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A PALAIS–SMALE APPROACH TO SOBOLEV SUBCRITICAL OPERATORS

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Abstract. In this article, we use Palais–Smale approaches to describe the achieved and nonachieved domains. We characterizes the achieved domain by the existence of a ground state solution for the energy functional J in Ω.

1. Introduction

By a reaction-diffusion system we mean a system of partial differential equations of the form

$$u_t = Du_{xx} + f(u), \quad U \in \mathbb{R}^N,$$

where D is a nonnegative, diagonal matrix, and $f\colon \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be smooth. These systems arise as models in various areas of mathematical biology and chemistry. Example include Fisher's equation in population genetics, the Hodgkin–Huxley equations as a model for the propagation of electrical impulses in a nerve axon, models for interacting species in ecology, and models for laminar flames in combustion. See Smoller ([18]) and Fife ([11]) and the reference cited there for a detailed description of there and other models.

Such a problem is very difficult. Even in the following steady state single equation, there is still a lot of open questions. In this article, we prove partially an open question.

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Let $N \geq 1$ and $2 < p < 2^*$, where $2^* = 2N/(N-2)$ for $N \geq 3$, $2^* = \infty$ for $N = 1, 2$. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where Ω is a domain in $\mathbb{R}^N$ and $H_0^1(\Omega)$ is the Sobolev space in Ω. For the general theory of the Sobolev space $H_0^1(\Omega)$, see Adams ([1]). Corresponding to equation (1), let the energy functionals a, b, and J in $H_0^1(\Omega)$ be given by

$$\begin{aligned} a(u) &= \int_\Omega (|\nabla u|^2 + u^2, \quad b(u) = \int_\Omega |u|^p, \\ J(u) &= \frac{1}{2}a(u) - \frac{1}{p}b(u). \end{aligned}$$

By Rabinowitz ([17, Proposition B.10]), a, b, and J are of class $C^{1,1}$.

Sobolev spaces $H_0^1(\Omega)$ provide the proper functional setting for the study of the partial differential equations and Sobolev imbedding theorems (Sobolev operators) provide the connection between Sobolev spaces and Lebesgue spaces.

(i) Standard books describe that the Sobolev critical operator $I\colon H_0^1(\Omega) \to L^{2^*}(\Omega)$ satisfies

$$\|u\|_{L^{2^*}} \leq c\|\nabla u\|_{L^2}.$$

Let $S^c(\Omega)$ be the best constant of the Sobolev critical operator,

$$S^c(\Omega) = \sup\left\{\frac{\|u\|_{L^{2^*}}}{\|\nabla u\|_{L^2}} \;\middle|\; u \in H_0^1(\Omega) \setminus \{0\}\right\}.$$

Then $S^c(\Omega)$ is independent of Ω, $S^c(\Omega)$ is achieved if and only if $\Omega = \mathbb{R}^N$.

(ii) Standard books also describe that the Sobolev subcritical operator I: $H_0^1(\Omega) \to L^p(\Omega)$ satisfies

$$\|u\|_{L^p(\Omega)} \leq c\|u\|_{H^1(\Omega)}. \tag{2}$$

Let $S(\Omega)$ be the best constant of the Sobolev subcritical operator,

$$S(\Omega) = \sup\left\{\frac{\|u\|_{L^p(\Omega)}}{\|u\|_{H^1(\Omega)}} \;\middle|\; u \in H_0^1(\Omega) \setminus \{0\}\right\}.$$

No standard books describe that for which domain Ω, $S(\Omega)$ is achieved.

For the convenience, an achieved domain is defined as follows.

Definition 1. We call that a domain Ω in $\mathbb{R}^N$ is an *achieved domain* if there is $u \in H_0^1(\Omega)$ such that $\|u\|_{L^p(\Omega)}/\|u\|_{H^1(\Omega)} = S(\Omega)$. Otherwise, we call that Ω is a *nonachieved domain*.

The open question is that for which domain Ω is $S(\Omega)$ achieved? Due to the lack of compactness, it is difficult to study the existence of solutions of equation (1) in unbounded domains. In this direction, the concentration-compactness principle of P. L. Lions ([2], [14] and [15]) made a breakthrough. Moreover, in 1982, Esteban and Lion ([10]) asserted that an Esteban–Lions domain is a nonachieved domain, where an Esteban–Lions domain is defined as follows.

Definition 2. We call that a proper unbounded domain Ω in $\mathbb{R}^N$ is an *Esteban–Lions domain* if there is $\chi \in \mathbb{R}^N$, $\|\chi\| = 1$ such that $n(x) \cdot \chi \geq 0$, $n(x) \cdot \chi \not\equiv 0$ on $\partial\Omega$, where $n(x)$ denotes the unit outward normal to $\partial\Omega$ at the point x.

After the Lions papers, there are nice tools from the books of Chabrowski (see [5]), Mawhin and Willem ([16]), Struwe ([19]), Willem ([22]). We have done some new analyses such as at Chen and Wang ([6]), Lien, Tzeng and Wang ([13]), and Wang ([21]). Now we are in the position to solve partially this open question. The paper is organized as follows: Section 2 defines the index of a domain. Section 3 presents some analyses. Section 4 describes various nonachieved domains. Section 5 describes achieved domains.

2. Indexes of domains

We define the (PS)-sequences for J.

Definition 3.

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ in $H_0^1(\Omega)$ is a $(\mathrm{PS})_\beta$*-sequence for* J if $J(u_n) \to \beta$ and $J'(u_n) \to 0$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$,

(ii) $\beta \in \mathbb{R}$ is a (PS)-*value for* J if there is a $(\mathrm{PS})_\beta$-sequence for J,

(iii) J satisfies the $(\mathrm{PS})_\beta$-condition if every $(\mathrm{PS})_\beta$-sequence for J contains a convergent subsequence,

(iv) J satisfies the (PS)-condition if J satisfies the $(\mathrm{PS})_\beta$-condition for every $\beta \in \mathbb{R}$.

A $(\mathrm{PS})_\beta$-sequence for J is bounded.

Lemma 4. *Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for J, then there is a positive bounded sequence $\{c_n(\beta)\}$ such that $\|u_n\|_{H^1} \leq c_n(\beta) \leq c$ for each n and $c_n(\beta) = \mathrm{o}(1)$ as $n \to \infty$ and $\beta \to 0$. Furthermore, $a(u_n) = b(u_n) + \mathrm{o}(1) = 2p\beta/(p-2) + \mathrm{o}(1)$ and $\beta \geq 0$.*

Proof. See Willem [22]. □

Consider the following four important positive values.

(i) Consider the constrained value $\alpha_\theta = (1/2 - 1/p) S(\Omega)^{2p/(2-p)}$, where

$$S(\Omega) = \sup\{\|u\|_{L^p(\Omega)} \mid u \in H_0^1(\Omega),\ a(u) = 1\}.$$

Clearly, α_θ is a positive value.

(ii) Consider the Nehari value $\alpha_{\mathbf{M}} = \inf_{u\in\mathbf{M}(\Omega)} J(u)$, where

$$\mathbf{M}(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b(u)\}.$$

As a consequence of the following lemma, $\alpha_{\mathbf{M}}$ is a positive value.

Lemma 5. *Let $\mathbf{U}(\Omega) = \{u \in H_0^1(\Omega) \mid \|u\|_{H^1} = 1\}$ be the unit sphere. Then there is a bijective $C^{1,1}$-map m from $\mathbf{U}(\Omega)$ to $\mathbf{M}(\Omega)$. Moreover, $\mathbf{M}(\Omega)$ is path-connected and there exists a constant $c > 0$ such that for $u \in \mathbf{M}(\Omega)$, $\|u\|_{H^1} \geq c$ and $J(u) \geq c$.*

Proof. See Chen and Wang ([6, Lemma 2.2]). □

(iii) Consider the minimax value $\alpha_\Gamma = \inf_{v\in\Gamma} \max_{t\in[0,1]} J(v(t))$, where

$$\Gamma = \{v \in C([0,1], H_0^1(\Omega)) \mid v(0) = 0, v(1) = e\} \quad \text{and} \quad J(e) = 0.$$

α_Γ is a positive value since J satisfies the mountain pass hypothesis: that is, there are r, $\delta > 0$ and $e \in H_0^1(\Omega)$ such that $e \notin \overline{B(0;r)}$, $J(e) = 0$, $J(u) \geq \delta > 0$ for each $u \in \partial B(0;r)$.

(iv) Consider the minimal value $\alpha_P = \inf_{\beta\in P(\Omega)} \beta$, where $P(\Omega)$ is the set of all positive (PS)-values for J in Ω. As a consequence of the following lemma, α_P is a positive value.

Lemma 6. *There is a $\beta_0 > 0$ such that $\beta \geq \beta_0$ for every positive* (PS)-*value β.*

Proof. Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(\mathrm{PS})_\beta$-sequence for J for $\beta > 0$. By Lemma 4, $a(u_n) \leq c_n(\beta)^2$. By the Sobolev embedding theorem, there is a constant $d > 0$ such that

$$b(u_n) \leq d a(u_n)^{p/2}.$$

By the above two inequalities, we have

$$\mathrm{o}(1) = a(u_n) - b(u_n) \geq a(u_n)[1 - dc_n(\beta)^{p-2}].$$

Take $\beta_0 > 0$ and $n_0 > 0$ such that if $\beta < \beta_0$ and $n \geq n_0$, then $1 - dc_n(\beta)^{p-2} > 1/2$. Consequently, $a(u_n) = b(u_n) + \mathrm{o}(1) = \mathrm{o}(1)$. Thus, $\beta = 0$, a contradiction. □

We find a couple of positive (PS)-values for J.

Lemma 7. *α_θ, $\alpha_{\mathbf{M}}$, α_Γ and α_P are positive* (PS)-*values for J.*

Proof. (i) By Lien, Tzeng nad Wang ([13, Theorem 2.1]), α_θ is a positive (PS)-value for J.

(ii) By Stuart ([20, Lemma 3.4]), $\alpha_{\mathbf{M}}$ is a positive (PS)-value for J. By Chen and Wang ([6, Lemma 2.1]), every minimizing sequence for $\alpha_{\mathbf{M}}$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}}$-sequence for J.

(iii) By Brezis and Nirenberg ([3]), α_Γ is a positive (PS)-value for J.

(iv) For each $n \in \mathbb{N}$, take $\{u_n\}$ in $H_0^1(\Omega)$ and $\beta_n \in P(\Omega)$ such that

$$|\beta_n - \alpha_P| < \frac{1}{2n}, \quad |J(u_n) - \beta_n| < \frac{1}{2n}, \quad \|J'(u_n)\|_{H^{-1}} < \frac{1}{2n}.$$

Then $J(u_n) = \alpha_P + \mathrm{o}(1)$ and $J'(u_n) = \mathrm{o}(1)$ strongly in $H^{-1}(\Omega)$. Thus, $\alpha_P \in P(\Omega)$. $\square$

In the following, we present a comparison lemma.

Lemma 8. *Let $\{u_n\} \subset H_0^1(\Omega)$ be a* (PS)$_\beta$*-sequence for J with $\beta > 0$. Then $\beta \geq \alpha_\theta$, $\beta \geq \alpha_{\mathbf{M}}$, $\beta \geq \alpha_\Gamma$ and $\beta \geq \alpha_P$.*

Proof. By Wang ([21, Lemma 9]), $\beta \geq \alpha_\theta$, $\beta \geq \alpha_{\mathbf{M}}$ and $\beta \geq \alpha_\Gamma$. Clearly, $\beta \geq \alpha_P$. $\square$

By Lemmas 7 and 8, we have the following interesting result.

Theorem 9. *Four important* (PS)*-values are equal $\alpha_\theta = \alpha_{\mathbf{M}} = \alpha_\Gamma = \alpha_P$.*

Remark. For the equality of the three important (PS)-values α_θ, $\alpha_{\mathbf{M}}$, and α_Γ, see Willem [22].

Definition 10. By Theorem 9, the positive (PS)-values α_θ, α_Γ, $\alpha_{\mathbf{M}}$ and α_P for J are the same. Any one of them is called the index of J in Ω and denoted by $\alpha(\Omega)$ (simply by α). By the definition of $\alpha_{\mathbf{M}}$, if u is a nonzero solution of equation (1), then $J(u) \geq \alpha$. Follows from Berestycki and Lions ([3]), we call that a solution u of equation (1) is a ground state solution if $J(u) = \alpha$ and is a higher energy solution if $J(u) > \alpha$.

3. Analyses

In this section, we presents analyses used for later sections.

By Lemma 4, a (PS)$_\alpha$-sequence for J possesses a weak limit u. Such u is a solution of equation (1).

Theorem 11. *Let $\{u_n\}$ in $H_0^1(\Omega)$ be a* (PS)$_\alpha$*-sequence for J satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then*

(i) *u is a solution of equation* (1)*.*
(ii) *If u is nonzero, then u is a positive ground state solution of equation* (1)*. Furthermore, we have $u_n \to u$ strongly in $H_0^1(\Omega)$, or the* (PS)$_\alpha$*-condition holds.*
(iii) *Suppose that the* (PS)$_\alpha$*-condition holds, then there is a positive ground state solution of equation* (1)*.*

Proof. (i) Clearly.

(ii) Since u is nonzero solution, by part (i) we have $u \in \mathbf{M}(\Omega)$. By Lemma 4, we have

$$a(u_n) = \frac{2p}{p-2}\alpha + \mathrm{o}(1). \tag{3}$$

Since a is weakly lower semicontinuous, we have

$$\alpha \le J(u) = \left(\frac{1}{2} - \frac{1}{p}\right)a(u) \le \left(\frac{1}{2} - \frac{1}{p}\right)\liminf_{n\to\infty} a(u_n) = \alpha,$$

or $J(u) = \alpha$. By the Lagrange multiplier theorem, it is known that every minimizer of the problem $\alpha = \inf_{u\in\mathbf{M}(\Omega)} J(u)$ is a critical point of J. Since $J(|u|) = J(u)$ and by the maximum principle, then $|u|$ is also a positive solution of equation (1). Thus we may assume that u is positive. Let $p_n = u_n - u$, then $\{p_n\}$ is a (PS)-sequence for J:

$$J(p_n) = J(u_n) - J(u) + \mathrm{o}(1) = \mathrm{o}(1), \quad J'(p_n) = \mathrm{o}(1).$$

Similarly to (3), we have

$$a(p_n) = \frac{2p}{p-2}J(p_n) + \mathrm{o}(1) = \mathrm{o}(1),$$

Thus $u_n \to u$ strongly in $H_0^1(\Omega)$.

(iii) Suppose that the $(\mathrm{PS})_\alpha$-condition holds, then there is a subsequence $\{u_n\}$ for J satisfying $u_n \to u$ strongly in $H_0^1(\Omega)$. Thus $J(u) = \alpha$. Hence, by part (ii), there is a positive ground state solution of equation (1). □

We need the Lions concentration-compactness principle as follows. Let

$$Q_n(t) = \sup_{y\in\mathbb{R}^N} \int_{y+B^N(0;t)} |u_n(x)|^2\,dx,$$

where $B^N(0;t)$ is the ball with the center at 0 and the radius t, we have

Lemma 12. *Let $\{u_n\}$ be bounded in $H^1(\Omega)$ where Ω is unbounded and for some $t_0 > 0$, $Q_n(t_0) \to 0$. Then*

(i) *$u_n \to 0$ strongly in $L^q(\Omega)$ for $2 < q < 2^*$.*

(ii) *If in addition u_n satisfies*

$$-\nabla u_n + u_n - |u_n|^{p-1}u_n = \varepsilon_n \to 0 \quad \text{in } H^{-1}(\Omega)$$

then $u_n \to 0$ strongly in $H^1(\Omega)$.

Proof. By Lions [14]. □

Let Ω_1 and Ω_2 be two domains in $\mathbb{R}^N$, $\alpha_i = \alpha(\Omega_i)$ the index of J in Ω_i and $\mathbf{M}_i = \{u \in H_0^1(\Omega_i) \setminus \{0\} \mid a(u) = b(u)\}$, where $i = 1, 2$.

Theorem 13. *Let* $\Omega_1 \subsetneqq \Omega_2$ *and* $J\colon H_0^1(\Omega_2) \to \mathbb{R}$ *be the energy functional. If* J *satisfies the* $(\mathrm{PS})_{\alpha_1}$*-condition or in particular* α_1 *is a critical value, then* $\alpha_2 < \alpha_1$.

Proof. $\Omega_1 \subset \Omega_2$, we have $H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$ and $\mathbf{M}_1 \subset \mathbf{M}_2$, for $i = 1, 2$. Thus, $\alpha_2 \le \alpha_1$. Suppose that J satisfies the $(\mathrm{PS})_{\alpha_1}$-condition, then there exists $u_0 \in \mathbf{M}_1$ such that $u_0 \ge 0$ and $J(u_0) = \alpha_1$. On the contrary, assume $\alpha_2 = \alpha_1$, then $J(u_0) = \alpha_2 = \inf_{u\in\mathbf{M}_2} J(u)$. By the Lagrange multiplier theorem, it is known that every minimizer of the problem $\alpha_2 = \inf_{u\in\mathbf{M}_2} J(u)$ is a critical point of J. Therefore, u_0 solves equation (1) in Ω_2. By the maximum principle, $u_0 > 0$ in Ω_2. This contradicts to that $u_0 \in H_0^1(\Omega_1)$. Therefore, $\alpha_2 < \alpha_1$. $\square$

Let $\Omega_0 = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2$ is bounded. Since $H_0^1(\Omega_i) \subset H_0^1(\Omega_0)$ and $\mathbf{M}_i \subset \mathbf{M}_0$, for $i = 1, 2$, we have $\alpha_0 \le \min\{\alpha_1, \alpha_2\}$. According to Del Pino and Felmer ([8], [9]), let

$$\begin{aligned}
\widetilde{\Omega}_n &= \Omega_0 \setminus \overline{B^N(0;n)},\\
\widetilde{\mathbf{M}}_n &= \{u \in H_0^1(\widetilde{\Omega}_n) \setminus \{0\} \mid a(u) = b(u)\},\\
\widetilde{\alpha}_n &= \alpha(\widetilde{\Omega}_n) = \inf_{u\in\widetilde{\mathbf{M}}_n} J(u).
\end{aligned}$$

Theorem 14. *The following five properties are equivalent.*

(i) *There is a* $(\mathrm{PS})_{\alpha_0}$*-sequence* $\{u_n\}$ *in* $H_0^1(\Omega_0)$ *for* J *such that* $u_n \rightharpoonup 0$ *weakly in* $H_0^1(\Omega_0)$.

(ii) *There is a* $(\mathrm{PS})_{\alpha_0}$*-sequence* $\{u_n\}$ *for* J *such that*

$$\int_{\Omega_n} |u_n|^p = \mathrm{o}(1), \quad \textit{where } \Omega_n = \Omega_0 \cap B^N(0;n).$$

(iii) $\alpha_0 = \min\{\alpha_1, \alpha_2\}$.

(iv) J *does not satisfy the* $(\mathrm{PS})_{\alpha_0}$*-condition.*

(v) $\alpha_0 = \widetilde{\alpha}_n$ *for each* $n \in \mathbb{N}$.

Proof. See Chen, Lin and Wang ([7, Theorem 23]). $\square$

4. Nonachieved domains

Let

$$\begin{aligned}
B^N(z;s) &= \{x \in \mathbb{R}^N \mid |x - z| < s\},\\
\mathbf{A}^r &= \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_1^2 + \dots + x_{N-1}^2 < r^2\},\\
\mathbf{A}^r_{s,t} &= \{(x_1, \dots, x_N) \in \mathbf{A}^r \mid s < x_N < t\},\\
\mathbf{A}^r_s &= \{(x_1, \dots, x_N) \in \mathbf{A}^r \mid s < x_N\},\\
\widetilde{\mathbf{A}^r_t} &= \{(x_1, \dots, x_N) \in \mathbf{A}^r \mid x_N < t\},\\
&\mathbf{A}^r \setminus \omega, \ \text{where } \omega \subset \mathbf{A}^r \text{ is a bounded domain,}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^{r_1,r_2} &= \{(x_1,\ldots,x_N)\in\mathbb{R}^N \mid r_1^2 < x_1^2+\ldots+x_{N-1}^2 < r_2^2\},\\
\mathbf{A}^{r_1,r_2}_s &= \{(x_1,\ldots,x_N)\in\mathbf{A}^{r_1,r_2} \mid s < x_N\},\\
\mathrm{C} &= \{(x_1,\ldots,x_N)\in\mathbb{R}^N \mid (x_1^2+\ldots+x_{N-1}^2)^{1/2} < x_N\},\\
\mathbb{R}^N_+ &= \{(x_1,\ldots,x_N)\in\mathbb{R}^N \mid x_N > 0\},\\
\mathrm{P} &= \{(x_1,\ldots,x_N)\in\mathbb{R}^N \mid x_1^2+\ldots+x_{N-1}^2 < x_N\}.
\end{aligned}$$

Esteban and Lions ([10, Theorem I.1]) proved the following:

Theorem 15. *Equation* (1) *in an Esteban–Lions domain* Ω *does not admit any nontrivial solution. In particular, equation* (1) *in either* $\mathbb{R}^N_+$ *or* $\mathbf{A}^r_s$ *does not admit any nontrivial solution.*

Theorem 16. *An Esteban–Lions domain is a nonachieved domain.*

Proof. By Theorems 22 and 15. □

Definition 17.

(i) We say that Ω is a *large subdomain of* $\mathbb{R}^N$ if for any $r > 0$ there exists $x\in\Omega$ such that $B^N(x;r)\subset\Omega$.
(ii) We call that Ω is a *large subdomain of* $\mathbf{A}^r$ if for any positive number m, there exist $s < t$ such that $t - s = m$ and $\mathbf{A}^r_{s,t}\subset\Omega$.

Example 18. The upper half strip $\mathbf{A}^r_s$, the upper half hollow strip $\mathbf{A}^{r_1,r_2}_s$ and the upper half space $\mathbb{R}^N_+$ are Esteban–Lions domains.

Example 19. The infinite cone C, the upper half space $\mathbb{R}^N_+$, and the paraboloid P are large subdomains of $\mathbb{R}^N$.

Example 20. $\mathbf{A}^r_s$ and $\mathbf{A}^r_s\setminus\omega_1$ are large subdomains of $\mathbf{A}^r$, and $\mathbf{A}^{r_1,r_2}_s$ and $\mathbf{A}^{r_1,r_2}_s\setminus\omega_2$ are large subdomains of $\mathbf{A}^{r_1,r_2}$, where $\omega_1\subset\mathbf{A}^r$ and $\omega_2\subset\mathbf{A}^{r_2}$ are bounded domains.

Theorem 21. *Let* Ω_2 *be either* $\mathbf{A}^r$, $\mathbf{A}^{r_1,r_2}$, *or* $\mathbb{R}^N$ *and* Ω_1 *a proper large subdomain of* Ω_2. *Then* $\alpha_1=\alpha_2$, *J does not satisfy the* $(\mathrm{PS})_{\alpha_1}$*-condition, and the only possible solutions of equation* (1) *in* Ω_1 *are higher energy solutions. In particular, a proper large subdomain* Ω_1 *of* Ω_2 *is nonachieved.*

Proof. It suffices to prove the case $\Omega_2=\mathbb{R}^N$. Since $\Omega_1\subset\Omega_2$, we have $\alpha_2\le\alpha_1$. Let $u\in H^1_0(\Omega_2)$ be a minimizer of α_2: $a(u)=b(u)$, $J(u)=\alpha_2$. Choose $\{x_n\}\subset\Omega_1$, $r_n\to\infty$, and $B^N(x_n;r_n)\subset\Omega_1$. Consider the cut-off function $\eta\in C^\infty_c([0,\infty))$ such that

$$0\le\eta\le1,\quad \eta(t)=\begin{cases}1 & \text{for } t\in[0,1],\\ 0 & \text{for } t\in[2,\infty).\end{cases}$$

Define

$$v_n(x) = \eta\left(\frac{2|x - x_n|}{r_n}\right) u(x - x_n).$$

Then

$$v_n(x) \in H_0^1(\Omega_1), \quad a(v_n) = a(u) + \mathrm{o}(1), \quad b(v_n) = b(u) + \mathrm{o}(1).$$

By Lemma 5, there exists $s_n > 0$ such that $a(s_n v_n) = b(s_n v_n)$. Therefore we have $s_n = 1 + \mathrm{o}(1)$. Then

$$J(s_n v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) s_n^2 a(v_n) = \left(\frac{1}{2} - \frac{1}{p}\right) a(u) + \mathrm{o}(1) = \alpha_2 + \mathrm{o}(1).$$

Therefore, $\alpha_1 \leq \alpha_2$. We conclude that $\alpha_1 = \alpha_2$. Then by Theorem 13, J does not satisfy the $(\mathrm{PS})_{\alpha_1}$-condition and the only possible solutions of equation (1) in Ω_1 are higher energy solutions. □

5. Achieved domains

We characterize achieved domains by the existence of a ground state solution for J in Ω.

Theorem 22. *Ω is an achieved domain if and only if there is a ground state solution for J in Ω.*

Proof. Recall that $\alpha(\Omega) = (1/2 - 1/p) S(\Omega)^{2p/(2-p)}$. Suppose that there is a $u \in H_0^1(\Omega)$ such that

$$J(u) = \alpha(\Omega), \quad \langle J(u), u\rangle = a(u) - b(u) = 0.$$

Then we have $a(u)^{(1/p - 1/2)} = S(\Omega)$. Let $v = u/\|u\|_{H^1}$. Then

$$\|v\|_{L^p} = \frac{b(u)^{1/p}}{a(u)^{1/2}} = a(u)^{1/p - 1/2} = S(\Omega).$$

Thus, $S(\Omega)$ is achieved by v. On the other hand, let $S(\Omega)$ be achieved by some function u where $a(u) = \|u\|_{H^1}^2 = 1$ and $b(u) = \|u\|_{L^p}^p = S(\Omega)^p$. By the Lagrange multiplier theorem there is a λ such that

$$b'(u) = \lambda a'(u).$$

It is easy to see that $\lambda = (p/2) S(\Omega)^p$, so we have

$$b'(u) = \frac{p}{2} S(\Omega)^p a'(u).$$

This implies

$$S(\Omega)^{-p}\left(\int |u|^{p-2} u\varphi\right) = \left(\int \nabla u \nabla \varphi + u\varphi\right).$$

Thus, u is a weak solution of

$$-\Delta u + u = S(\Omega)^{-p} |u|^{p-2} u.$$

Let $v = S(\Omega)^{p/(2-p)}u$, then $-\Delta v + v = |v|^{p-2}v$. We have $a(v) = b(v) = S(\Omega)^{2p/(2-p)}$, $\langle J'(v), \varphi\rangle = 0$ for each $\varphi \in C_c^\infty(\Omega)$, and

$$J(v) = (1/2 - 1/p)S(\Omega)^{2p/(2-p)} = \alpha(\Omega). \qquad \square$$

Remark. Note that if u is a ground state solution for J in Ω, then u solves the semilinear elliptic equation (1) and $J(u) = \alpha(\Omega)$. By the Kato regularity, L^p-regularity and Schauder regularity, the ground state solution u of equation (1) is classical.

A bounded domain Ω is an achieved domain.

Theorem 23. *A bounded domain Ω is an achieved domain.*

Proof. This is well-known. For the convenience of readers, we give the proof. Let $\{u_n\}$ be a minimizing sequence for $\alpha(\Omega)$. Since $\alpha(\Omega) = \alpha_M(\Omega)$, by Chen and Wang (see [6, Lemma 2.1]), $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_{\mathbf{M}}}$-sequence for J such that

$$J(u_n) = \alpha(\Omega) + \mathrm{o}(1), \quad a(u_n) = b(u_n) + \mathrm{o}(1).$$

Take a subsequence $\{u_n\}$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_n &\to u \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Suppose $u = 0$, then $a(u_n) = b(u_n) = \mathrm{o}(1)$. Thus, $J(u_n) = \mathrm{o}(1)$, a contradiction. By Theorem 11, u is a ground state solution of J in Ω. By Theorem 22, Ω is an achieved domain. $\square$

Definition 24. A domain in $\mathbb{R}^N$ is *periodic* if there exist a partition $\{P_n\}$ of Ω and points $\{y_n\}$ in $\mathbb{R}^N$ satisfying the following conditions:

(i) $\{y_n\}$ forms a subgroup of $\mathbb{R}^N$,
(ii) P_0 is bounded,
(iii) $P_n = y_n + P_0$.

Typical examples of periodic domains are the infinite strip $\mathbf{A}^r$, the infinite hollow strip $\mathbf{A}^{r_1,r_2}$, and the whole space $\mathbb{R}^N$. In Theorem 11, we proved that if a $(\mathrm{PS})_\alpha$-sequence for J admits a nonzero weak limit u, then u is a ground state solution for J. However, though the weak limit is zero but if the domain is periodic, then we can still obtain a ground state solution for J.

Theorem 25. *A periodic domain Ω is an achieved domain. In particular, the infinite strip $\mathbf{A}^r$, the infinite hollow strip $\mathbf{A}^{r_1,r_2}$ and the whole space $\mathbb{R}^N$ are achieved domains.*

PROOF. It suffices to prove the case $\Omega = \mathbf{A}^r$. Let $\{u_n\}$ in $H_0^1(\mathbf{A}^r)$ be a $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-sequence for J

$$J(u_n) = \alpha(\mathbf{A}^r) + \mathrm{o}(1), \quad J'(u_n) = \mathrm{o}(1) \quad \text{strongly in } H^{-1}(\mathbf{A}^r).$$

By Lemma 4, $\|u_n\|_{H^1} \le c$ for each n. There is a subsequence $\{u_n\}$ and $u \in H_0^1(\mathbf{A}^r)$ such that $u_n \rightharpoonup u$ weakly in $H_0^1(\mathbf{A}^r)$.

Suppose that u is nonzero, then by Theorem 11, we are done. Suppose that u is zero, then $u_n \not\to 0$ strongly in $H_0^1(\mathbf{A}^r)$. Otherwise, $\alpha(\mathbf{A}^r) = 0$, a contradiction. Since $u_n \not\to 0$ strongly in $H_0^1(\mathbf{A}^r)$, apply Lemma 12 to the infinite strip $\mathbf{A}^r$, there is a subsequence $\{u_n\}$, a constant $d > 0$ such that

$$Q_n(1) = \sup_{y\in\mathbb{R}} \int_{(0,y)+L} |u_n(z)|^2\,dz > d > 0 \quad \text{for } n \in \mathbb{N},$$

where $L = \{(x,y) \in \mathbf{A}^r \mid 0 \le y \le 1\}$. Take $\{z_n = (0, y_n)\}$ in $\mathbf{A}^r$ such that $\int_{z_n+L} |u_n(x)|^2\,dx \ge d/2$, and let $w_n(z) = u_n(z + z_n)$. Then, for $n \in \mathbb{N}$,

$$\int_L |w_n(z)|^2\,dz = \int_L |u_n(z+z_n)|^2\,dz = \int_{z_n+L} |u_n(z)|^2\,dz \ge \frac{d}{2},$$

and $\|w_n\|_{H^1(\mathbf{A}^r)} = \|u_n\|_{H^1(\mathbf{A}^r)} \le c$ for $n \in \mathbb{N}$, so there is $w \in H_0^1(\mathbf{A}^r)$ such that $w_n \rightharpoonup w$ weakly in $H_0^1(\mathbf{A}^r)$. We have

$$\begin{aligned} u_n &\rightharpoonup 0 && \text{weakly in } H_0^1(\mathbf{A}^r), \\ w_n &\rightharpoonup w && \text{weakly in } H_0^1(\mathbf{A}^r), \\ \int_L |w_n|^2 &\ge d/2 && \text{for } n \in \mathbb{N}. \end{aligned}$$

Hence,

(i) $|z_n| \to \infty$. On the contrary, there is r, $s > 0$ such that $z_n + L \subset G = \{(x,y) \in \mathbf{A}^r \mid r \le y \le s\}$ for $n \in \mathbb{N}$. Then

$$0 = \lim_{n\to\infty} \int_G |u_n|^2 \ge \lim_{n\to\infty} \int_{z_n+L} |u_n|^2 \ge \frac{d}{2},$$

a contradiction.

(ii) $w \not\equiv 0$. By the Rellich–Kondrakov theorem

$$\int_L |w|^2 = \lim_{n\to\infty} \int_L |w_n|^2 \ge d/2.$$

(iii) $\{w_n\}$ is a $(\mathrm{PS})_{\alpha(\mathbf{A}^r)}$-sequence for J.

By Theorem 11, w is a ground state solution of equation (1). By Theorem 22, Ω is an achieved domain. $\square$

Moreover, $\mathbb{R}$ is an achieved domain. As a matter of fact, by Theorem 22, $\mathbb{R}$ is an achieved domain if and only if there is a classical solution u of equation

$$u'' = u - S(\mathbb{R})^{-p}|u|^{p-2}u. \tag{4}$$

By Berestycki and Lions ([3]), such a solution is unique. Actually, such a solution is constructed as follows by routine computations.

Theorem 26. *With $\mu = 2/(p-2)$, we have*

$$\begin{aligned} u(r) &= \left(\frac{pS(\mathbb{R})^p}{2}\right)^{\mu/2}\left\{\cosh\left(\frac{r}{\mu}\right)\right\}^{-\mu}, \\ S(\mathbb{R}) &= \frac{\|u\|_{L^p}}{\|u\|_{H^1}} = \left[\frac{(2\mu+1)\Gamma(2\mu)}{\mu\Gamma(\mu)^2}\right]^{1/2-1/p}\left(\frac{\mu}{4}\right)^{1/p}(\mu+1)^{-1/2}, \end{aligned}$$

solve equation (4)*. In particular, $\mathbb{R}$ is an achieved domain.*

Next we present achieved domains from the perturbations of nonachieved domains. By Theorem 16, the upper half strip $\mathbf{A}_0^r$ and the upper half hollow strip $\mathbf{A}_0^{r_1,r_2}$ are nonachieved. However, the perturbed domains of $\mathbf{A}_0^r$ and $\mathbf{A}_0^{r_1,r_2}$ may be achieved. Let $\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0;s)$ be an interior flask domain. For large s, interior flask domains are achieved domains.

Theorem 27. *There exists $s_0 > 0$ such that the interior flask domain $\mathbf{F}_s^r$ is achieved if $s > s_0$, but is nonachieved if $s < s_0$.*

Proof. Let $\Omega_0 = \mathbf{F}_s^r$, $\Omega_1 = \mathbf{A}_0^r$, and $\Omega_2 = B^N(0;s)$. By Theorems 13 and 25, $\alpha(\mathbf{A}^{\ r}) > \alpha(\mathbb{R}^N)$. Note that, by Theorem 21, $\alpha(\mathbf{A}^r) = \alpha(\mathbf{A}_0^r)$ and $\lim_{s\to\infty}\alpha(B^N(0;s)) = \alpha(\mathbb{R}^N)$. Take s large enough such that

$$\alpha(B^N(0;s)) < \alpha(\mathbf{A}^r) = \alpha(\mathbf{A}_0^r).$$

By Theorem 23, $B^N(0;s)$ is an achieved domain. By Theorem 13, we have

$$\alpha(\Omega_0) = \alpha(\mathbf{F}_s^r) < \alpha(B^N(0;s)).$$

We conclude that

$$\alpha(\Omega_0) = \alpha(\mathbf{F}_s^r) < \alpha(B^N(0;s)) = \alpha(\Omega_2) < \alpha(\mathbf{A}_0^r) = \alpha(\Omega_1).$$

By Theorem 14, equation (1) admits a ground state solution in $\mathbf{F}_s^r$ for large s. Let

$$s_0 = \inf\{s > r \mid \text{equation (1) admits a ground state solution in } \mathbf{F}_s^r\}.$$

By Theorem 23, Theorem 13 and 14, equation (1) admits a ground state solution in $\mathbf{F}_s^r$ if $s > s_0$ and equation (1) admits no ground state solution in $\mathbf{F}_s^r$ if $s < s_0$. This theorem follows from Theorem 22 $\square$

REMARK. In Theorem 27, we have asserted that the interior flask domains $\mathbf{F}_s^r = \mathbf{A}_0^r \cup B^N(0;s)$ is achieved if $s > s_0$. As a matter of fact, replace $\mathbf{A}_0^r \cup B^N(0;s)$ by $\mathbf{A}_0^r \cup \Omega$, where Ω is a bounded domain containing $B^N(0;s)$, the theorem still holds.

CONJECTURE. In Theorem 27, $s_0 = r$.

For $\delta > 0$, there is $\varepsilon(\delta) > 0$, such that a flat interior flask domain Ω_ε is an achieved domain, where

$$\begin{aligned} E_\varepsilon &= \{(x,y) \in \mathbb{R}^N \mid (x, \varepsilon y) \in \mathrm{B}(0; r+\delta)\}, \\ \Omega_\varepsilon &= \mathbf{A}_0^r \cup E_\varepsilon. \end{aligned}$$

THEOREM 28. *Given $\delta > 0$, there exists $\varepsilon_0 > 0$, such that if $\varepsilon \le \varepsilon_0$, then the flat interior flask domain Ω_ε is an achieved domain.*

PROOF. By Theorem 25, the infinite strip $\mathbf{A}^r$ admits a ground state solution. Since $\mathbf{A}^r \subsetneq \mathbf{A}^{r+\delta}$, by Theorem 13, we have $\alpha(\mathbf{A}^{r+\delta}) < \alpha(\mathbf{A}^r)$. Since $E_\varepsilon \subset \mathbf{A}^{r+\delta}$ and $\lim_{\varepsilon \to 0} \alpha(E_\varepsilon) = \alpha(\mathbf{A}^{r+\delta})$, there exists $\varepsilon_0 > 0$, such that if $\varepsilon \le \varepsilon_0$, then $\alpha(E_\varepsilon) < \alpha(\mathbf{A}^r)$. Fix ε, $\varepsilon \le \varepsilon_0$, there exists a large $N \in \mathbb{N}$ such that

$$\alpha((\widetilde{\Omega_\varepsilon})_N) = \alpha(\mathbf{A}_N^r) = \alpha(\mathbf{A}^r).$$

Thus

$$\alpha(\Omega_\varepsilon) \le \alpha(E_\varepsilon) < \alpha(\mathbf{A}^r) = \alpha((\widetilde{\Omega_\varepsilon})_N).$$

By Theorem 14, there exists a ground state solution u of equation (1). By Theorem 22, Ω_ε is an achieved domain. □

We know that the upper half strip $\mathbf{A}_s^r$ is Esteban–Lions domain, Wang ([21]) proved that there exists a higher energy solution of equation (1) in the upper half strip with a hole. Here we prove that there exists a ground state solution of the equation (1) in the manger domains. For $0 < r_1 < r_2$, we define

$$\begin{aligned} \Omega_t^{r_1,r_2} &= \mathbf{A}_0^{r_1} \cup \mathbf{A}_{0,t}^{r_2}, \\ \Theta_t^{r_1,r_2} &= \mathbf{A}_{0,t}^{r_1} \cup \mathbf{A}_0^{r_1,r_2}, \\ \Theta_t^{r_1,r_2,r_3} &= \mathbf{A}_0^{r_1,r_2} \cup \mathbf{A}_{0,t}^{r_2} \cup \widetilde{\mathbf{A}}_t^{r_3}. \end{aligned}$$

By Lien, Tzeng and Wang ([13]), we have

LEMMA 29.

(i) $\lim_{r\to\infty} \alpha(\mathrm{A}^r) = \alpha(\mathbb{R}^N)$,
(ii) $\lim_{r\to 0^+} \alpha(\mathbf{A}^r) = \infty$,
(iii) $\lim_{t\to 0^+} \alpha(\mathbf{A}_{0,t}^r = \infty$ *for* $r > 0$,
(iv) $\lim_{t\to\infty} \alpha(\mathbf{A}_{0,t}^r) = \alpha(\mathbf{A}^r) = \alpha(\mathbf{A}_0^r)$.

Theorem 30. *For $r_1 < r_2$ there is a $t_0 > 0$ such that if $t \geq t_0$ then $\Omega_t^{r_1,r_2}$ is an achieved domain.*

Proof. Note that

$$\begin{aligned} \alpha(\mathbf{A}_0^{r_2}) &= \alpha(\mathbf{A}^{r_2}) < \alpha(\mathbf{A}^{r_1}) = \alpha(\mathbf{A}_0^{r_1}), \\ \alpha(\mathbf{A}_0^{r_2}) &\leq \alpha(\mathbf{A}_{0,t}^{r_2}) \quad \text{for each } t > 0, \end{aligned}$$

and

$$\lim_{t\to\infty} \alpha(\mathbf{A}_{0,t}^{r_2}) = \alpha(\mathbf{A}_0^{r_2}).$$

We conclude that there is $t_0 > 0$ such that if $t \geq t_0$, then $\alpha(\mathbf{A}_{0,t}^{r_2}) < \alpha(\mathbf{A}_0^{r_1})$. The theorem follows from Theorems 14 and 22. $\square$

Similarly, we have the following two results

Theorem 31. *Given $0 < r_1 < r_2$ there is a $t_0(r_1) > 0$ such that if $t \geq t_0$, then the manger $\Theta_t^{r_1,r_2}$ is an achieved domain.*

Theorem 32. *Given r_1, r_2, r_3 with $r_1 < r_2$, $r_3 < r_2$, there is a $t_0 > 0$ such that if $t \geq t_0$ then $\Theta_t^{r_1,r_2,r_3}$ is an achieved domain.*

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