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PARAMETER DEPENDENT PULL-BACK OF CLOSED DIFFERENTIAL FORMS AND INVARIANT INTEGRALS

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Dedicated to the memory of Olga Ladyzhenskaya

Abstract. We prove, given a closed differential k-form ω in an arbitrary open set $D \subset \mathbb{R}^n$, and a parameter dependent smooth map $F(\,\cdot\,,\lambda)$ from an arbitrary open set $G \subset \mathbb{R}^m$ into D, that the derivative with respect to λ of the pull-back $F(\,\cdot\,,\lambda)^*\omega$ is exact in G. We give applications to various theorems in topology, dynamics and hydrodynamics.

1. Introduction

It is well known that a closed differential form (cocycle) on a set $D \subset \mathbb{R}^n$ needs not be exact (coboundary) on D [8], [15]. The converse of Poincaré's lemma implies that it is the case if D is simply connected. In recent papers [9], [10], it has been shown that given a differential n-form ω on $D \subset \mathbb{R}^n$, which necessarily is a cocycle, the derivative with respect to λ of its pull-back $F(\,\cdot\,,\lambda)^*\omega$ by a C^2 parameter dependent mapping $F(\,\cdot\,,\lambda)\colon G \subset \mathbb{R}^n \to D \subset \mathbb{R}^n$ is always a coboundary. This result allows a simple and complete proof of a lemma on the invariance of an integral stated and proved in a special case by Tartar [16] and reproduced in [2]. This lemma was used in [9] to obtain the homotopy invariance of Brouwer degree, and in [10] to give elementary proofs of various existence and fixed point theorems.

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In this paper, we want to show that the above mentioned property holds indeed for any k-cocycle on $D \subset \mathbb{R}^n$ and any C^2 parameter dependent mapping $F(\cdot, \lambda): G \subset \mathbb{R}^m \to D \subset \mathbb{R}^n$ (Theorem 61). The given proof is a lengthy and tedious computation, which is substantially shorter only for $k = 1$ and for $k = n$. For the readers uniquely interested in those situations, we have explicited the proof for $k = 1$ (Theorem 4.1) and reproduced, for the sake of completeness, the proof for $k = n$ given in [10] (Theorem 5.1).

For $k = 1$, we give as direct applications simple proofs of the n-dimensional generalization of a theorem on the invariance of the circulation of a perfect fluid due to Lord Kelvin [17] (see also [6]), and of Cauchy integral theorem for holomorphic functions. For $k = n - 1$, Theorem 61 generalizes a result of Hatziafratis and Tsarpalias [3] obtained for the $(n-1)$ solid angle form occuring in the definition of Kronecker's index. For $k = n$, we complete the applications given in [10] by an elementary proof of a Poincaré–Krasnosel'skiĭ bifurcation theorem in finite dimension.

In some physical situations, the family of pull-back transformations is parametrized by time and is given by the flow associated to an evolution equation. We show in two classical examples, Liouville's theorem in dynamics [7] and Helmholtz theorem in hydrodynamics [4] (see also [14]), how those classical results follow from the same type of reasonings (Theorems 7.1 and 8.2). Those results belong of course to Poincaré's theory of integral invariants (see [12] and [13]), which also can be related to the considerations developed here.

2. Parameter dependent differential forms

We first recall a few elementary facts and results on differential forms [8], [15].

If $D \subset \mathbb{R}^n$ is open and $0 \le k \le n$ is an integer, we consider the differential k-form of class C^l in D $(l \ge 0)$

$$\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} w_{i_1 \ldots i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where the real functions $w_{i_1 \ldots i_k}$ are of class C^l on D. If $G \subset \mathbb{R}^m$ is open and $T: G \to D$ is of class C^1, the *pull-back* $T^*\omega$ is the differential k-form in G defined by

$$T^*\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} (w_{i_1 \ldots i_k} \circ T) \, dT_{i_1} \wedge \ldots \wedge dT_{i_k},$$

where dT_i is the differential 1-form on G defined by $dT_i = \sum_{j=1}^m \partial_j T_i \, dy_j$. If ω is of class C^1, the *exterior differential* $d\omega$ of ω is the differential $(k+1)$-form in D defined by

$$d\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} dw_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where $dw_{i_1\ldots i_k} = \sum_{j=1}^n \partial_j w_{i_1\ldots i_k}\, dx_j$. Explicitely, with

$$1 \le i_1, \ldots, i_k, j_1, \ldots, j_{k+1}, j \le n,$$

we have

$$d\omega = \sum_{j_1<\ldots<j_{k+1}} \left[\sum_{l=1}^{k+1} (-1)^{l-1} \partial_{j_l} w_{j_1\ldots\widehat{j_l}\ldots j_{k+1}} \right] dx_{j_1} \wedge \ldots \wedge dx_{j_{k+1}},$$

where the symbol $\widehat{\ }$ means that the corresponding term is missing. When ω is of class C^1, ω is *closed* or is a k-*cocycle* if $d\omega = 0$, which, by the computation above, is equivalent to the set of conditions

$$\sum_{l=1}^{k+1} (-1)^{l-1} \partial_{j_l} w_{j_1\ldots\widehat{j_l}\ldots j_{k+1}} = 0 \quad (1 \le j_1 < j_2 < \ldots < j_{k+1} \le n). \tag{2.1}$$

Consider now a *parameter dependent* differential k-form in $D \subset \mathbb{R}^n$

$$\mu(\lambda) = \sum_{1\le i_1<\ldots<i_k\le n} m_{i_1\ldots i_k}(\,\cdot\,,\lambda)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

whose coefficients

$$m_{i_1\ldots i_k} \colon D \times [a,b] \to \mathbb{R}, \quad (x,\lambda) \mapsto m_{i_1\ldots i_k}(x,\lambda)$$

are of class C^1 on $D \times [a,b]$.

Definition 2.1. The *partial derivative* $\partial_\lambda \mu$ of $\mu(\lambda)$ with respect to λ is the differential k-form in D

$$\partial_\lambda \mu(\lambda) := \sum_{1\le i_1<\ldots<i_k\le n} \partial_\lambda m_{i_1\ldots i_k}(\,\cdot\,,\lambda)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

It follows easily from this definition that if $f\colon D \times [a,b] \to \mathbb{R}$ and

$$\nu(\lambda) = \sum_{1\le j_1<\ldots<j_l\le n} n_{j_1\ldots j_l}(\,\cdot\,,\lambda)\, dx_{j_1} \wedge \ldots \wedge dx_{j_l},$$

are of class C^1 on $D \times [a,b]$, then

$$\partial_\lambda[f(\,\cdot\,,\lambda)\mu(\lambda)] = \partial_\lambda f(\,\cdot\,,\lambda)\mu(\lambda) + f(\,\cdot\,,\lambda)\partial_\lambda \mu(\lambda), \tag{2.2}$$

$$\partial_\lambda[\mu(\lambda) \wedge \nu(\lambda)] = \partial_\lambda \mu(\lambda) \wedge \nu(\lambda) + \mu(\lambda) \wedge \partial_\lambda \nu(\lambda), \tag{2.3}$$

and if $\mu(\lambda)$ is of class C^2, then

$$\partial_\lambda[d\mu(\lambda)] = d[\partial_\lambda \mu(\lambda)]. \tag{2.4}$$

3. Parameter dependent pullback of a differential form

If $D \subset \mathbb{R}^n$ is open and $0 \le k \le n$ is an integer, let us consider the differential k-form in D

$$\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} w_{i_1 \ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

If $G \subset \mathbb{R}^m$ is open and if $F\colon G \times [a,b] \mapsto D$ is of class C^2, we consider for each $\lambda \in [a,b]$ the pull-back $F(\,\cdot\,,\lambda)^*\omega$ of ω by $F(\,\cdot\,,\lambda)$

$$F(\,\cdot\,,\lambda)^*\omega := \sum_{1 \le i_1 < \ldots < i_k \le n} (w_{i_1 \ldots i_k} \circ F)(\,\cdot\,,\lambda)\, dF_{i_1} \wedge \ldots \wedge dF_{i_k}, \tag{3.1}$$

where we write

$$dF_i = dF_i(\,\cdot\,,\lambda) = \sum_{l=1}^{m} \partial_l F_i(\,\cdot\,,\lambda)\, dy_l.$$

Notice that, by formula (2.4), we have

$$\partial_\lambda(dF_i) = d(\partial_\lambda F_i). \tag{3.2}$$

Lemma 3.1. *If the differential k-form ω is of class C^1 on D, and $F\colon G \times [a,b] \to D$ is of class C^2, then, with $1 \le i_1, \ldots, i_k \le n$,*

$$\begin{aligned}\partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] = {} & \sum_{i_1 < \ldots < i_k} \sum_{j=1}^{n} (\partial_j w_{i_1 \ldots i_k} \circ F)\partial_\lambda F_j\, dF_{i_1} \wedge \ldots \wedge dF_{i_k} \\ & + \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^{k} (-1)^{l-1} d[\partial_\lambda F_{i_l}\, dF_{i_1} \wedge \ldots \wedge \widehat{dF_{i_l}} \wedge \ldots \wedge dF_{i_k}].\end{aligned} \tag{3.3}$$

Proof. Using formulas (2.2) and (3.2), we get, if ω is of class C^1 in D, and $1 \le i_1, \ldots, i_k \le n$, $1 \le j_1, \ldots, j_{k+1} \le n$,

$$\begin{aligned}\partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] &= \partial_\lambda\Big[\sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F)\, dF_{i_1} \wedge \ldots \wedge dF_{i_k}\Big] \\ &= \sum_{i_1 < \ldots < i_k} \partial_\lambda(w_{i_1 \ldots i_k} \circ F)\, dF_{i_1} \wedge \ldots \wedge dF_{i_k} \\ &\quad + \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^{k} dF_{i_1} \wedge \ldots \wedge d(\partial_\lambda F_{i_l}) \wedge \ldots \wedge dF_{i_k} \\ &= \sum_{i_1 < \ldots < i_k} \sum_{j=1}^{n} (\partial_j w_{i_1 \ldots i_k} \circ F)\partial_\lambda F_j\, dF_{i_1} \wedge \ldots \wedge dF_{i_k} \\ &\quad + \sum_{i_1 < \ldots < i_k} (w_{i_1 \ldots i_k} \circ F) \sum_{l=1}^{k} (-1)^{l-1} d[\partial_\lambda F_{i_l}\, dF_{i_1} \wedge \ldots \wedge \widehat{dF_{i_l}} \wedge \ldots \wedge dF_{i_k}]. \quad \square\end{aligned}$$

4. The case of 1-cocycle

Let the differential 1-form

$$\omega = \sum_{j=1}^{n} w_j \, dx_j \tag{4.1}$$

be of class C^1 on D. By formula (2.1), ω is a 1-cocycle if and only if

$$\partial_i w_j = \partial_j w_i \quad (1 \le i < j \le n). \tag{4.2}$$

Let $G \subset \mathbb{R}^m$ be open and $F\colon G \times [a,b] \to D$, $(y,\lambda) \mapsto F(y,\lambda)$ be of class C^2.

Theorem 4.1. *If ω is a 1-cocycle of class C^1 on D, then*

$$\partial_\lambda [F(\,\cdot\,,\lambda)^*\omega] := \partial_\lambda \Bigg[\sum_{j=1}^{n} (w_j \circ F) \, dF_j \Bigg] = d \Bigg[\sum_{j=1}^{n} (w_j \circ F) \partial_\lambda F_j \Bigg].$$

Proof. We have, using formulas (3.3) and (4.2),

$$\begin{aligned}
\partial_\lambda [F(\,\cdot\,,\lambda)^*\omega] &= \sum_{j=1}^{n}\sum_{k=1}^{n} (\partial_k w_j \circ F) \partial_\lambda F_k \, dF_j + \sum_{j=1}^{n} (w_j \circ F) \, d(\partial_\lambda F_j) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} (\partial_j w_k \circ F) \, \partial_\lambda F_k \, dF_j + \sum_{k=1}^{n} (w_k \circ F) \, d(\partial_\lambda F_k) \\
&= \sum_{k=1}^{n} d(w_k \circ F) \partial_\lambda F_k + \sum_{k=1}^{n} (w_k \circ F) \, d(\partial_\lambda F_k) \\
&= d \Bigg[\sum_{j=1}^{n} (w_j \circ F) \partial_\lambda F_j \Bigg]. \qquad \square
\end{aligned}$$

We now show how Theorem 4.1 imply some classical conservation theorems.

The first result for $n = 3$ is due to Lord Kelvin [17], in the context of hydrodynamics of perfect fluids. Recall that the *circulation* of the differential 1-form ω along the 1-simplex $\varphi\colon [0,T] \to D$ of class C^1 is defined by the integral

$$\int_\varphi \omega = \int_0^T \varphi^* \omega = \int_0^T \Bigg[\sum_{j=1}^{n} u_j(\varphi(s)) \varphi_j'(s) \, ds \Bigg]. \tag{4.4}$$

φ is called a 1-*cycle* if $\varphi(0) = \varphi(T)$.

Corollary 4.2. *If $\omega = \sum_{j=1}^{n} w_j \, dx_j$ is a 1-cocycle of class C^1 on D, and for each $\lambda \in [a,b]$, $F(\,\cdot\,,\lambda)\colon [0,T] \to D$ is a 1-cycle of class C^2 in D, then the circulation of ω along $F(\,\cdot\,,\lambda)$*

$$\int_{F(\,\cdot\,,\lambda)} \omega = \int_0^T \sum_{j=1}^{n} (w_j \circ F)(y,\lambda) \, \partial_y F_j(y,\lambda) \, dy \tag{4.5}$$

is independent of λ on $[a,b]$.

PROOF. Using Leibniz' rule and Theorem 4.1, we obtain

$$\partial_\lambda\bigg[\int_{F(\,\cdot\,,\lambda)}\omega\bigg] = \int_0^T \partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] = \int_0^T d\bigg[\sum_{j=1}^n (w_j\circ F)\,\partial_\lambda F_j\bigg] = 0,$$

as $F(\,\cdot\,,\lambda)$ is a 1-cycle. □

REMARK 4.3. If $n=3$ and if (w_1,w_2,w_3) denotes the field of velocities of the irrotational motion of a perfect fluid, if λ denotes the time and if $F([a,b],\lambda)$ denotes the time evolution of a closed curve under the motion of the fluid, Corollary 4.2 expresses the constancy of the circulation of the velocity around the closed curve.

A second consequence of Theorem 4.1 is a version of Cauchy's theorem in complex functions theory [8]. Let $D\subset\mathbb{C}$ be open, $f\colon D\to\mathbb{C}$ holomorphic and let

$$\Gamma_j\colon[0,T]\times[a,b]\to D,\quad (y,\lambda)\mapsto\Gamma_j(y,\lambda),\quad (1\le j\le m)$$

be of class C^2 and such that

$$\Gamma_j(T,\lambda)=\Gamma_{j+1}(0,\lambda),\quad (j=1,\dots,m-1),\quad \Gamma_m(T,\lambda)=\Gamma_1(0,\lambda),\quad \lambda\in[a,b].$$

So, when λ varies, the family of the $\Gamma_j(\,\cdot\,,\lambda)$ represents a continuous deformation of a piecewise-C^2 1-cycle in D.

COROLLARY 4.4. *The expression*

$$\sum_{j=1}^m\int_{\Gamma_j(\,\cdot\,,\lambda)} f(z)\,dz$$

is independent of λ on $[a,b]$.

PROOF. We have, using Leibniz rule and Theorem 4.1,

$$\begin{aligned}\partial_\lambda\bigg(\sum_{j=1}^m\int_{\Gamma_j(\,\cdot\,,\lambda)} f(z)\,dz\bigg) &= \sum_{j=1}^m\int_0^T \partial_\lambda[\Gamma_j^*(\,\cdot\,,\lambda)(f(z)\,dz)]\\ &= \sum_{j=1}^m\int_0^T d[(f\circ\Gamma_j)(\,\cdot\,,\lambda)\,\partial_\lambda\Gamma_j]\\ &= \sum_{j=1}^m[(f\circ\Gamma_j)(T,\lambda)\,\partial_\lambda\Gamma_j(T,\lambda)-(f\circ\Gamma_j)(0,\lambda)\,\partial_\lambda\Gamma_j(0,\lambda)]\\ &= (f\circ\Gamma_m)(T,\lambda)\partial_\lambda\Gamma_m(T,\lambda)-(f\circ\Gamma_1)(0,\lambda)\partial_\lambda\Gamma_1(0,\lambda)=0.\end{aligned}$$ □

5. The case of a differential n-form

Let the differential n-form

$$\omega = w\, dx_1 \wedge \ldots \wedge dx_n,$$

be of class C^1 in D. Notice that ω is always a n-cocycle in D, as $d\omega$ is a differential $(n+1)$-form in $\mathbb{R}^n$. Let $G \subset \mathbb{R}^m$ be open and $F: G \times [a,b] \to D$, $(y,\lambda) \mapsto G(y,\lambda)$ be of class C^2.

THEOREM 5.1. *If ω is a differential n-form of class C^1 in D, then*

$$\begin{aligned}\partial_\lambda[F^*(\,\cdot\,,\lambda)\omega] &:= \partial_\lambda[(w\circ F)\, dF_1 \wedge \ldots \wedge dF_n] \\ &= d\Big[(w\circ F)\Big(\sum_{j=1}^n (-1)^{j-1}\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\Big)\Big].\end{aligned} \tag{5.1}$$

PROOF. We have, using formula (3.3)

$$\begin{aligned}\partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] &= \Big[\sum_{j=1}^n (\partial_j w\circ F)\,\partial_\lambda F_j\Big]\, dF_1 \wedge \ldots \wedge dF_n \\ &\quad + (w\circ F)\Big[\sum_{j=1}^n (-1)^{j-1} d(\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n)\Big] \\ &= \sum_{j=1}^n (-1)^{j-1} (\partial_j w\circ F)\, dF_j \wedge \partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\ &\quad + (w\circ F)\Big[\sum_{j=1}^n (-1)^{j-1} d(\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n)\Big] \\ &= \sum_{j=1}^n (-1)^{j-1} \Big[\sum_{k=1}^n (\partial_k w\circ F)\, dF_k\Big] \\ &\quad \wedge \partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n \\ &\quad + (w\circ F)\Big[\sum_{j=1}^n (-1)^{j-1} d(\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n)\Big] \\ &= d(w\circ F) \wedge \Big(\sum_{j=1}^n (-1)^{j-1}\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\Big) \\ &\quad + (w\circ F)\, d\Big(\sum_{j=1}^n (-1)^{j-1}\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\Big) \\ &= d\Big[(w\circ F)\Big(\sum_{j=1}^n (-1)^{j-1}\partial_\lambda F_j\, dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\Big)\Big]. \ \square\end{aligned}$$

Like in the previous section, one deduces from Theorem 5.1 the following invariance result.

Corollary 5.2. *Let $\omega = w\,dx_1 \wedge \ldots \wedge dx_n$ be a differential n-form of class C^1 in the open set $D \subset \mathbb{R}^n$, $G \subset \mathbb{R}^n$ be open and bounded and $F\colon \overline{G} \times [a,b] \to D$ be of class C^2. If, for each $\lambda \in [a,b]$, one has*

$$\operatorname{supp}\omega \cap F(\,\cdot\,,\lambda)(\partial G) = \emptyset, \tag{5.2}$$

then the integral

$$\int_G F(\,\cdot\,,\lambda)^*\omega = \int_G [w \circ F(y,\lambda)] \operatorname{Jac} F(y,\lambda)\,dy \tag{5.3}$$

is independent of λ on $[a,b]$.

As an application of Corollary 5.2, let us give an elementary proof of a fundamental bifurcation result which can be traced to Poincaré [11] and Krasnosel'skiĭ [5]. Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be continuous and such that $f(0,\lambda) = 0$ for each $\lambda \in \mathbb{R}$, and consider the family of equations

$$f(x,\lambda) = 0. \tag{5.4}$$

Definition 5.3. $(0,\lambda_0)$ is a *bifurcation point* for (5.4) if

$$(\forall\ r > 0)(\exists (x,\lambda) \in (B[0,r] \setminus \{0\}) \times [\lambda_0 - r, \lambda_0 + r]) : f(x,\lambda) = 0. \tag{5.5}$$

Theorem 5.4. *Let $A\colon \mathbb{R} \to \mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$ be continuous and $R\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be continuous and such that*

$$\lim_{x\to 0} \frac{R(x,\lambda)}{\|x\|} = 0, \tag{5.6}$$

uniformly on compact intervals of $\mathbb{R}$. Assume that there exists $a < b$ such that

$$\det A(a) \det A(b) < 0. \tag{5.7}$$

Then (5.4) *with*

$$f(x,\lambda) := A(\lambda)x + R(x,\lambda)$$

has a bifurcation point in $\{0\} \times [a,b]$.

Proof. Notice first that if $(0,\lambda_0)$ is not a bifurcation point for (5.4), then there exists $r = r(\lambda_0) > 0$ such that $f(x,\lambda) \neq 0$ for all $x \in B[0,r] \setminus \{0\}$ and all $\lambda \in [\lambda_0 - r, \lambda_0 + r]$. Hence, an easy compactness argument implies that if (5.4) has no bifurcation point in $\{0\} \times [a,b]$, then

$$(\exists\ r > 0)(\forall\ x \in B[0;r] \setminus \{0\})(\forall\ \lambda \in [a,b]) : f(x,\lambda) \neq 0. \tag{5.8}$$

Now, by assumptions, there exists $\alpha > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\|A(a)x\| \geq \alpha\|x\|, \quad \|A(b)x\| \geq \alpha\|x\|,$$

and there exists $r_1 \in]0, r]$ such that, for all $x \in B[0, r_1]$ and all $\lambda \in [a, b]$, one has

$$\|R(x, \lambda)\| \leq \frac{\alpha}{2}\|x\|.$$

Consequently, for all $x \in \partial B(0, r_1)$, and all $\mu \in [0, 1]$, one has

$$\|g_c(x, \mu)\| := \|A(c)x + \mu R(x, c)\| \geq \frac{\alpha}{2} r_1 := \alpha_1, \quad c = a, b. \tag{5.9}$$

Now, it follows from relation (5.8) that there exists $\alpha_2 > 0$ such that, for all $x \in \partial B(0, r_1)$ and all $\lambda \in [a, b]$, one has

$$\|f(x, \lambda)\| \geq \alpha_1.$$

Let $\alpha_3 := \min\{\alpha_1, \alpha_2\}$ and $w \in C^1(\mathbb{R}^n, \mathbb{R}_+)$ be such that $\operatorname{supp} w \subset B(0, \alpha_3)$ and

$$\int_{\mathbb{R}^n} w(x)\, dx = 1. \tag{5.11}$$

A first application of Corollary 5.2 to the family of pull-backs $f(\,\cdot\,, \lambda)$, $\lambda \in [a, b]$ implies that

$$\int_{B(0,r)} (w \circ f)(y, a) \operatorname{Jac} f(y, a)\, dy = \int_{B(0,r)} (w \circ f)(y, b) \operatorname{Jac} f(y, b)\, dy. \tag{5.12}$$

A second application of Corollary 5.2 to the families of pull-backs $g_a(\,\cdot\,, \mu)$, $g_b(\,\cdot\,, \mu)$, $\mu \in [0, 1]$ implies that

$$\begin{aligned} \int_{B(0,r)} & (w \circ f)(y, a) \operatorname{Jac} f(y, a)\, dy \\ &= \int_{B(0,r)} (w \circ g_a)(y, 1) \operatorname{Jac} g_a(\,\cdot\,, 1)\, dy \\ &= \int_{B(0,r)} (w \circ g_a)(y, 0) \operatorname{Jac} g_a(\,\cdot\,, 0)\, dy \\ &= \int_{B(0,r)} (w \circ A(a))(y) \det A(a)\, dy = \operatorname{sign} \det A(a), \end{aligned} \tag{5.13}$$

$$\begin{aligned} \int_{B(0,r)} & (w \circ f)(y, b) \operatorname{Jac} f(y, b)\, dy \\ &= \int_{B(0,r)} (w \circ g_b)(y, 1) \operatorname{Jac} g_b(\,\cdot\,, 1)\, dy \\ &= \int_{B(0,r)} (w \circ g_b)(y, 0) \operatorname{Jac} g_b(\,\cdot\,, 0)\, dy \\ &= \int_{B(0,r)} (w \circ A(b))(y) \det\, A(b)\, dy = \operatorname{sign} \det A(b). \end{aligned} \tag{5.14}$$

where we have used the change of variables rule in a multiple integral and condition (5.11). The contradiction follows from relations (5.12)–(5.14) and assumption (5.7). □

6. The case of a k-cocycle

Let the differential k-form

$$\omega = \sum_{1\le i_1<\ldots<i_k\le n} w_{i_1\ldots i_k}\, dx_{i_1}\wedge\ldots\wedge dx_{i_k}$$

be of class C^1 in an open set $D\subset\mathbb{R}^n$. Recall that ω is a k-cocycle if and only if relations (2.1) hold. Let $G\subset\mathbb{R}^m$ be open and let $F\colon G\times[a,b]$, $(y,\lambda)\mapsto F(y,\lambda)$ be of class C^2.

Theorem 6.1. *If ω is a k-cocycle in D, then, with $1\le i_1,\ldots,i_k\le n$,*

$$\begin{aligned}(6.1)\quad \partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] &:= \partial_\lambda\bigg[\sum_{i_1<\ldots<i_k}(w_{i_1\ldots i_k}\circ F)\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\bigg]\\ &= d\bigg[\sum_{i_1<\ldots<i_k}(w_{i_1\ldots i_k}\circ F)\sum_{j=1}^k(-1)^{j-1}\partial_\lambda F_{i_j}\, dF_{i_1}\wedge\ldots\wedge\widehat{dF_{i_j}}\wedge\ldots\wedge dF_{i_k}\bigg].\end{aligned}$$

Proof. To simplify some heavy notations in this proof, we write

$$\sum_I \quad\text{for}\quad \sum_{1\le i_1<\ldots<i_k\le n},\qquad \sum_J\quad\text{for}\quad\sum_{1\le j_1<\ldots<j_{k+1}\le n}$$

and, for $1\le i_1,\ldots,i_l,\ldots,i_k\le n$ and $1\le j_1,\ldots,j_l,\ldots,j_{k+1}\le n$, we set

$$[\widehat{dF_{i_l}}] = dF_{i_1}\wedge\ldots\wedge\widehat{dF_{i_l}}\wedge\ldots\wedge dF_{i_k},\quad [\widehat{dF_{j_l}}] = dF_{j_1}\wedge\ldots\wedge\widehat{dF_{j_l}}\wedge\ldots\wedge dF_{j_{k+1}}.$$

We have, using formula (3.1),

$$\begin{aligned}(6.2)\qquad \partial_\lambda[F(\,\cdot\,,\lambda)^*\omega] = d\bigg[&\sum_I(w_{i_1\ldots i_k}\circ F)\sum_{l=1}^k(-1)^{l-1}\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\bigg]\\ &+\sum_I\sum_{j=1}^n(\partial_j w_{i_1\ldots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\\ &-\sum_I d(w_{i_1\ldots i_k}\circ F)\wedge\bigg[\sum_{l=1}^k(-1)^{l-1}\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\bigg].\end{aligned}$$

Now

$$\begin{aligned}&\sum_I\sum_{j=1}^n(\partial_j w_{i_1\ldots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\\ &\qquad=\sum_I\sum_{j<i_1}(\partial_j w_{i_1\ldots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\\ &\qquad\quad+\sum_I(\partial_{i_1} w_{i_1\ldots i_k}\circ F)\partial_\lambda F_{i_1}\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\\ &\qquad\quad+\sum_I\sum_{i_1<j<i_2}(\partial_j w_{i_1\ldots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\ldots\wedge dF_{i_k}\end{aligned}$$

$$+\sum_I (\partial_{i_2} w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_2}\, dF_{i_1}\wedge\dots\wedge dF_{i_k}+\dots$$
$$+\sum_I \sum_{i_{k-1}<j<i_k} (\partial_j w_{i_1\dots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\dots\wedge dF_{i_k}$$
$$+\sum_I (\partial_{i_k} w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_k}\, dF_{i_1}\wedge\dots\wedge dF_{i_k}$$
$$+\sum_I \sum_{i_k<j} (\partial_j w_{i_1\dots i_k}\circ F)\partial_\lambda F_j\, dF_{i_1}\wedge\dots\wedge dF_{i_k}.$$

Grouping the terms of similar nature and renaming the multi-indices, we obtain

$$\begin{aligned}(6.3)\qquad &\sum_I\sum_{j=1}^{n}(\partial_j w_{i_1\dots i_k}\circ F)\partial_\lambda F_j dF_{i_1}\wedge\dots\wedge dF_{i_k}\\ &=\sum_J\sum_{l=1}^{k+1}(\partial_{j_l} w_{j_1\dots\widehat{j_l}\dots j_{k+1}}\circ F)\partial_\lambda F_{j_l}\,[\widehat{dF_{j_l}}]\\ &\quad+\sum_I\sum_{l=1}^{k}(\partial_{i_l} w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_l}\, dF_{i_1}\wedge\dots\wedge dF_{i_k}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\sum_I d(w_{i_1\dots i_k}\circ F)\wedge\Big[\sum_{l=1}^{k}(-1)^{l-1}\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\Big]\\ &=\sum_I\sum_j(\partial_j w_{i_1\dots i_k}\circ F)\, dF_j\wedge\Big[\sum_{l=1}^{k}(-1)^{l-1}\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\Big]\\ &=\sum_I\sum_{l=1}^{k}(-1)^{l-1}\Big(\sum_{j<i_1}+\sum_{i_1<j<i_2}+\dots+\sum_{i_{l-1}<j<i_l}\Big)(\partial_j w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\\ &\quad+\sum_I\sum_{l=1}^{k}(-1)^{l-1}(\partial_{i_l} w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\\ &\quad+\sum_I\sum_{l=1}^{k}(-1)^{l-1}\Big(\sum_{i_l<j<i_{l+1}}+\dots+\sum_{i_k<j}\Big)(\partial_j w_{i_1\dots i_k}\circ F)\partial_\lambda F_{i_l}[\widehat{dF_{i_l}}].\end{aligned}$$

Renaming the indices, we obtain

$$\begin{aligned}&\sum_I d(w_{i_1\dots i_k}\circ F)\wedge\Big[\sum_{l=1}^{k}(-1)^{l-1}\partial_\lambda F_{i_l}\,[\widehat{dF_{i_l}}]\Big]\\ &=\sum_{j_2<\dots<j_{k+1}}\sum_{l=1}^{k}(-1)^{l-1}\sum_{j_1<j_2}(\partial_{j_1} w_{\widehat{j_1}j_2\dots j_{k+1}}\circ F)\,\partial_\lambda F_{j_{l+1}}\,[\widehat{dF_{j_{l+1}}}]\\ &\quad+\sum_{j_1<j_3<\dots<j_{k+1}}\sum_{l=1}^{k}(-1)^{l-1}\sum_{j_1<j_2<j_3}(\partial_{j_2} w_{j_1\widehat{j_2}j_3\dots j_{k+1}}\circ F)\end{aligned}$$

$$
\begin{aligned}
&\cdot \partial_\lambda F_{j_{l+1}}\,(-1)\,[\widehat{dF_{j_{l+1}}}] + \dots \\
&+ \sum_{j_1<\dots<\widehat{j_l}<\dots<j_{k+1}} \sum_{l=1}^{k} (-1)^{l-1} \sum_{j_{l-1}<j_l<j_{l+1}} (\partial_{j_l} w_{j_1\dots\widehat{j_l}\dots j_{k+1}} \circ F) \\
&\cdot \partial_\lambda F_{j_{l+1}} (-1)^{l-1}\,[\widehat{dF_{j_{l+1}}}] \\
&+ \sum_I \sum_{l=1}^{k} (\partial_{i_l} w_{i_1\dots i_k} \circ F) \partial_\lambda F_{i_l}\, dF_{i_1} \wedge \dots \wedge dF_{i_k} \\
&+ \sum_{j_1<\dots<\widehat{j_{l+1}}<\dots j_{k+1}} \sum_{l=1}^{k} (-1)^{l-1} \sum_{j_l<j_{l+1}<j_{l+2}} (\partial_{j_{l+1}} w_{j_1\dots\widehat{j_{l+1}}\dots<j_{k+1}} \circ F)\cdot \\
&\cdot \partial_\lambda F_{j_{l+1}} (-1)^{l-1}\,[\widehat{dF_{j_{l+1}}}] + \dots \\
&+ \sum_{j_1<\dots<j_k} \sum_{l=1}^{k} (-1)^{l-1} \sum_{j_k<j_{k+1}} (\partial_{j_{l+1}} w_{j_1\dots j_k} \circ F) \partial_\lambda F_{j_{l+1}} (-1)^{k-1}\,[\widehat{dF_{j_{l+1}}}] \\
= &\sum_{l=1}^{k} (-1)^{l-1} \sum_J \sum_{s=1}^{l} (-1)^{s-1} (\partial_{j_s} w_{j_1\dots\widehat{j_s}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_{l+1}} [\widehat{dF_{j_{l+1}}}] \\
&+ \sum_I \sum_{l=1}^{k} (\partial_{i_l} w_{i_1\dots i_k} \circ F) \partial_\lambda F_{i_l}\, dF_{i_1} \wedge \dots \wedge dF_{i_k} \\
&+ \sum_{l=1}^{k} (-1)^{l-1} \sum_J \sum_{s=l+2}^{k+1} (-1)^{s-1} (\partial_{j_s} w_{j_1\dots\widehat{j_s}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_{l+1}} [\widehat{dF_{j_{l+1}}}] \\
= &(-1)^{k-1} \sum_J \sum_{s=1}^{k} (-1)^{s-1} (\partial_{j_s} w_{j_1\dots\widehat{j_s}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_{k+1}} [\widehat{dF_{j_{k+1}}}] \\
&+ \sum_I \sum_{l=1}^{k} (\partial_{i_l} w_{i_1\dots i_k} \circ F) \partial_\lambda F_{i_l}\, dF_{i_1} \wedge \dots \wedge dF_{i_k} \\
&+ \sum_J \sum_{s=2}^{k+1} (-1)^{s-2} (\partial_{j_s} w_{j_1\dots\widehat{j_s}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_1}\, [\widehat{dF_{j_1}}] \\
&+ \sum_{l=1}^{k} (-1)^{l-1} \sum_J \sum_{s=1}^{k+1} (\partial_{j_s} w_{j_1\dots\widehat{j_s}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_{l+1}} [\widehat{dF_{j_{l+1}}}] \\
&- \sum_{l=1}^{k-1} \sum_J (-1)^{2l+1} (\partial_{j_{l+1}} w_{j_1\dots\widehat{j_{l+1}}\dots j_{k+1}} \circ F) \partial_\lambda F_{j_{l+1}} [\widehat{dF_{j_{l+1}}}].
\end{aligned}
$$

Using relations (2.1), this implies that

$$
\begin{aligned}
&\sum_I d(w_{i_1\dots i_k} \circ F) \wedge \left[\sum_{l=1}^{k} (-1)^{l-1} \partial_\lambda F_{i_l}\, [\widehat{dF_{i_l}}] \right] \\
&\qquad\qquad = (-1)^{k-1} \sum_J (-1)^{k+1} (\partial_{j_{k+1}} w_{j_1\dots j_k} \circ F) \partial_\lambda F_{j_{k+1}} [\widehat{dF_{j_{k+1}}}]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_I \sum_{l=1}^{k} (\partial_{i_l} w_{i_1\dots i_k} \circ F)\partial_\lambda F_{i_l}\, dF_{i_1} \wedge \dots \wedge dF_{i_k} \\
&+\sum_J (\partial_{j_1} w_{j_2\dots j_{k+1}} \circ F)\partial_\lambda F_{j_1}\, [\widehat{dF_{j_1}}] \\
&+\sum_{l=1}^{k-1} \sum_J (\partial_{j_{l+1}} w_{j_1\dots \widehat{j_{l+1}}\circ F)\dots j_{k+1}} \partial_\lambda F_{j_{l+1}} [\widehat{dF_{j_{l+1}}}].
\end{aligned}
$$

Regrouping the terms, we find

$$
\begin{aligned}
(6.4)\qquad \sum_I d(w_{i_1\dots i_k} \circ F) \wedge &\Big[\sum_{l=1}^{k} (-1)^{l-1} \partial_\lambda F_{i_l}\, [\widehat{dF_{i_l}}]\Big] \\
&= \sum_J \sum_{l=1}^{k+1} (\partial_{j_l} w_{j_1\dots\widehat{j_l}\dots j_{k+1}} \circ F)\partial_\lambda F_{j_l}\, [\widehat{dF_{j_l}}] \\
&\quad + \sum_I \sum_{l=1}^{k} (\partial_{i_l} w_{i_1\dots i_k} \circ F)\partial_\lambda F_{i_l}\, dF_{i_1} \wedge \dots \wedge dF_{i_k}.
\end{aligned}
$$

Comparing formulas (6.3) and (6.4) finishes the proof. □

An interesting consequence of Theorem 6.1 is the following result on the invariance of an integral. For the differential k-form

$$\omega = \sum_{1\le i_1<\dots<i_k\le n} w_{i_1\dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

define the *support* of ω by

$$\operatorname{supp}\omega = \bigcup_{1\le i_1<\dots<i_k\le n} \operatorname{supp} w_{i_1\dots i_k}.$$

Corollary 6.2. *Let ω be a differential k-cocycle of class C^1 in the open set $D \subset \mathbb{R}^n$, $G \subset \mathbb{R}^k$ be open and bounded and $F\colon \overline{G} \times [a,b] \to D$ be of class C^2. If, for each $\lambda \in [a,b]$, one has*

$$\operatorname{supp}\omega \cap F(\,\cdot\,,\lambda)(\partial G) = \emptyset, \tag{6.5}$$

then the integral

$$\int_G F(\,\cdot\,,\lambda)^*\omega \tag{6.6}$$

is independent of λ on $[a,b]$.

Proof. Using Leibniz rule, Theorem 6.1 and Stokes theorem, we get, with

$$\alpha = \sum_{i_1<\ldots<i_k} (w_{i_1\ldots i_k} \circ F) \sum_{j=1}^{k} (-1)^{j-1} \partial_\lambda F_{i_j}\, dF_{i_1} \wedge \ldots \wedge \widehat{dF_{i_j}} \wedge \ldots \wedge dF_{i_k},$$

$$\partial_\lambda \bigg[\int_G F(\,\cdot\,,\lambda)^* \omega \bigg] = \int_G \partial_\lambda [F(\,\cdot\,,\lambda)^*\omega] = \int_G d\alpha = \int_{\partial G} \alpha = 0. \qquad \square$$

7. Liouville theorem

Let $v\colon \mathbb{R}^n \to \mathbb{R}^n$ be of class C^1 and, for each $y \in \mathbb{R}^n$, let $x\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $(t,y) \mapsto x(t,y)$ be the unique solution of the Cauchy problem

$$\frac{dx}{dt} = v(t,x), \quad x(0) = y, \tag{7.1}$$

so that, for each $(t,y) \in [0,T] \times \mathbb{R}^n$, we have

$$\partial_t x(t,y) = v[t, x(t,y)]. \tag{7.2}$$

If $\omega = dy_1 \wedge \ldots \wedge dy_n$ is the volume n-form, then, for each $t \in [0,T]$,

$$[x(t,\,\cdot\,)]^*\omega = dx_1(t,\,\cdot\,) \wedge \ldots \wedge dx_n(t,\,\cdot\,) = \operatorname{Jac} x(t,\,\cdot\,)(y)\, dy_1 \wedge \ldots \wedge dy_n, \tag{7.3}$$

where, for each fixed $t \in [0,T]$, $\operatorname{Jac} x(t,\,\cdot\,)$ is the Jacobian of $x(t,\,\cdot\,)$. For each fixed t, $\operatorname{div} v(t,\,\cdot\,) = \sum_{j=1}^n \partial_j v_j(t,x)$. The following result can be traced to Liouville [7] (see also [1]).

Theorem 7.1. *For each $t \in [0,T]$, we have*

$$\partial_t\{[x(t,\,\cdot\,)]^*\omega\} = [x(t,\,\cdot\,)]^*[\operatorname{div} v(t,\,\cdot\,)\, dy_1 \wedge \ldots \wedge dy_n] \tag{7.4}$$

or equivalently

$$\partial_t[dx_1(t,\,\cdot\,) \wedge \ldots \wedge dx_n(t,\,\cdot\,)] = \operatorname{div} v[t, x(t,\,\cdot\,)]\, dx_1(t,\,\cdot\,) \wedge \ldots \wedge dx_n(t,\,\cdot\,), \tag{7.5}$$

or equivalently

$$\partial_t \operatorname{Jac} x(t,y) = \operatorname{div} v[t, x(t,y)] \operatorname{Jac} x(t,y). \tag{7.6}$$

Proof. Using formulas (3.2) and (7.2), we get

$$\begin{aligned}
\partial_t\{[x(t,\cdot)]^*\omega\} &= \partial_t[dx_1(t,\,\cdot\,)\wedge\ldots\wedge dx_n(t,\,\cdot\,)]\\
&=\sum_{j=1}^n dx_1(t,\,\cdot\,)\wedge\ldots\wedge d[\partial_t x_j(t,\,\cdot\,)]\wedge\ldots\wedge dx_n(t,\,\cdot\,)\\
&=\sum_{j=1}^n dx_1(t,\,\cdot\,)\wedge\ldots\wedge dv_j[t,x_j(t,\,\cdot\,)]\wedge\ldots\wedge dx_n(t,\,\cdot\,)\\
&=\sum_{j=1}^n dx_1(t,\,\cdot\,)\wedge\ldots\wedge\left[\sum_{k=1}^n \partial_k v_j[t,x_j(t,\,\cdot\,)]\,dx_k(t,\,\cdot\,)\right]\wedge\ldots\wedge dx_n(t,\,\cdot\,)\\
&=\left[\sum_{j=1}^n \partial_j v_j[t,x_j(t,\,\cdot\,)]\right]dx_1(t,\,\cdot\,)\wedge\ldots\wedge dx_n(t,\,\cdot\,)\\
&=[x(t,\,\cdot\,)]^*[\operatorname{div} v(t,\,\cdot\,)\,dy_1\wedge\ldots\wedge dy_n]\\
&=\operatorname{div} v[t,x(t,\,\cdot\,)]\,dx_1(t,\,\cdot\,)\wedge\ldots\wedge dx_n(t,\,\cdot\,)\\
&=\operatorname{div} v[t,x(t,\,\cdot\,)]\operatorname{Jac} x(t,\,\cdot\,)\,dy_1\wedge\ldots\wedge dy_n.
\end{aligned}$$

and the three formulas easily follow. □

8. Helmholtz theorem

We present here a n-dimensional version of Helmholtz theorem in hydrodynamics [4], [6], [11]. Let

$$x\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}^n,\quad (t,y)\mapsto x(t,y) \tag{8.1}$$

be of class C^2. For $n=3$, in the hydrodynamics setting, it represents the position at time t of a particule located at y for $t=0$ (Lagrange's notations). Let

$$u\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}^n,\quad (t,x)\mapsto u(t,x) \tag{8.2}$$

be of class C^1. For $n=3$, in the hydrodynamics setting, it represents the velocity of a point of the fluid located in x at time t (Euler's notations). Consequently, we have, for all $(t,y)\in[0,T]\times\mathbb{R}^n$,

$$u[t,x(t,y)]=\partial_t x(t,y), \tag{8.3}$$

Assume that there exists a function $\psi\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}$ of class C^1 such that, for all $(t,y)\in[0,T]\times\mathbb{R}^n$, one has

$$\frac{d}{dt}\{u[t,x(t,y)]\}=\nabla_x\psi[t,x(t,y)]. \tag{8.4}$$

For $n=3$, in the hydrodynamics setting, those are the equations of motion of the fluid, under the assumption that the external forces depend upon a potential and that the density depends only of the pressure.

Lemma 8.1. *For each $t \in [0,T]$, one has*

$$\partial_t\bigg\{[x(t,\,\cdot\,)]^*\bigg[\sum_{j=1}^n u_j(t,\,\cdot\,)\,dy_j\bigg]\bigg\} = \partial_t\bigg[\sum_{j=1}^n u_j[t,x(t,\,\cdot\,)]\,dx_j(t,\,\cdot\,)\bigg] = d\bigg[\psi(t,\,\cdot\,) + \frac{1}{2}\sum_{j=1}^n u_j^2[t,x(t,\,\cdot\,)]\bigg]. \tag{8.5}$$

Proof. Using formulations (3.2), (8.3) and (8.4), we get

$$\begin{aligned}
&\partial_t\bigg[\sum_{j=1}^n u_j[t,x(t,\,\cdot\,)]\,dx_j(t,\,\cdot\,)\bigg] \\
&\quad = \sum_{j=1}^n \bigg[\frac{d}{dt}\{u_j[t,x(t,\,\cdot\,)]\}\,dx_j(t,\,\cdot\,) + u_j[t,x(t,\,\cdot\,)]\,\partial_t[dx_j(t,\,\cdot\,)]\bigg] \\
&\quad = \sum_{j=1}^n [\partial_j\psi[t,x(t,\,\cdot\,)]\,dx_j(t,\,\cdot\,) + u_j[t,x(t,\,\cdot\,)]\,d\{\partial_t x_j(t,\,\cdot\,)\}] \\
&\quad = d\psi(t,\,\cdot\,) + \sum_{j=1}^n u_j(t,x(t,\,\cdot\,)\,du_j[t,x(t,\,\cdot\,)] \\
&\quad = d\bigg[\psi(t,\,\cdot\,) + \frac{1}{2}\sum_{j=1}^n u_j^2[t,x(t,\,\cdot\,)]\bigg].
\end{aligned}$$

□

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a 1-cycle of class C^2 (i.e. $\gamma(0) = \gamma(1)$), so that, for each fixed $t \in [0,T]$, $x(t,\gamma(\,\cdot\,))$ is the 1-cycle of class C^2 which is the image of $\gamma([a,b])$ at time t under the motion of the fluid. Let us consider now the circulation of the velocity field along $x(t,\gamma(\,\cdot\,))$,

$$C(t) := \int_{x(t,\gamma(\,\cdot\,))} \sum_{j=1}^n u_j\,dy_j. \tag{8.6}$$

Theorem 8.2. *The integral* (8.6) *is constant on* $[0,T]$.

Proof. We have, from Leibniz' rule and formula (8.5),

$$\begin{aligned}
C'(t) &= \int_0^T \partial_t\bigg[\sum_{j=1}^n u_j[t,x(t,\gamma(s))]\,dx_j[t,\gamma(s)]\bigg] \\
&= \int_0^T d\bigg[\psi[t,\gamma(s)] + \sum_{j=1}^n \frac{u_j^2[t,x(t,\gamma(s))]}{2}\bigg] = 0.
\end{aligned}$$

□

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