\mathbf{C}^n) := \{u \in \mathrm{PSH}(\mathbf{C}^n) ; \sup\{u(z) - \log^+ |z| ; z \in \mathbf{C}^n\} < +\infty\}. \tag{2.5}$$

The *global extremal function* with logarithmic growth at infinity associated to a Borel subset $K \Subset \mathbf{C}^n$ is defined by

$$V_K(z) := \sup\{u(z) ; u \in \mathcal{L}(\mathbf{C}^n) \text{ and } u|_K \leq 0\}, \quad z \in \mathbf{C}^n, \tag{2.6}$$

and its upper semi-continuous regularization V_K^* in $\mathbf{C}^n$ is the pluricomplex Green function with logarithmic singularities at infinity associated to K (see [Z] and [Si1]). Recall that if $n=1$ and $K\subset\mathbf{C}$ is not polar, then V_K^* coincides with the Green function of the unbounded component D_∞ of $\mathbf{C}\setminus K$ with pole at infinity extended by 0 on the set $\widehat{K}:=\mathbf{C}\setminus D_\infty$.

It is well known that V_K is locally bounded on $\mathbf{C}^n$ if and only if K is non-pluripolar in $\mathbf{C}^n$ (see [Si1] and [Si2]).

By a theorem of Siciak [Si2], we know that if $K\subset\mathbf{C}^n$ is a compact set, then

$$T_B(K)=\exp\Bigl(-\max_B V_K^*\Bigr) \tag{2.7}$$

The formula (2.7) allows us to extend the definition of the set function $T_B(\cdot)$ to Borel subsets of $\mathbf{C}^n$. Moreover the extended set function is a generalized Choquet capacity on any bounded domain in $\mathbf{C}^n$, which is outer regular (see [Si2]). When $K\subset B$, the constant $T_B(K)$ will be called the *relative logarithmic capacity* of K with respect to B in $\mathbf{C}^n$.

It is also well known that the null sets for this capacity are precisely the pluripolar subsets of $\mathbf{C}^n$ (see [Si2]).

Thus if $K\subset\mathbf{C}^n$ is non-pluripolar then $-\log T_B(K)=\max_B V_K^*$ $(<+\infty)$ is the best constant for which the following Bernstein–Walsh inequality holds

$$\sup_B u\le\sup_K u-\log T_B(K)\quad\text{for all } u\in\mathcal{L}(\mathbf{C}^n). \tag{2.8}$$

There is another relative capacity defined using the Monge–Ampère operator (see [BT1]). Here we choose a normalisation of the usual differential operators on $\mathbf{C}^n$ so that

$$dd^c:=\frac{i}{\pi}\partial\bar\partial.$$

Let $\Omega\Subset\mathbf{C}^n$ be an open set and $K\subset\Omega$ a compact subset. Then the *relative Monge–Ampère capacity* of the condenser (K,Ω) is defined by the formula (see [BT1])

$$\operatorname{cap}(K;\Omega):=\sup\biggl\{\int_K (dd^c u)^n\,;u\in\operatorname{PSH}(\Omega)\text{ and } -1\le u\le 0\biggr\}. \tag{2.9}$$

This capacity is related to the so called *plurisubharmonic measure* associated to the condenser (K,Ω) defined by

$$h_K(z):=\sup\{u(z)\,;u\in\operatorname{PSH}(\Omega), u\le 0 \text{ and } u|_K\le -1\},\quad z\in\Omega. \tag{2.10}$$

Then if $\Omega\Subset\mathbf{C}^n$ is a hyperconvex open set and $K\subset\Omega$ is a compact subset, it follows from [BT1] that

$$\operatorname{cap}(K;\Omega)=\int_K (dd^c h_K^*)^n=\int_\Omega (dd^c h_K^*)^n. \tag{2.11}$$

We will need the following Alexander–Taylor comparison inequality (see [AT]). For a fixed bounded domain $\Omega\Subset\mathbf{C}^n$ and a fixed euclidean ball $\mathbf{B}\subset\mathbf{C}^n$ such that $\Omega\subset\mathbf{B}$,

$$T_{\mathbf{B}}(E)\le\exp(-\operatorname{cap}(E;\Omega)^{-1/n}) \tag{2.12}$$

for any Borel subset $E\subset\Omega$.

We will also need to define the *Cegrell class* of plurisubharmonic functions. Let $\Omega\Subset\mathbf{C}^n$ be a hyperconvex open set. Denote by $\mathcal{F}(\Omega)$ the class of negative plurisubharmonic functions φ on Ω such that there exists a decreasing sequence $(\varphi_j)_{j=1}^\infty$ of bounded plurisubharmonic functions on Ω with boundary values 0 which converges to φ on Ω and satisfies $\sup_j\int_\Omega (dd^c\varphi_j)^n<+\infty$.

By Cegrell [C2], for $\varphi\in\mathcal{F}(\Omega)$, the Monge–Ampère measure $(dd^c\varphi)^n$ is a well defined Borel measure of finite mass on Ω as the weak limit of the sequence of measures $(dd^c\varphi_j)^n$, where $(\varphi_j)_{j=1}^\infty$ is any decreasing sequence converging to φ on Ω and satisfying all the requirements of the definition.

3. Relative Polya's inequalities

Here we want to compare the relative Lebesgue measure on a generic subspace $\mathbf{G}\subset\mathbf{C}^n$ with respect to a real euclidean ball in $\mathbf{G}$ with the relative logarithmic capacity in $\mathbf{C}^n$ with respect to the same ball.

First recall some definitions. A real subspace $\mathbf{G}\subset\mathbf{C}^n$ is said to be a *generic subspace* of $\mathbf{C}^n$ if $\mathbf{G}+J\mathbf{G}=\mathbf{C}^n$, where J is the complex structure on $\mathbf{C}^n$. We denote by $\mathbf{G}^c:=\mathbf{G}\cap J\mathbf{G}$ the maximal complex subspace of $\mathbf{C}^n$ contained in $\mathbf{G}$ and set $m:=\dim_{\mathbf{C}}\mathbf{G}^c$, which will be called the *complex dimension* of $\mathbf{G}$. Then it is clear that $\dim_{\mathbf{R}}\mathbf{G}=n+m$.

If $m=0$ which means that $\mathbf{G}^c=\{0\}$, the subspace $\mathbf{G}$ is said to be *totally real*. If $m=n$ then $\mathbf{G}=\mathbf{C}^n$.

It is easy to see that $\mathbf{G}\subset\mathbf{C}^n$ is a generic subspace of complex dimension m if and only if there is a unitary automorphism $U\colon\mathbf{C}^n\to\mathbf{C}^n$ such that $U(\mathbf{G})=\mathbf{C}^m\times\mathbf{R}^{n-m}\subset\mathbf{C}^m\times\mathbf{C}^{n-m}=\mathbf{C}^n$.

Observe that the subspace $\mathbf{G}\subset\mathbf{C}^n$ is non-pluripolar in $\mathbf{C}^n$ precisely when $\mathbf{G}$ is a generic subspace.

The subspace $\mathbf{G}\subset\mathbf{C}^n$ will be endowed with the induced euclidean structure and the corresponding Lebesgue measure which will be denoted by λ_{n+m}.

Now we can state our version of Polya's inequality which is the main result of this section.

Theorem 3.1. (1) *For any complex euclidean closed ball* $\mathbf{B}\subset\mathbf{C}^n$ *and any Borel subset* $K\subset\mathbf{B}$,

$$\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})}\le c_n T_{\mathbf{B}}(K)^2, \tag{3.1}$$

where

$$c_n:=\frac{4^n(n!)^2}{(2n-1)!}. \tag{3.2}$$

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $0\le m\le n-1$. *Then for any real euclidean closed ball* $B\subset\mathbf{G}$ *and any Borel subset* $K\subset B$,

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le 8(n+m)T_B(K). \tag{3.3}$$

We will see below that these inequalities are sharp as far as the exponents are concerned (see Remarks 3.6). For the proof of relative Polya's inequalities, we start to look at the simplest case where $n=1$.

Lemma 3.2. (1) *For any closed disc* $\mathbf{D}\subset\mathbf{C}$ *and any Borel subset* $K\subset\mathbf{D}$,

$$\frac{\lambda_2(K)}{\lambda_2(\mathbf{D})}\le 4T_{\mathbf{D}}(K)^2. \tag{3.4}$$

(2) *For any real closed interval* $\mathbf{I}\subset\mathbf{R}$ *and any Borel subset* $K\subset\mathbf{I}$

$$\frac{\lambda_1(K)}{\lambda_1(\mathbf{I})}\le 4T_{\mathbf{I}}(K). \tag{3.5}$$

We do not know if 4 is the best constant in these inequalities.

Proof. (1) By regularity of the Lebesgue measure and the relative logarithmic capacity in $\mathbf{C}$, we can assume that K is a non-polar compact subset. We can also assume that $\mathbf{C}\setminus K$ is connected since $\lambda_2(K)\le\lambda_2(\widehat{K})$ and $T_{\mathbf{D}}(K)=T_{\mathbf{D}}(\widehat{K})$. Then the extremal function V_K^* is a subharmonic function on $\mathbf{C}$ which coincides with the

Green function of $\mathbf{C}\setminus K$ with a pole at infinity. Therefore it can be represented by the formula

$$V_K^*(z) = \int_K \log|z-\zeta|\, d\mu(\zeta) - \log c(K), \quad z\in\mathbf{C},$$

where $\mu := (1/2\pi)\Delta V_K^*$ is the normalized equilibrium measure of K. From this representation formula, we get the estimate

$$\max_{\mathbf{D}} V_K^* \le \log(2R) - \log c(K),$$

where R is the radius of the disc $\mathbf{D}\subset\mathbf{C}$. This inequality implies that

$$c(K) \le 2R\mathrm{T}_{\mathbf{D}}(K). \tag{3.6}$$

Therefore using the inequality (2.2), we get from (3.6) the estimate

$$\lambda_2(K) \le 4\lambda_2(\mathbf{D})\mathrm{T}_{\mathbf{D}}(K)^2,$$

which is the required estimate.

(2) In the real case we prove in the same way that

$$c(K) \le 2R\mathrm{T}_{\mathbf{I}}(K),$$

where R is the radius of the interval $\mathbf{I}$. Therefore using the inequality (2.3), we get

$$\lambda_1(K) \le 4\lambda_1(\mathbf{I})\mathrm{T}_{\mathbf{I}}(K),$$

which is the required inequality. □

To prove our theorem in higher dimension, we need the following elementary slicing lemma.

Lemma 3.3. (1) *Let $\mathbf{B}\subset\mathbf{C}^n$ be any complex euclidean closed ball, $K\subset\mathbf{B}$ be a Lebesgue measurable subset and $a\in\partial\mathbf{B}$. Then there exists a complex line $L_a\subset\mathbf{C}^n$ passing through the point a such that $\lambda_2(\mathbf{B}\cap L_a)>0$ and*

$$\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})} \le c_n' \frac{\lambda_2(K\cap L_a)}{\lambda_2(\mathbf{B}\cap L_a)}, \tag{3.7}$$

where $c_n' = \frac{1}{4}c_n = 4^{n-1}(n!)^2/(2n-1)!$.

(2) *Let $B\subset\mathbf{R}^N$ be any euclidean ball, $K\subset B$ be any Lebesgue measurable subset and $a\in B$. Then there exists a real line $l_a\subset\mathbf{R}^N$ passing through the point a such that $\lambda_1(B\cap l_a)>0$ and*

$$\frac{\lambda_N(K)}{\lambda_N(B)} \le 2N\frac{\lambda_1(K\cap l_a)}{\lambda_1(B\cap l_a)}. \tag{3.8}$$

Observe that $c_n \sim 2\sqrt{\pi}\, n^{3/2}$ as $n \to +\infty$. We conjecture that the inequality (3.7) is true with the constant $c'_n = n$. The inequality (3.8) could be deduced from [BG], Lemma 3, with the constant N but the proof given there is not clear for us. So we decided to give another proof which uses the same idea of symmetrisation but leads to the constant $2N$ instead of N, unless the point a in the lemma coincides with the center of the ball B.

Proof. (1) We can of course assume that $n \geq 2$. Since our inequality is invariant under translation, we can also assume that $a = 0 \in \partial \mathbf{B}$ is the origin and $\lambda_{2n}(K) > 0$.

Now assume by contradiction that the inequality (3.7) is not true. Then we will have

$$\lambda_2(K \cap L) < \frac{\lambda_{2n}(K)}{c'_n \lambda_{2n}(\mathbf{B})} \lambda_2(\mathbf{B} \cap L), \tag{3.9}$$

for any complex line L passing through the origin $a = 0$ such that $\lambda_2(\mathbf{B} \cap L) > 0$.

Since relative volume and relative area are invariant under non-singular affine transformations, we can assume that

$$\mathbf{B} = \{z = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n \,;\, |z_1 - R|^2 + |z_2|^2 + \dots + |z_n|^2 < R^2\}$$

and $L_w = \{\zeta w; \zeta \in \mathbf{C}\}$, where $w = (w_1, \dots, w_n) \in S^{2n-1}$. Then

$$L_w \cap \mathbf{B} = \{\zeta w \,;\, |\zeta|^2 < 2R \operatorname{Re} \zeta w_1\}$$

is the disc centred at $R\bar{w}_1$ of radius $R|w_1|$ which by the last inequality leads to

$$\lambda_2(K \cap L_w) < \frac{\lambda_{2n}(K)}{c'_n \lambda_{2n}(\mathbf{B})} \pi R^2 |w_1|^2, \quad w \in S^{2n-1}, \ w_1 \neq 0. \tag{3.10}$$

Now, integrating in polar coordinates and using the invariance of the sphere S^{2n-1} by rotation, we obtain the formula

$$\begin{aligned} \lambda_{2n}(K) &= \frac{1}{2\pi} \int_{S^{2n-1}} \int_{|\zeta| < 2R|\zeta w_1|} |\zeta|^{2n-2} \chi_K(\zeta w)\, d\lambda_2(\zeta)\, d\sigma_{2n-1}(w) \\ &\leq \frac{2^{2n-2} R^{2n-2}}{2\pi} \int_{S^{2n-1}} |w_1|^{2n-2} \int_{|\zeta| \leq 2R|w_1|} \chi_K(\zeta w)\, d\lambda_2(\zeta)\, d\sigma_{2n-1}(w), \end{aligned}$$

where χ_K is the characteristic function of the set K.

Using inequality (3.10), we deduce from the last inequality that

$$\lambda_{2n}(K) < 2^{2n-2} R^{2n} \frac{\lambda_{2n}(K)}{2c'_n \lambda_{2n}(\mathbf{B})} \int_{S^{2n-1}} |w_1|^{2n}\, d\sigma_{2n-1}(w). \tag{3.11}$$

Now, an elementary computation using spherical coordinates leads to the formula

$$\int_{S^{2n-1}} |w_1|^{2n}\, d\sigma_{2n-1}(w) = \frac{2(n!)^2}{(2n-1)!}\tau_{2n}, \tag{3.12}$$

where τ_{2n} is the volume of the euclidean unit ball in $\mathbf{R}^{2n}$.

The last formula (3.12) combined with (3.11) leads finally to the inequality

$$\lambda_{2n}(K) < \frac{2^{2n-2}R^{2n}\lambda_{2n}(K)}{2c'_n\lambda_{2n}(B)} 2n \frac{(n!)^2}{(2n-1)!}\tau_{2n} = \lambda_{2n}(K),$$

which yields a contradiction.

(2) As in the complex case, we assume that $a=0$ is the origin in $\mathbf{R}^N$, $\lambda_N(K)>0$ and the ball B is of radius 1.

First, observe that $\lambda_1(B\cap l_a)\le 2$ for any real line l_a passing through the point a, then to show (3.8) it is enough to prove that

$$\frac{1}{N}\frac{\lambda_N(K)}{\lambda_N(B)} \le \lambda_1(K\cap l_a)$$

for some real line l_a.

Assume by contradiction that the last inequality is not true. Then we will have

$$\lambda_1(K\cap l) < \frac{1}{N}\frac{\lambda_N(K)}{\lambda_N(B)} \tag{3.13}$$

for any real line l passing through the origin $a=0$.

Let $\widetilde{K}$ be the annulus with the same center x_0 as B and of radii r and 1 ($r<1$) such that $\lambda_N(\widetilde{K})=\lambda_N(K)$.

Then we deduce the formula

$$r = \left(1-\frac{\lambda_N(K)}{\lambda_N(B)}\right)^{1/N}.$$

Denote by $e(\widetilde{K}):=1-r$ the depth of the annulus $\widetilde{K}$ and observe that

$$e(\widetilde{K}) = 1-\left(1-\frac{\lambda_N(K)}{\lambda_N(B)}\right)^{1/N} \ge \frac{1}{N}\frac{\lambda_N(K)}{\lambda_N(B)}.$$

The last inequality together with (3.13) lead to

$$e(\widetilde{K}) > \lambda_1(K\cap l) \tag{3.14}$$

for any real line l passing through a.

Now, observe that, if l is any real line passing through the origin such that $l \cap B(x_0, r) \neq \emptyset$, then $\lambda_1(\widetilde{K} \cap l) \geq 2e(\widetilde{K})$ and hence from (3.14) we derive the inequality

$$\lambda_1(\widetilde{K} \cap l) > 2\lambda_1(K \cap l) \tag{3.15}$$

for any real line l passing through the origin $a=0$.

Now to get a contradiction with the fact that $\lambda_N(\widetilde{K}) = \lambda_N(K)$, it is enough to construct a Borel set $K^{(s)} \subset \widetilde{K}$ such that $\lambda_N(K) \leq \lambda_N(K^{(s)}) < \lambda_N(\widetilde{K})$. For the construction of the set $K^{(s)}$, we will use the inequality (3.15) and a special symmetrisation process following an idea of [BG]. Indeed, let l be a given real line passing through the point $a=0$. Then, it follows from (3.15) that the segment $B \cap l$ contains an extreme segment $I(K \cap l)$ (i.e. issued from the boundary of $B \cap l$) with length equal to $\lambda_1(K \cap l)$ and of maximal distance from the origin $a=0$. Then from the inequality (3.15), it follows that $I(K \cap l) \subset \widetilde{K} \cap l$. Now denote by $K^{(s)}$ the union of all the segments $I(K \cap l)$ when l runs over all the real lines passing through the origin. Then $K^{(s)} \subset \widetilde{K}$ and

$$\lambda_N(K^{(s)}) < \lambda_N(\widetilde{K}). \tag{3.16}$$

On the other hand, from the construction of the set $K^{(s)}$, we see that if l is a real line passing through the origin $a=0$, then for any $\tau \in (K \cap l) \setminus K^{(s)}$ and any $t \in (K^{(s)} \cap l) \setminus K$ we have $|\tau| \leq |t|$. Since $\lambda_1(K \cap l) = \lambda_1(K^{(s)} \cap l)$, it follows that

$$\int_{K \cap l} |\tau|^{N-1}\, d\tau \leq \int_{K^{(s)} \cap l} |t|^{N-1}\, dt \tag{3.17}$$

for any real line l passing through the origin $a=0$.

Now, integrating in polar coordinates we obtain

$$\begin{aligned} \lambda_N(K) &= \frac{1}{2} \int_{S^{N-1}} \left(\int_{\mathbf{R}} |\tau|^{N-1} \chi_K(\tau w)\, d\tau \right) d\sigma_{N-1}(w) \\ &= \frac{1}{2} \int_{S^{N-1}} \left(\int_{K \cap l_w} |\tau|^{N-1}\, d\tau \right) d\sigma_{N-1}(w), \end{aligned}$$

where χ_K is the characteristic function of the set K and $l_w = \{t \cdot w; t \in \mathbf{R}\}$.

Using the last formula and the inequality (3.17), we obtain

$$\begin{aligned} \lambda_N(K) &\leq \frac{1}{2} \int_{S^{N-1}} \left(\int_{K^{(s)} \cap l_w} |t|^{N-1}\, dt \right) d\sigma_{N-1}(w) \\ &\leq \frac{1}{2} \int_{S^{N-1}} \left(\int_{\mathbf{R}} |t|^{N-1} \chi_{K^{(s)}}(tw)\, dt \right) d\sigma_{N-1}(w) \leq \lambda_N(K^{(s)}), \end{aligned}$$

which proves that the set $K^{(s)}$ satisfies the required properties. □

Now we are ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. (1) By outer regularity of the Lebesgue measure and the relative logarithmic capacity, we can assume that $K\subset\mathbf{B}$ is a compact set with non-empty interior in $\mathbf{C}^n$ so that $\lambda_{2n}(K)>0$ and $T_{\mathbf{B}}(K)>0$. Therefore $V_K^*\in\mathcal{L}(\mathbf{C}^n)$ and by the maximum principle there exists $a\in\partial\mathbf{B}$ such that $V_K^*(a)=\sup_{\mathbf{B}} V_K^*$. By translation we can assume that $a=0$ is the origin in $\mathbf{C}^n$. Now the key of the proof is contained in the following fundamental observation: For any complex line L passing through the origin $a=0$, $K\cap L$ is a compact subset of the complex disc $\mathbf{B}\cap L$ in L and $T_{\mathbf{B}\cap L}(K\cap L)\leq T_{\mathbf{B}}(K)$. Indeed identifying L with the complex line $\mathbf{C}$, we see that any function $u\in\mathcal{L}(\mathbf{C}^n)$ with $u|_K\leq 0$ satisfies $u^+|_L\in\mathcal{L}(\mathbf{C})$ with $u^+|_{(K\cap L)}\leq 0$. Then from the definition of $V_{K\cap L}$, it follows that $V_K\leq V_{K\cap L}$ on L. Since $a\in\mathbf{B}\cap L$ and $\max_{\mathbf{B}} V_K^*=V_K^*(a)$, we deduce that $\max_{\mathbf{B}} V_K^*\leq\max_{\mathbf{B}\cap L} V_{K\cap L}^*$, which implies that $T_{\mathbf{B}\cap L}(K\cap L)\leq T_{\mathbf{B}}(K)$.

Now by the complex slicing lemma, we can find a complex line $L\subset\mathbf{C}^n$ passing through the point $a=0$ such that $\lambda_2(K\cap L)>0$ and

$$\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})}\leq c_n'\frac{\lambda_2(K\cap L)}{\lambda_2(\mathbf{B}\cap L)}. \tag{3.18}$$

Therefore from (3.18) and (3.4) we finally deduce that

$$\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})}\leq 4c_n' T_{\mathbf{B}}(K)^2, \tag{3.19}$$

which is exactly the required inequality (3.1).

(2) We assume for simplicity that $\mathbf{G}\subset\mathbf{C}^n$ is a generic subspace of complex dimension $1\leq m\leq n-1$ (the totally real case $m=0$ can be treated in the same way). By the invariance of the Lebesgue measure and the relative capacity T_B by unitary transformations, we can assume that $\mathbf{G}=\mathbf{C}^m\times\mathbf{R}^{n-m}$. By outer regularity of the Lebesgue measure and the relative capacity T_B, we can assume that $K\subset B$ is a compact subset with non-empty interior in $\mathbf{G}$ so that $\lambda_{n+m}(K)>0$. Let us prove that $T_B(K)>0$. Indeed, since K is a compact subset with non-empty interior in $\mathbf{G}$, there exists an interval $I\subset\mathbf{R}$ of positive length and a disc $\mathbf{D}\subset\mathbf{C}$ with positive radius such that $\mathbf{D}^m\times I^{n-m}\subset K$. Then by the product property of the extremal function (see [Si1]), we get

$$V_K(z,\zeta)\leq\max\{V_{\mathbf{D}}(z_i),V_I(\zeta_j)\,;1\leq i\leq m \text{ and } 1\leq j\leq n-m\},$$

for any $z=(z_1,\dots,z_m)\in\mathbf{C}^m$ and $\zeta\in\mathbf{C}^{n-m}$. Therefore V_K is locally bounded on $\mathbf{C}^n$ and then $T_B(K)>0$. Then $V_K^*\in\mathcal{L}(\mathbf{C}^n)$ and there exists $a\in B$ such that $V_K^*(a)=\sup_B V_K^*$.

By translation we may assume that $a=0$ is the origin in $\mathbf{G}$. Then by the real slicing lemma, there exists a real line $l\subset\mathbf{G}$ passing through the point $a=0$ such that $\lambda_1(K\cap l)>0$ and

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le 2(n+m)\frac{\lambda_1(K\cap l)}{\lambda_1(B\cap l)}. \tag{3.20}$$

Let $L:=l+il$ be the complex line in $\mathbf{C}^n$ generated by the real line l. As in the complex case we see easily that $T_{B\cap l}(K\cap l)\le T_B(K)$ and then from (3.5) and (3.20) we deduce that

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le 8(n+m)T_B(K), \tag{3.21}$$

which is exactly the required inequality (3.3). $\square$

It is interesting to observe that from the formula (2.7) it follows that our relative Polya's inequalities leads to the following quantitative version of the Bernstein–Walsh inequality.

Corollary 3.4. (1) *For any closed complex euclidean ball $\mathbf{B}\subset\mathbf{C}^n$, any Borel subset $K\subset\mathbf{B}$ and any function $u\in\mathcal{L}(\mathbf{C}^n)$,*

$$\sup_{\mathbf{B}} u\le\sup_K u+\frac{1}{2}\log c_n-\frac{1}{2}\log\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})}, \tag{3.22}$$

where c_n is the constant given by the formula (3.2).

(2) *Let $\mathbf{G}\subset\mathbf{C}^n$ be any generic subspace of complex dimension $m\le n-1$. Then for any closed real euclidean ball $B\subset\mathbf{G}$, any Borel subset $K\subset B$ and any function $u\in\mathcal{L}(\mathbf{C}^n)$,*

$$\sup_B u\le\sup_K u+\log 8(n+m)-\log\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}. \tag{3.23}$$

Let us mention that in the totally real case $\mathbf{G}=\mathbf{R}^n$, inequalities like (3.23) were obtained earlier by A. Brudnyi (see [B1], [B2]).

From the relative Polya's inequalities (3.1), (3.3) and Alexander–Taylor's inequality (2.12), we deduce the following interesting comparison inequalities between the relative volume and the relative Monge–Ampère capacity.

Corollary 3.5. (1) *For any complex euclidean ball $\mathbf{B}\subset\mathbf{C}^n$ and any Borel subset $K\subset\mathbf{B}$,*

$$\frac{\lambda_{2n}(K)}{\lambda_{2n}(\mathbf{B})}\le c_n\exp(-2\operatorname{cap}(K;\mathbf{B})^{-1/n}), \tag{3.24}$$

where c_n is the constant given by (3.2).

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $0\le m\le n-1$. *Then for any euclidean ball* $B\subset\mathbf{G}$ *and any Borel subset* $K\subset B$,

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le 8(1+\sqrt{2})(n+m)\exp(-\operatorname{cap}(K;\mathbf{B})^{-1/n}), \tag{3.25}$$

where $\mathbf{B}$ *is the euclidean ball in* $\mathbf{C}^n$ *such that* $\mathbf{B}\cap\mathbf{G}=B$.

Proof. (1) The inequality (3.24) is a direct consequence of (2.12) and (3.1).

(2) Let us prove the inequality (3.25). Since both the relative volume and the relative capacity are invariant under non-singular affine transformations, we can assume that $\mathbf{G}=\mathbf{C}^m\times\mathbf{R}^{n-m}$, B is the unit real euclidean ball in $\mathbf{G}$ and $\mathbf{B}$ is the unit complex euclidean ball in $\mathbf{C}^n$. Then by (3.3), we have

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le 8(n+m)T_B(K). \tag{3.26}$$

On the other hand, by (2.12), we have

$$T_{\mathbf{B}}(K)\le\exp(-\operatorname{cap}(K;\mathbf{B})^{-1/n}).$$

So to prove (3.25), it remains to estimate $T_B(K)$ from above by $T_{\mathbf{B}}(K)$. Indeed, from the definition of the extremal function V_B, it follows that

$$V_K(z)\le\max_B V_K+V_B(z),\quad z\in\mathbf{C}^n.$$

Therefore, we get

$$T_B(K)\le\exp\Big(\max_{\mathbf{B}}V_B\Big)T_{\mathbf{B}}(K). \tag{3.27}$$

It remains to estimate $\max_{\mathbf{B}}V_B$. Since $\mathbf{R}^n\subset\mathbf{G}$, the euclidean unit ball B in $\mathbf{G}$, contains the euclidean unit ball D of $\mathbf{R}^n$ and then $V_B\le V_D$ on $\mathbf{C}^n$, which implies that $\max_{\mathbf{B}}V_B\le\max_{\mathbf{B}}V_D$. Now by Lundin's formula (see [Lu], [S2] and [Kl]), we have

$$V_D(z)=\max\{\log|h(\xi\cdot z)|\,;\xi\in S^{n-1}\},\quad z\in\mathbf{C}^n, \tag{3.28}$$

where $h(\zeta):=\zeta+\sqrt{\zeta^2-1}$ for $\zeta\in\mathbf{C}$, with the right branch of the square root, $S^{n-1}=\partial D$ is the euclidean unit sphere of $\mathbf{R}^n\subset\mathbf{C}^n$ and $\xi\cdot z=\sum_{1\le j\le n}\xi_j\cdot z_j$. It is easy to see from the formula (3.28) that

$$\max_{\mathbf{B}}V_D=\max_{|z|=1}V_D(z)=\max_{|\zeta|=1}\log|h(\zeta)|=\log(1+\sqrt{2})$$

and then $\exp(\max_{\mathbf{B}}V_B)\le\exp(\max_{\mathbf{B}}V_D)=1+\sqrt{2}$, which by the inequalities (3.27) and (3.26) implies the required inequality (3.25). $\square$

It is interesting to observe that the inequality (3.25) shows that the Lebesgue measure on any generic subspace restricted to any hyperconvex domain $\Omega\Subset\mathbf{C}^n$ is dominated by capacity in a strong sense and then by a result of S. Kolodziej, it belongs to the image of the complex Monge–Ampère operator acting on the class of bounded plurisubharmonic functions on Ω (see [K1], [K2] and [C1]).

Remarks 3.6. (1) Polya's inequalities (3.1) and (3.3) can be stated in one formula as follows. Given a generic subspace $\mathbf{G}\subset\mathbf{C}^n$ of complex dimension $0\le m\le n$, then for any euclidean ball $B\subset\mathbf{G}$ and any Borel subset $K\subset B$, we have

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B)}\le c_{n,m}T_B(K)^{1+[m/n]}, \tag{3.29}$$

where $c_{n,m}:=8(n+m)$ if $0\le m\le n-1$ and $c_{n,n}:=c_n$.

We can deduce from the general relative Polya's inequality (3.29) analogous inequalities in terms of relative volume and relative logarithmic capacity with respect to balls associated to any fixed real norm on the generic space $\mathbf{G}$. Indeed, if we denote by $|\cdot|$ the euclidean norm and we are given another real norm $\|\cdot\|$ on $\mathbf{G}$, then there exist two constants $\alpha,\beta>0$ such that

$$\alpha\|z\|\le|z|\le\beta\|z\|,\quad z\in\mathbf{G}.$$

Then given a ball B' for the norm $\|\cdot\|$, there exists a ball B for the norm $|\cdot|$ such that $\alpha B\subset B'\subset\beta B$. Then it follows easily from (3.29) that for any Borel set $K\subset B'$, we have

$$\frac{\lambda_{n+m}(K)}{\lambda_{n+m}(B')}\le c_{n,m}(\beta/\alpha)^{n+m}T_{B'}(K)^{1+[m/n]}. \tag{3.30}$$

(2) Observe that the relative Polya's inequalities proved above are optimal as far as the exponents are concerned. Indeed we will use inequality (3.30) for the sup-norm, since in this case, explicit computations can be made using the product formula for the relative logarithmic capacity. Let $B_1,\dots,B_n$ be regular sets in $\mathbf{C}$, $K_1,\dots,K_n$ Borel subsets such that $K_j\subset B_j$ for $j=1,\dots,n$ and set $K:=K_1\times\dots\times K_n$ and $B:=B_1\times\dots\times B_n$. Then using the product property for the extremal function (see [Si1]), we get the formula

$$T_B(K)=\min_{1\le j\le n}T_{B_j}(K_j). \tag{3.31}$$

In the case where $\mathbf{G}=\mathbf{C}^n$, take B' to be the closed unit polydisc Δ^n in $\mathbf{C}^n$ and $K_r:=\{z\in\Delta^n;|z_1|\le r\}$. Then the relative volume of K_r with respect to Δ^n is

$\lambda_{2n}(K_r)/\lambda_{2n}(\Delta^n)=r^2$ while its relative logarithmic capacity is $T_{\Delta^n}(K_r)=r$. By (3.30) this proves that the exponent 2 in the complex Polya's inequality (3.1) is the best possible.

In the totally real case, we can assume that $\mathbf{G}=\mathbf{R}^n$ and consider an analogous example with intervals. Take B' to be the unit n-cube $\mathbf{I}^n$, where $\mathbf{I}:=[-1,+1]$ is the closed unit real interval, and define $I^n(r):=\{x\in\mathbf{I}^n; |x_1|\le r\}$. Then it is easy to see that

$$T_{\mathbf{I}^n}(I^n(r))=\frac{r}{1+\sqrt{1-r^2}}\sim\frac{r}{2},\quad \text{as } r\to 0,$$

while the relative n-volume of $I^n(r)$ with respect to $\mathbf{I}^n$ is equal to r, which proves by (3.30) that the exponent 1 in Polya's inequality (3.3) is the best possible in this case.

Now if $\mathbf{G}=\mathbf{C}^m\times\mathbf{R}^{n-m}$ with $1\le m\le n-1$, it is enough to take $B'=\Delta^m\times\mathbf{I}^{n-m}$ and $K_r:=\Delta^m\times I^{n-m}(r)$. Then $T_{B'}(K_r)\sim\frac{1}{2}r$ as $r\to 0$, while $\lambda_{n+m}(K_r)/\lambda_{n+m}(B')=r$, which prove again by (3.30) that the exponent 1 in Polya's inequality (3.3) is the best possible in this case.

4. The relative size of plurisubharmonic lemniscates

Here we want to deduce from the relative Polya's inequalities an estimate on the relative size of plurisubharmonic lemniscates (i.e. sublevel sets) associated to two important classes of plurisubharmonic functions.

Let us start with estimating precisely the size of the plurisubharmonic lemniscates associated to the Lelong class $\mathcal{L}(\mathbf{C}^n)$.

Theorem 4.1. (1) *For any complex euclidean closed ball* $\mathbf{B}\subset\mathbf{C}^n$ *and any* $u\in\mathcal{L}(\mathbf{C}^n)$ *with* $\max_{\mathbf{B}} u=0$,

$$\text{(4.1)}\qquad \frac{\lambda_{2n}(\{z\in\mathbf{B}\,;u(z)\le -s\})}{\lambda_{2n}(\mathbf{B})}\le c_n e^{-2s},\quad s>0,$$

where c_n *is the constant given by* (3.2).

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $m\le n-1$. *Then for any real euclidean closed ball* $B\subset\mathbf{G}$ *and any* $u\in\mathcal{L}(\mathbf{C}^n)$ *with* $\max_B u=0$,

$$\text{(4.2)}\qquad \frac{\lambda_{n+m}(\{x\in B\,;u(z)\le -s\})}{\lambda_{n+m}(B)}\le 8(n+m)e^{-s},\quad s>0.$$

Proof. (1) Let $\mathbf{B}\subset\mathbf{C}^n$ be an arbitrary complex ball and let $u\in\mathcal{L}(\mathbf{C}^n)$ with $\max_{\mathbf{B}} u=0$. Set $E_t(u):=\{z\in\mathbf{B};u(z)\le t\}$ for $t<0$. Then $u-t\le V_{E_t(u)}$ on $\mathbf{C}^n$ and then $-t=\max_{\mathbf{B}} u-t\le\max_{\mathbf{B}} V_{E_t(u)}$. This implies that $T_{\mathbf{B}}(E_t(u))\le e^t$ for any $t<0$. Now in order to get the estimate (4.1), it is enough to apply the complex Polya's inequality (3.1) to the Borel set $E_t(u)$ with $s=-t$. To prove the estimate (4.2), we proceed in the same way using the real Polya's inequality (3.3). □

Observe that estimates of plurisubharmonic lemniscates were obtained in the complex case earlier by the third author in a more general context but with less precise exponents (see [Z2] and [Z3]).

In particular, observing that $(1/d)\log|P|\in\mathcal{L}(\mathbf{C}^n)$ for any polynomial $P\in\mathbf{C}[z]$ with degree $d\ge1$, we obtain the following precise estimate for polynomial lemniscates.

Corollary 4.2. (1) *For any complex ball $\mathbf{B}\subset\mathbf{C}^n$ and any polynomial $P\in\mathbf{C}[z]$ of degree $d\ge1$ satisfying $\|P\|_{\mathbf{B}}=1$, we have*

$$\frac{\lambda_{2n}(\{z\in\mathbf{B}\,;|P(z)|\le\varepsilon^d\})}{\lambda_{2n}(\mathbf{B})}\le c_n\varepsilon^2,\quad \varepsilon\in]0,1], \tag{4.3}$$

where c_n is the constant given by (3.2).

(2) *Let $\mathbf{G}\subset\mathbf{C}^n$ be a generic subspace of complex dimension $0\le m\le n-1$. Then for any real euclidean ball $B\subset\mathbf{G}$ and any polynomial $P\in\mathbf{C}[z]$ of degree $d\ge1$ satisfying $\|P\|_B=1$, we have*

$$\frac{\lambda_{n+m}(\{z\in B\,;|P(z)|\le\varepsilon^d\})}{\lambda_{n+m}(B)}\le 8(n+m)\varepsilon,\quad \varepsilon\in]0,1]. \tag{4.4}$$

All these estimates are optimal as far as the exponents are concerned (see Remarks 3.6 above). The first inequality is an improvement of previous results (see [CDL], [Z2] and [Z3]) and answers a question asked by the third author in [Z2]. In the totally real case where $\mathbf{G}=\mathbf{R}^n$, the second inequality appears also in [BG].

Now let us estimate the size of plurisubharmonic lemniscates associated to the Cegrell class $\mathcal{F}(\Omega)$. These estimates are important in the study of the complex Monge–Ampère equation (see [K1] and [K2]).

Theorem 4.3. *Let $\Omega\Subset\mathbf{C}^n$ be a hyperconvex open set. Then for any plurisubharmonic function $\varphi\in\mathcal{F}(\Omega)$ with $\int_\Omega(dd^c\varphi)^n\le1$, we have*

$$\lambda_{2n}(\{z\in\Omega\,;\varphi(z)\le-s\})\le c_n\tau_{2n}(\Omega)e^{-2s},\quad s>0, \tag{4.5}$$

where $\tau_{2n}(\Omega)$ is the volume of the smallest euclidean ball of $\mathbf{C}^n$ containing Ω and c_n is the constant given by (3.2).

Moreover, if $\mathbf{G}\subset\mathbf{C}^n$ *is a generic subspace of complex dimension* $m\leq n-1$ *such that* $D:=\Omega\cap\mathbf{G}\neq\emptyset$, *then for any* $s>0$,

$$\lambda_{n+m}(\{z\in D\,;\varphi(z)\leq -s\})\leq 8\big(1+\sqrt{2}\big)(n+m)\tau_{n+m}(D)e^{-s}, \tag{4.6}$$

where $\tau_{n+m}(D)$ *is the volume of the smallest euclidean ball of* $\mathbf{G}$ *containing* D.

Observe that our estimates are sharp as far as the exponents of decrease are concerned and improve previous estimates obtained in the complex case by Kolodziej and the third author (see [K1], [K2] and [Z2]).

For the proof of this theorem, we will need the following elementary lemma.

Lemma 4.4. *Let* $\Omega\Subset\mathbf{C}^n$ *be a hyperconvex open set. Then for any* $\varphi\in\mathcal{F}(\Omega)$,

$$\operatorname{cap}(\{z\in\Omega\,;\varphi(z)\leq -s\};\Omega)\leq \frac{1}{s^n}\int_\Omega (dd^c\varphi)^n,\quad s>0. \tag{4.7}$$

Proof. (1) Assume first that φ is a bounded plurisubharmonic function on Ω with boundary values 0 and finite Monge–Ampère mass on Ω. Let $s>0$ be fixed and $K\subset\Omega(\varphi;s):=\{z\in\Omega;\varphi(z)\leq -s\}$ be any fixed regular compact set in the sense that the plurisubharmonic measure h_K of the condenser (K,Ω) is continuous on Ω. Since h_K and φ have boundary values 0, from the comparison principle (see [BT1] and [Kl]) it follows that

$$\operatorname{cap}(K;\Omega)=\int_K (dd^c h_K)^n\leq\int_{\{s^{-1}\varphi<h_K\}}(dd^c h_K)^n\leq\frac{1}{s^n}\int_\Omega (dd^c\varphi)^n.$$

Taking an exhaustive sequence of regular compact subsets of the open set $\Omega(s;\varphi)$ and using interior regularity of the capacity we obtain our inequality in this case.

(2) Now for an arbitrary given function $\varphi\in\mathcal{F}(\Omega)$, there exists a decreasing sequence $(\varphi_j)_{j=1}^{+\infty}$ of bounded plurisubharmonic functions with boundary values 0 which converges to φ such that

$$\int_\Omega (dd^c\varphi)^n=\lim_{j\to+\infty}\int_\Omega (dd^c\varphi_j)^n$$

(see [C2] and [CZ]). Then the estimate (4.7) follows from the first case and the lemma is proved. □

Now we can prove the theorem.

Proof of Theorem 4.3. (1) Let $\mathbf{B}$ be the smallest euclidean ball of $\mathbf{C}^n$ containing Ω. Let $\varphi\in\mathcal{F}(\Omega)$ be as in the theorem and set $\Omega(\varphi;s):=\{z\in\Omega;\varphi(z)\leq -s\}$

and $c(s)=c_\Omega(s,\varphi):=\operatorname{cap}(\Omega(\varphi;s);\Omega)$ for $s>0$. Then applying inequality (3.24), we obtain

$$\lambda_{2n}(\Omega(\varphi;s))\le c_n\lambda_{2n}(\mathbf{B})\exp(-2c_\Omega(s)^{-1/n}),\quad s>0. \tag{4.8}$$

Now the estimate (4.5) follows from the estimate (4.8) using the estimate (4.7).

The estimate (4.6) is proved in the same way using the inequalities (3.25) and (4.7). □

Remarks. The exponent of decrease of the volumes in the last theorem is sharp as simple examples show (see Remarks 3.6). Up to the normalization factor 2π, the estimate (4.5) with an exponent arbitrary close to 2 was obtained in [K2] (see also [Z2]).

5. Global behaviour of the Lelong class

The next application of our theorems from the last section will concern the Lelong class of plurisubharmonic functions with logarithmic singularities at infinity defined by the formula (2.5).

The Lelong class of plurisubharmonic functions is known to play an important role in pluripotential theory (see [L1], [BT2], [Si1], [Si2], [S1], [Z], [Z1] and [Z2]).

Here we want to prove new general uniform integrability theorems for the Lelong class of plurisubharmonic functions.

Let $g\colon \mathbf{R}^+\to\mathbf{R}^+$ be an increasing function such that $g(0)=0$ and $\lim_{t\to+\infty} g(t)=+\infty$. For $\delta>0$, consider the following Riemann–Stieltjes integral

$$I_\delta(g):=\int_0^{+\infty} e^{-\delta t}\,dg(t). \tag{5.1}$$

Then we have the following result.

Theorem 5.1. (1) *For any complex euclidean closed ball* $\mathbf{B}\subset\mathbf{C}^n$ *and any function* $u\in\mathcal{L}(\mathbf{C}^n)$

$$\frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}} g\Big(\max_{\mathbf{B}} u-u\Big)\,d\lambda_{2n}\le c_n I_2(g), \tag{5.2}$$

provided that $I_2(g)<+\infty$, *where* c_n *is the constant given by* (3.2).

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* m. *Then for any real euclidean closed ball* $B\subset\mathbf{G}$ *and any function* $u\in\mathcal{L}(\mathbf{C}^n)$

$$\frac{1}{\lambda_{n+m}(B)}\int_B g\Big(\max_B u-u\Big)\,d\lambda_{n+m}\le 8(n+m)I_1(g) \tag{5.3}$$

provided that $I_1(g)<+\infty$.

Proof. We can assume g to be strictly increasing. Let μ be any Borel measure on $\mathbf{C}^n$ and $K\Subset\mathbf{C}^n$ any Borel set. Then for any function $u\in\mathcal{L}(\mathbf{C}^n)$ with $u|_K\le 0$, we have

$$\int_K g(-u)\,d\mu=\int_0^{+\infty}\mu(\{z\in K:g(-u(z))\ge t\})\,dt \tag{5.4}$$

$$=\int_0^{+\infty}\mu(\{z\in K:u(z)\le -s\})\,dg(s). \tag{5.5}$$

(1) Assume that $\mu:=\chi_{\mathbf{B}}\lambda_{2n}$, where $\mathbf{B}\subset\mathbf{C}^n$ is a complex euclidean closed ball and $u\in\mathcal{L}(\mathbf{C}^n)$ with $\max_{\mathbf{B}} u=0$. Then by (5.5), we get

$$\int_{\mathbf{B}} g(-u)\,d\lambda_{2n}=\int_0^{+\infty}\lambda_{2n}(\{z\in\mathbf{B}:u(z)\le -s\})\,dg(s). \tag{5.6}$$

Applying the estimates (4.1) to the formula (5.6), we obtain the inequality

$$\int_{\mathbf{B}} g(-u)\,d\lambda_{2n}\le c_n\lambda_{2n}(\mathbf{B})\int_0^{+\infty}e^{-2s}\,dg(s). \tag{5.7}$$

If $I_2(g)<+\infty$, we easily see that $\lim_{t\to+\infty}g(t)e^{-2t}=0$ and by integration by parts, it follows that $\int_0^{+\infty}e^{-2s}\,dg(s)=I_2(g)$, which implies the required inequality thanks to the inequality (5.7).

(2) Assume that $\mu:=\chi_B\lambda_{n+m}$, where $B\subset\mathbf{G}$ is a real euclidean closed ball and $u\in\mathcal{L}(\mathbf{C}^n)$ with $\max_B u=0$. Then applying the estimates (4.2) to the formula (5.6), we obtain the inequality

$$\int_B g(-u)\,d\lambda_n\le 8(n+m)\lambda_{n+m}(B)\int_0^{+\infty}e^{-s}\,dg(s). \tag{5.8}$$

If $I_1(g)<+\infty$, then as in the first case the required inequality follows from the inequality (5.8) by integration by parts. $\square$

From this general result we derive the following corollaries which will be useful later.

Corollary 5.2. (1) *For any complex euclidean ball $\mathbf{B}\subset\mathbf{C}^n$, any function $u\in\mathcal{L}(\mathbf{C}^n)$ and any $0<\alpha<2$,*

$$\frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}}e^{-\alpha u}\,d\lambda_{2n}\le\Bigl(1+c_n\frac{\alpha}{2-\alpha}\Bigr)\exp\Bigl(-\alpha\max_{\mathbf{B}}u\Bigr), \tag{5.9}$$

where c_n is the constant given by (3.2).

(2) *Let $\mathbf{G}\subset\mathbf{C}^n$ be a generic real subspace of complex dimension $m\le n-1$. Then for any real euclidean ball $B\subset\mathbf{G}$, any function $u\in\mathcal{L}(\mathbf{C}^n)$ and any $0<\alpha<1$,*

$$\frac{1}{\lambda_{n+m}(B)}\int_B e^{-\alpha u}\,d\lambda_{n+m}\le\Bigl(1+8(n+m)\frac{\alpha}{1-\alpha}\Bigr)\exp\Bigl(-\alpha\max_B u\Bigr). \tag{5.10}$$

Proof. (1) Indeed, it is enough to apply Theorem 5.1 with the increasing function $g(t):=e^{\alpha t}-1$, with $0<\alpha<2$ in the complex case and $0<\alpha<1$ in the real generic case. □

Corollary 5.3. (1) *For any complex euclidean ball* $\mathbf{B}\subset\mathbf{C}^n$, *any function* $u\in\mathcal{L}(\mathbf{C}^n)$ *and any real number* $p>0$,

$$\frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}}\Big(\max_{\mathbf{B}} u-u\Big)^p\,d\lambda_{2n}\le 2pc_n2^{-p}\Gamma(p+1), \tag{5.11}$$

where Γ *is the gamma function and* c_n *is the constant given by the formula* (3.2).

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $m\le n-1$. *Then for any real euclidean ball* $B\subset\mathbf{G}$, *any function* $u\in\mathcal{L}(\mathbf{C}^n)$ *and any real number* $p>0$,

$$\frac{1}{\lambda_{n+m}(B)}\int_{B}\Big(\max_{B} u-u\Big)^p\,d\lambda_{n+m}\le 8(n+m)p\Gamma(p+1). \tag{5.12}$$

Proof. Indeed, it is enough to apply Theorem 5.1 with the increasing function $g(t):=t^p$, $t\ge0$, which clearly satisfies the required conditions. □

Now we want to study the global behaviour of the Lelong class $\mathcal{L}(\mathbf{C}^n)$, estimating uniformly the size of the deviation between a function and its mean values on complex or real euclidean balls.

Let us recall the general definition of the space BMO. Let $\mathbf{G}$ be a real euclidean space of dimension $k\ge1$ and let λ_k be the Lebesgue measure on $\mathbf{G}$. For a locally integrable function $f:\mathbf{G}\to\overline{\mathbf{R}}$ and any euclidean ball $B\subset\mathbf{G}$, define the mean value of f on B by

$$f_B:=\frac{1}{\lambda_k(B)}\int_B f\,d\lambda_k.$$

Then we say that $f\in\mathrm{BMO}(\mathbf{G})$ if and only if

$$\|f\|_{\mathrm{BMO}(\mathbf{G})}:=\sup_B\frac{1}{\lambda_k(B)}\int_B|f-f_B|\,d\lambda_k<+\infty,$$

where the supremum is taken over all the euclidean balls $B\subset\mathbf{G}$.

Let us first prove the following result which can be considered as a quantitative version for the Lelong class $\mathcal{L}(\mathbf{C}^n)$ of the classical John–Nirenberg inequality for BMO-functions (see [St]).

Theorem 5.4. (1) *For any complex euclidean ball* $\mathbf{B}\subset\mathbf{C}^n$, *any function* $u\in\mathcal{L}(\mathbf{C}^n)$ *and any real number* $\alpha<2$,

$$\frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}}e^{\alpha|u-u_{\mathbf{B}}|}\,d\lambda_{2n}\le\Big(1+c_n\frac{\alpha}{2-\alpha}\Big)\exp\frac{\alpha c_n}{2}, \tag{5.13}$$

where $u_{\mathbf{B}}:=(1/\lambda_{2n}(\mathbf{B}))\int_{\mathbf{B}}u\,d\lambda_{2n}$ *and* c_n *is the constant given by* (3.2).

(2) *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $0\le m\le n-1$. *Then for any real euclidean ball* $B\subset\mathbf{G}$, *any function* $u\in\mathcal{L}(\mathbf{C}^n)$ *and any real number* $\alpha<1$,

$$(5.14)\qquad \frac{1}{\lambda_{n+m}(B)}\int_B e^{\alpha|u-u_B|}\,d\lambda_{n+m}\le\Big(1+8(n+m)\frac{\alpha}{1-\alpha}\Big)\exp 8\alpha(n+m),$$

where $u_B:=(1/\lambda_{n+m}(B))\int_B u\,d\lambda_{n+m}$.

Proof. (1) From Corollary 5.2, it follows that for a fixed function $u\in\mathcal{L}(\mathbf{C}^n)$ and any euclidean ball $\mathbf{B}\subset\mathbf{C}^n$,

$$(5.15)\qquad \frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}} e^{\alpha(\max_{\mathbf{B}} u-u)}\,d\lambda_{2n}\le 1+c_n\frac{\alpha}{2-\alpha}.$$

Now, from Corollary 5.3, we get

$$(5.16)\qquad \max_{\mathbf{B}} u-u_{\mathbf{B}}\le\tfrac{1}{2}c_n$$

Therefore by (5.15) and (5.16) we get

$$\frac{1}{\lambda_{2n}(\mathbf{B})}\int_{\mathbf{B}} e^{\alpha|u-u_{\mathbf{B}}|}\,d\lambda_{2n}\le\Big(1+c_n\frac{\alpha}{2-\alpha}\Big)e^{c_n\alpha/2}.$$

The real case is proved in the same way. □

Observe that in the complex case, a better estimate can be obtained using a refined version of the inequality (5.16) due to Lelong (see [L2], [D] and [Si2]).

From the last theorem we deduce the following result which is an effective version of a result by E. Stein (see [St], [B2]).

Corollary 5.5. *Let* $\mathbf{G}\subset\mathbf{C}^n$ *be a generic real subspace of complex dimension* $m\le n$. *Then for any function* $u\in\mathcal{L}(\mathbf{C}^n)$, $u|_{\mathbf{G}}\in\mathrm{BMO}(\mathbf{G})$ *and*

$$\|u\|_{\mathrm{BMO}(\mathbf{G})}\le\sigma_{n,m}.$$

In particular, for any polynomial $P\in\mathbf{C}[z]$, *with* $\deg P=d\ge1$,

$$(5.17)\qquad \big\|\log|P|\big\|_{\mathrm{BMO}(\mathbf{G})}\le\sigma_{n,m}d.$$

Here $\sigma_{n,m}:=2\log(1+8(n+m))+8(n+m)$ *if* $0\le m\le n-1$ *and* $\sigma_{n,n}:=\log(1+c_n)+\frac{1}{2}c_n$, *where* c_n *is the constant given by* (3.2).

In the totally real case where $\mathbf{G}=\mathbf{R}^n$, the existence of a (non-effective) uniform bound for the $\mathrm{BMO}(\mathbf{R}^n)$-norm of plurisubharmonic functions of logarithmic singularities on $\mathbf{C}^n$ was proved earlier by A. Brudnyi with a different proof (see [B1], [B2]). Our proof gives a precise quantitative estimate of the uniform bound.

6. Global uniform integrability of plurisubharmonic functions

Here we want to give a sufficient condition for global integrability of plurisubharmonic functions in terms of the relative Monge–Ampère capacity of their sublevel sets. Then we will deduce a global integrability theorem for the class of plurisubharmonic functions with uniformly bounded Monge–Ampère masses.

For any $u \in \mathrm{PSH}^-(\Omega)$ and any Borel subset $E \subset \Omega$ we define the truncated plurisubharmonic lemniscates associated to u as $E(s,u) := \{z \in E; u(z) < -s\}$ for $s > 0$, and the corresponding capacity function

$$c_E(s,u) = \mathrm{Cap}(E(s,u);\Omega).$$

Let $\mathcal{U} \subset \mathrm{PSH}^-(\Omega)$ be a class of plurisubharmonic functions on Ω and define

$$c_E(s,\mathcal{U}) := \sup\{c_E(s,u)\,;\, u \in \mathcal{U}\}, \quad s > 0.$$

Let $g\colon \mathbf{R}^+ \to \mathbf{R}^+$ be an increasing function such that $g(0)=0$ and $\lim_{t\to+\infty} g(t) = +\infty$. As in the last section, consider the following Riemann–Stieltjes' integral for $\delta > 0$,

$$I_\delta(g) := \int_0^{+\infty} e^{-\delta t}\, dg(t). \tag{6.1}$$

The main result of this section is the following theorem.

Theorem 6.1. *Let $\mathcal{U} \subset \mathrm{PSH}^-(\Omega)$ be a class of plurisubharmonic functions on Ω and $E \subset \Omega$ a Borel subset such that*

$$\eta = \eta(E;\mathcal{U}) := \sup_{s \geq 0} s\, c_E(s,\mathcal{U})^{1/n} < +\infty.$$

Then the following properties hold:

(1) *For any function $u \in \mathcal{U}$,*

$$\int_E g(-u)\, d\lambda_{2n} \leq c_n \tau_{2n}(E) I_{2/\eta}(g),$$

provided that $I_{2/\eta}(g) < +\infty$, where $\tau_{2n}(E)$ is the $2n$-volume of the smallest complex euclidean ball of $\mathbf{C}^n$ containing E and c_n is the constant given by (3.2).

(2) *Let $\mathbf{G} \subset \mathbf{C}^n$ be a generic real subspace of complex dimension $m \leq n-1$ such that $\Omega \cap \mathbf{G} \neq \emptyset$ and $E \subset \Omega \cap \mathbf{G}$. Then for any function $u \in \mathcal{U}$,*

$$\int_E g(-u)\, d\lambda_{n+m} \leq 8\big(1+\sqrt{2}\,\big)(n+m)\tau_{n+m}(E) I_{1/\eta}(g),$$

provided that $I_{1/\eta}(g) < +\infty$, where $\tau_{n+m}(E)$ is the $(n+m)$-volume of the smallest euclidean ball in $\mathbf{G}$ which contains E.

Proof. By approximation we can assume that g is strictly increasing. Let μ be any positive Borel measure on Ω and $u\in\mathrm{PSH}^-(\Omega)$. Then

$$\int_\Omega g(-u)\,d\mu=\int_0^{+\infty}\mu(\{z:u(z)\le -s\})\,dg(s). \tag{6.2}$$

Now let $\mu=\chi_E\lambda_{2n}$ and $\mathbf{B}$ be a complex euclidean ball of $\mathbf{C}^n$ containing E. Then by (3.24) we get

$$\lambda_{2n}(\{z\in E\,;u(z)\le -s\})\le c_n\lambda_{2n}(\mathbf{B})\exp(-2c_E(s,u)^{-1/n}).$$

Therefore from (6.2) we conclude that

$$\int_E g(-u)\,d\lambda_{2n}\le c_n\lambda_{2n}(\mathbf{B})\int_0^{+\infty}\exp(-2c_E(s,u)^{-1/n})\,dg(s). \tag{6.3}$$

From this and the hypothesis, we deduce that

$$\int_E g(-u)\,d\lambda_{2n}\le c_n\lambda_{2n}(\mathbf{B})\int_0^{+\infty}\exp(-2s/\eta)\,dg(s),$$

which proves the required estimate. The real generic case is proved in the same way. $\square$

From this result we can deduce the following corollaries.

Corollary 6.2. *Let $\mathcal{U}\subset\mathrm{PSH}^-(\Omega)$ be a class of plurisubharmonic functions on Ω and $E\subset\Omega$ be a Borel subset such that*

$$\eta=\eta(E;\mathcal{U}):=\sup_{s\ge 0} s c_E(s,\mathcal{U})^{1/n}<+\infty.$$

Then the following properties hold:

(1) *For any function $u\in\mathcal{U}$ and any exponent $0<\alpha<2/\eta$,*

$$\int_E e^{-\alpha u}\,d\lambda_{2n}\le\lambda_{2n}(E)+c_n\tau_{2n}(E)\frac{\alpha\eta}{2-\alpha\eta},$$

where $\tau_{2n}(E)$ is the $2n$-volume of the smallest complex euclidean ball of $\mathbf{C}^n$ containing E, and c_n is the constant given by (3.2).

(2) *Moreover if $\mathbf{G}\subset\mathbf{C}^n$ is a generic real subspace of complex dimension $m\le n-1$ such that $\Omega\cap\mathbf{G}\ne\emptyset$ and $E\subset\Omega\cap\mathbf{G}$, then for any function $u\in\mathcal{U}$ and any real number $0<\alpha<1/\eta$,*

$$\int_D e^{-\alpha u}\,d\lambda_{n+m}\le\lambda_{n+m}(D)+8(1+\sqrt{2})(n+m)\tau_{n+m}(D)\frac{\alpha\eta}{1-\alpha\eta},$$

where $\tau_{n+m}(D)$ is the $(n+m)$-volume of the smallest euclidean ball of $\mathbf{G}$ containing D.

From the last result we can also deduce the following consequence.

Corollary 6.3. *Let $\mathcal{U}\subset \mathrm{PSH}^-(\Omega)$ be a class of plurisubharmonic functions on Ω. Then the following properties hold*:

(1) *If*

$$\gamma := \limsup_{s\to+\infty} sc_\Omega(s,\mathcal{U})^{1/n} < +\infty,$$

then for any exponent $0<\alpha<2/\gamma$, there exists a constant $A_{2n}=A_{2n}(\alpha,\delta,\Omega,\mathcal{U})>0$ such that

$$\int_\Omega e^{-\alpha u}\,d\lambda_{2n} \le A_{2n}, \quad u\in\mathcal{U}.$$

(2) *If $\mathbf{G}\subset\mathbf{C}^n$ is a generic real subspace of complex dimension $m\le n-1$ such that $D:=\Omega\cap\mathbf{R}^n\ne\emptyset$ and*

$$\delta := \limsup_{s\to+\infty} sc_D(s,\mathcal{U})^{1/n} < +\infty,$$

then for any $\alpha<1/\delta$, there is a constant $A_{n,m}=A_{n,m}(\alpha,\delta,D,\mathcal{U})>0$ such that

$$\int_D e^{-\alpha u}\,d\lambda_{n+m} \le A_{n,m}, \quad u\in\mathcal{U}.$$

Proof. (1) If $\gamma<+\infty$, then for any $\alpha<2/\gamma$, there is $s_0>0$ and $\gamma_0>0$ such that $\alpha<2/\gamma_0$ and

$$sc_\Omega(s,u)^{1/n} \le \gamma_0 \quad \text{for all } s\ge s_0 \text{ and } u\in\mathcal{U}.$$

Then if we define the class $\mathcal{V}:=\mathcal{U}+s_0$, it follows that

$$tc_\Omega(t,v)^{1/n} \le \gamma_0 \quad \text{for all } t\ge 0 \text{ and } v\in\mathcal{V},$$

which implies that $\eta:=\eta(\Omega,\mathcal{V})\le\gamma_0$. Therefore, since $\alpha<2/\gamma_0\le 2/\eta$, we can apply Theorem 6.1 to the class $\mathcal{V}$ and get the estimate

$$\int_\Omega e^{-\alpha v}\,d\lambda_{2n} \le \lambda_{2n}(\Omega)+c_n\tau_{2n}(\Omega)\frac{\alpha\eta}{2-\alpha\eta}.$$

This inequality implies clearly that

$$\int_\Omega e^{-\alpha u}\,d\lambda_{2n} \le \lambda_{2n}(\Omega)+c_n\tau_{2n}(\Omega)e^{\alpha s_0}\frac{\alpha\eta}{2-\alpha\eta}, \quad u\in\mathcal{U},$$

which proves the first estimate of the theorem. The second estimate is proved in the same way. □

Now we will give an application of Corollary 6.2 to the global uniform integrability of the Cegrell class of plurisubharmonic functions of uniformly bounded Monge–Ampère mass on a bounded hyperconvex domain.

Corollary 6.4. (1) *For any* $\alpha<2$ *and any* $\varphi\in\mathcal{F}(\Omega)$ *with* $\int_\Omega (dd^c\varphi)^n\le 1$,

$$\int_\Omega e^{-\alpha\varphi(z)}\,d\lambda_{2n}(z)\le\lambda_{2n}(\Omega)+c_n\tau_{2n}(\Omega)\frac{\alpha}{2-\alpha}, \tag{6.4}$$

where c_n *is the constant given by* (3.2).

(2) *If* $\mathbf{G}\subset\mathbf{C}^n$ *is a generic real subspace of complex dimension* $m\le n-1$ *such that* $D:=\Omega\cap\mathbf{G}\ne\emptyset$, *then for any* $\alpha<1$ *and any* $\varphi\in\mathcal{F}(\Omega)$ *with* $\int_\Omega (dd^c\varphi)^n\le 1$,

$$\int_D e^{-\alpha\varphi(z)}\,d\lambda_{n+m}(z)\le\lambda_{n+m}(D)+8\big(1+\sqrt{2}\big)(n+m)\tau_{n+m}(D)\frac{\alpha}{1-\alpha}. \tag{6.5}$$

Proof. Consider the class $\mathcal{U}$ of plurisubharmonic functions $\varphi\in\mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c\varphi)^n\le 1$. By Lemma 4.4, we get the inequality $\eta=\eta(E,\mathcal{U})\le 1$ for any Borel subset $E\subset\Omega$. Hence the results above follows immediately from Corollary 6.2. □

A uniform estimate of type (6.4) was obtained recently in [CZ] with a different method and a non-explicit uniform constant, while the estimate (6.5) seems to be new.

As in Section 5, from Theorem 6.1 we can deduce uniform L^p estimates for functions from the class $\mathcal{F}(\Omega)$.

Corollary 6.5. (1) *For any* $\varphi\in\mathcal{F}(\Omega)$ *and any real number* $p>0$,

$$\int_\Omega(-\varphi)^p\,d\lambda_{2n}\le c_n\tau_{2n}(\Omega)2^{-p}\Gamma(p+1)\bigg(\int_\Omega (dd^c\varphi)^n\bigg)^{p/n},$$

where c_n *is the constant given by* (3.2).

(2) *If* $\mathbf{G}\subset\mathbf{C}^n$ *is a generic real subspace of complex dimension* $m\le n-1$ *such that* $D:=\Omega\cap\mathbf{G}\ne\emptyset$, *then for any* $\varphi\in\mathcal{F}(\Omega)$ *and any real number* $p>0$,

$$\int_D(-\varphi)^p\,d\lambda_{n+m}\le 8\big(1+\sqrt{2}\big)(n+m)\tau_{n+m}(D)\Gamma(p+1)\bigg(\int_\Omega (dd^c\varphi)^n\bigg)^{p/n}.$$

Proof. Indeed, by Lemma 4.4 the real number $\eta=\eta(E,\mathcal{U})$ for the class $\mathcal{U}$ of plurisubharmonic functions $\varphi\in\mathcal{F}(\Omega)$ such that $\int_\Omega (dd^c\varphi)^n\le 1$ and any subset $E\subset\Omega$ satisfies the inequality $\eta\le 1$. Since the function $I_\delta(g)$ is decreasing in δ, we easily see that the corollary is an easy consequence of Theorem 6.1 with the function $g(t)=t^p$. □

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References

[AT] Alexander, H. J. and Taylor, B. A., Comparison of two capacities in $\mathbf{C}^n$, *Math. Z.* **186** (1984), 407–417.

[BT1] Bedford, E. and Taylor, B. A., A new capacity for plurisubharmonic functions, *Acta Math.* **149** (1982), 1–40.

[BT2] Bedford, E. and Taylor, B. A., Plurisubharmonic functions with logarithmic singularities, *Ann. Inst. Fourier* (*Grenoble*) **38** (1988), 133–171.

[BJ] Benelkourchi, S. and Jennane, B., Intégrabilité uniforme semi-globale pour une classe de fonctions plurisousharmoniques, *C. R. Math. Acad. Sci. Paris* **337** (2003), 239–242.

[B1] Brudnyi, A., A Bernstein-type inequality for algebraic functions, *Indiana Univ. Math. J.* **46** (1997), 93–116.

[B2] Brudnyi, A., Local inequalities for plurisubharmonic functions, *Ann. of Math.* **149** (1999), 511–533.

[BG] Brudnyi, Yu. A. and Ganzburg, M. I., On an extremal problem for polynomials in n variables, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 344–355 (Russian). English transl.: *Math. USSR-Izv.* **7** (1973), 345–357.

[C1] Cegrell, U., Pluricomplex energy, *Acta Math.* **180** (1998), 187–217.

[C2] Cegrell, U., The general definition of the complex Monge–Ampère operator, *Ann. Inst. Fourier* (*Grenoble*) **54** (2004), 159–179.

[CZ] Cegrell, U. and Zeriahi, A., Subextension of plurisubharmonic functions with bounded Monge–Ampère mass, *C. R. Math. Acad. Sci. Paris* **336** (2003), 305–308.

[CDL] Cuyt, A., Driver, K. A. and Lubinsky, D. S., On the size of lemniscates of polynomials in one and several variables, *Proc. Amer. Math. Soc.* **124** (1996), 2123–2136.

[D] Demailly, J.-P., Potential theory in several complex variables, `http://www-fourier.ujf-grenoble.fr/~demailly/lectures.html`.

[FG] Favre, C. and Guedj, V., Dynamique des applications rationnelles des espaces multiprojectifs, *Indiana Univ. Math. J.* **50** (2001), 881–934.

[Ki] Kiselman, C. O., Ensembles de sous-niveau et images inverses des fonctions plurisousharmoniques, *Bull. Sci. Math.* **124** (2000), 75–92.

[Kl] Klimek, M., *Pluripotential Theory*, Oxford Univ. Press, New York, 1991.

[K1] Kolodziej, S., The complex Monge–Ampère equation, *Acta Math.* **180** (1998), 69–117.

[K2] Kolodziej, S., Equicontinuity of families of plurisubharmonic functions with bounds on their Monge–Ampère masses, *Math. Z.* **240** (2002), 835–847.

[L1] Lelong, P., Théorème de Banach–Steinhaus pour les polynômes; applications entières d'espaces vectoriels complexes, in *Séminaire P. Lelong* (*Analyse*), Lecture Notes in Math. **205**, pp. 87–112, Springer-Verlag, Berlin, 1971.

[L2] Lelong, P., Mesures de Malher et calcul de constantes universelles pour les polynômes de N variables, *Math. Ann.* **299** (1994), 673–695.

[Lu] LUNDIN, M., The extremal PSH for the complement of convex, symmetric subsets of $\mathbf{R}^n$, *Michigan Math. J.* **32** (1985), 197–201.

[P] PLESNIAK, W., Volume of polynomial lemniscates in $\mathbf{C}^n$, *Numer. Algorithms* **33** (2003), 415–420.

[R] RANSFORD, T., *Potential Theory in the Complex Plane*, Cambridge Univ. Press, Cambridge, 1995.

[S1] SADULLAEV, A., An estimate for polynomials on analytic sets, *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), 524–534, 671 (Russian). English transl.: *Math. USSR-Izv.* **20** (1983), 493–502.

[S2] SADULLAEV, A., The extremal plurisubharmonic function of the unit ball $\overline{B}\subset\mathbf{R}^n$, *Ann. Polon. Math.* **46** (1985), 433–437 (Russian).

[Si1] SICIAK, J., Extremal plurisubharmonic functions in $\mathbf{C}^n$, *Ann. Polon. Math.* **39** (1981), 175–211.

[Si2] SICIAK, J., *Extremal plurisubharmonic functions and capacities in* $\mathbf{C}^N$, Sophia Kokyuroku in Math. **14**, Sophia University, Tokyo, 1982.

[St] STEIN, E. M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. **43**, Princeton Univ. Press, Princeton, NJ, 1993.

[T] TSUJI, M., *Potential Theory in Modern Function Theory*, Chelsea, New York, 1975.

[Z] ZAHARIUTA, V. A., Extremal plurisubharmonic functions, orthogonal polynomials and Bernstein–Walsh theorem for analytic functions of several variables, *Ann. Polon. Math.* **33** (1976), 137–148 (Russian).

[Z1] ZERIAHI, A., A criterion of algebraicity for Lelong classes and analytic sets, *Acta Math.* **184** (2000), 113–143.

[Z2] ZERIAHI, A., Volume and capacity of sublevel sets of a Lelong class of plurisubharmonic functions, *Indiana Univ. Math. J.* **50** (2001), 671–703.

[Z3] ZERIAHI, A., The size of plurisubharmonic lemniscates in terms of Hausdorff–Riesz measures and capacities, *Proc. London Math. Soc.* **89** (2004).

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