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UPPER SEMICONTINUITY OF GLOBAL ATTRACTORS FOR THE PERTURBED VISCOUS CAHN–HILLIARD EQUATIONS

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Abstract. It is known that the semigroup generated by the initial-boundary value problem for the perturbed viscous Cahn–Hilliard equation with $\varepsilon > 0$ as a parameter admits a global attractor $\mathcal{A}_\varepsilon$ in the phase space $X^{1/2} = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, $n \leq 3$ (see [14]). In this paper we show that the family $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ is upper semicontinuous at 0, which means that the Hausdorff semidistance

$$d_{X^{1/2}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \equiv \sup_{\psi \in \mathcal{A}_\varepsilon} \inf_{\phi \in \mathcal{A}_0} \|\psi - \phi\|_{X^{1/2}},$$

tends to 0 as $\varepsilon \to 0^+$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set with the boundary $\partial\Omega$ of class C^4. In this paper we consider two equations: the *perturbed viscous Cahn–Hilliard equation*

$$\varepsilon u_{tt}^\varepsilon + u_t^\varepsilon + \Delta(\Delta u^\varepsilon + f(u^\varepsilon) - \delta u_t^\varepsilon) = 0, \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u^\varepsilon(0,x) = u_0(x), \qquad u_t^\varepsilon(0,x) = v_0(x) \qquad \text{for } x \in \Omega, \tag{1.2}$$

$$u^\varepsilon(t,x) = 0, \qquad \Delta u^\varepsilon(t,x) = 0 \qquad \text{for } x \in \partial\Omega, \tag{1.3}$$

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and the *viscous Cahn–Hilliard equation*

$$u_t + \Delta(\Delta u + f(u) - \delta u_t) = 0, \quad x \in \Omega,\ t > 0, \tag{1.4}$$

$$u(0,x) = u_0(x) \qquad \text{for } x \in \Omega, \tag{1.5}$$

$$u(t,x) = 0, \quad \Delta u(t,x) = 0 \qquad \text{for } x \in \partial\Omega, \tag{1.6}$$

where $\varepsilon, \delta \in (0,1]$ and $n \leq 3$. Without loss of generality we can assume that $f(0) = 0$; otherwise we replace $f(s)$ by $\widetilde{f}(s) := f(s) - f(0)$, and then the equations (1.1) and (1.4) will not change. To emphasize the dependence of solutions, semigroups and attractors of the considered problems on the perturbation parameter ε we use the subscript or superscript ε, ε_k or k.

The initial-boundary value problem for the perturbed viscous Cahn–Hilliard equation was studied in [18] in one space dimension and in [14] for $n \leq 3$. In the aforementioned second paper it was assumed that $f\colon \mathbb{R} \to \mathbb{R}$ is a real function satisfying the following assumptions:

$$f \in C^2(\mathbb{R},\mathbb{R}), \tag{1.7}$$

$$|f'(s)| \leq \widehat{C}(1 + |s|^q), \quad s \in \mathbb{R}, \tag{1.8}$$

where $q > 0$ can be arbitrarily large if $n = 1, 2$ and $0 < q < 2$ if $n = 3$,

$$\mathop{\exists}_{\overline{C}\in\mathbb{R}^+} \ \mathop{\forall}_{s\in\mathbb{R}} \quad \overline{F}(s) := \int_0^s f(z)\,dz \leq \overline{C}, \tag{1.9}$$

$$\mathop{\exists}_{\sigma \geq (2+\lambda_1)/(3\lambda_1\sqrt{\varepsilon})} \ \mathop{\exists}_{C_\sigma\in\mathbb{R}^+} \ \mathop{\forall}_{s\in\mathbb{R}} \quad sf(s) - \frac{4}{3}\overline{F}(s) \leq -\sigma s^2 + C_\sigma, \tag{1.10}$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian in Ω. Denote by $\{S_\varepsilon(t)\}$ the C^0-semigroup of global solutions to $((1.1)\text{–}(1.3))_\varepsilon$ defined on $X^{1/2} := (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ via the relation

$$\mathcal{S}_\varepsilon(t)(u_0, v_0) = (u^\varepsilon(t), u_t^\varepsilon(t)), \quad t \geq 0.$$

We showed (see [14, Theorem 4.1]) that for any $0 < \varepsilon \leq 1$ the semigroup $\{S_\varepsilon(t)\}$ has a global attractor $\mathcal{A}_\varepsilon$ in the phase space $X^{1/2}$. Analysing the calculations in [14], it is easy to see that the same result holds, when the condition (1.10) will be replaced by a more general condition

$$\mathop{\exists}_{\sigma>0} \ \mathop{\exists}_{0<\mu\leq\min\{2/3;(2\sigma\lambda_1\sqrt{\varepsilon})/(\lambda_1+2)\}} \ \mathop{\exists}_{C_\sigma>0} \ \mathop{\forall}_{s\in\mathbb{R}} \quad sf(s) - 2\mu\overline{F}(s) \leq -\sigma s^2 + C_\sigma. \tag{1.11}$$

The limit problem (1.4)–(1.6), with equation (1.4) written in the form

$$(1-\nu)u_t = -\Delta(\Delta u + f(u) - \nu u_t), \quad t \geq 0,\ x \in \Omega,\ \nu \in [0,1],$$

was studied recently in [3] under the following assumptions on the nonlinear term $f \in C^2(\mathbb{R},\mathbb{R})$

$$\underset{\kappa \geq 0}{\exists} \ \underset{d \geq 0}{\exists} \ \underset{s \in \mathbb{R}}{\forall} \qquad sf(s) \leq (\lambda_1 - \kappa)s^2 + d,$$

and

$$\lim_{|s| \to \infty} \frac{|f'(s)|}{|s|^{q-1}} = 0.$$

It was shown in [3] that there exists a global attractor for the semigroup of global ε-regular solutions of this problem in $H_0^1(\Omega)$ (see also [9]).

Our main goal here is to study upper semicontinuity of the family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ as the perturbation parameter ε goes to 0. Some well-known papers concerning the upper semicontinuity of the family of attractors are [2], [11] and [12]. Following [6] we say that *the family of attractors* $\mathcal{A}_\lambda \subset E$ (E — complete metric space) is *upper semicontinuous* at the point λ_0 if

$$d_E(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) = \sup\{\operatorname{dist}(\psi, \mathcal{A}_{\lambda_0}) \colon \psi \in \mathcal{A}_\lambda\} \to 0, \tag{1.12}$$

as $\lambda \to \lambda_0$. We remind that the distance between an element ψ and a set $\mathcal{A}_{\lambda_0}$ is defined by the equality

$$\operatorname{dist}(\psi, \mathcal{A}_{\lambda_0}) = \inf\{\rho\,(\psi, \phi) \colon \phi \in \mathcal{A}_{\lambda_0}\},$$

where ρ denotes the distance in E. The distance from the set $\mathcal{A}_\lambda$ to the set $\mathcal{A}_{\lambda_0}$ defined in (1.12) is known as the *Hausdorff semidistance*. Note that this is different from the Hausdorff metric, which is defined as

$$\sup\{d_E(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}); d_E(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda)\}.$$

1.1. Plan of the paper. Our paper is organized as follows. In Section 2 we prove that the family of global attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in (0,1]}$ for $((1.1)–(1.3))_\varepsilon$ is bounded in $X^1 := H_B^3(\Omega) \times H_0^1(\Omega)$ uniformly with respect to the perturbation parameter ε, where $H_B^3(\Omega) := \{\phi \in H^3(\Omega) : \phi = \Delta\phi = 0 \text{ on } \partial\Omega\}$. In Section 3 we study existence of the global attractor $\widetilde{\mathcal{A}}_0$ for (1.4)–(1.6) in the phase space $H^2(\Omega) \cap H_0^1(\Omega)$. Finally, in Section 4, using the results of previous sections, we show that the family $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0,1]}$ is upper semicontinuous at 0, i.e.

$$d_{X^{1/2}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = \sup_{\psi \in \mathcal{A}_\varepsilon} \inf_{\phi \in \mathcal{A}_0} \|\psi - \phi\|_{X^{1/2}} \to 0,$$

as $\varepsilon \to 0^+$, where the set

$$\mathcal{A}_0 := \{(\phi, \psi) : \phi \in \widetilde{\mathcal{A}}_0,\ \psi = -\Delta(I - \delta\Delta)^{-1}(f(\phi) + \Delta\phi)\},$$

is a "natural" embedding of $\widetilde{\mathcal{A}}_0$ into $X^{1/2}$. This embedding is introduced to make possible a comparison of the attractor of the perturbed viscous Cahn–Hilliard equation, which lies in $(H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$ with the attractor for the viscous Cahn–Hilliard equation lying in $H^2(\Omega) \cap H_0^1(\Omega)$.

1.2. Notations. In this article all the Sobolev spaces $H^k(\Omega)$ are considered for functions defined on fixed domain $\Omega \subset \mathbb{R}^n$, so we use the simplified notation $H^k = H^k(\Omega)$ throughout. The norm in L^2 is denoted by $\|\cdot\|$ and the scalar product on this space by $\langle\,\cdot\,,\,\cdot\,\rangle$. We reserve the letter K with suitable subscripts to denote the constants appearing in the embedding estimates.

We denote by $(-\Delta)$ the minus Laplace operator with domain $D(-\Delta) = H^1_0$, extended to the space H^{-1}. Simultaneously we consider the L^2-realization, $(-\Delta_{L^2})$, of $(-\Delta)$ with Dirichlet condition (see [1]) that is the linear operator in L^2 defined by

$$D(-\Delta_{L^2}) := \{u \in L^2 \cap D(-\Delta) : (-\Delta)u \in L^2\}, \quad (-\Delta_{L^2})u := (-\Delta)u.$$

We preserve the notation $(-\Delta)$ for such L^2-realization. Since $(-\Delta)$ is an unbounded, closed, positive self-adjoint linear operator with compact resolvent in L^2, we can define for $s \in \mathbb{R}$ its fractional powers $(-\Delta)^s$. The domain $D((-\Delta)^s)$ of $(-\Delta)^s$ endowed with the scalar product and the norm

$$\begin{cases} \langle u, v\rangle_{D((-\Delta)^s)} = \langle(-\Delta)^s u, (-\Delta)^s v\rangle, \\ \|u\|_{D((-\Delta)^s)} = (\langle u, u\rangle_{D((-\Delta)^s)})^{1/2}, \end{cases}$$

is a Hilbert space for any $s > 0$. Let $D((-\Delta)^{-s})$ denote the dual spaces of $D((-\Delta)^s)$ $(s > 0)$. These Hilbert spaces can be endowed with the product and the norm like above, with s replaced by $-s$ (see [15, Section 2.1]). Moreover, we infer from [13, Section 1.4] that for $\alpha > 0$, $H^\alpha \supset D((-\Delta)^{\alpha/2})$, since the space on the right-hand side contains some boundary conditions. The inner product on H^{-1} will be introduced as

$$\langle\phi, \varphi\rangle_{H^{-1}} = \langle(-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\varphi\rangle, \quad \varphi, \phi \in H^{-1}.$$

Using Poicaré's inequality, for $\phi \in H^1_0$, we have

$$\|\phi\|_{H^{-1}} \le \frac{1}{\sqrt{\lambda_1}}\|(-\Delta)^{1/2}(-\Delta)^{-1/2}\phi\| = \frac{1}{\sqrt{\lambda_1}}\|\phi\| \le \frac{1}{\lambda_1}\|(-\Delta)^{1/2}\phi\| = \frac{1}{\lambda_1}\|\phi\|_{H^1_0}.$$

2. Uniform boundedness of the global attractors

We will study properties of global solutions of the Cauchy problem for the perturbed viscous Cahn–Hilliard equation

$$\text{(2.1)} \qquad \begin{cases} \varepsilon u^\varepsilon_{tt} + u^\varepsilon_t + \Delta(\Delta u^\varepsilon + f(u^\varepsilon) - \delta u^\varepsilon_t) = 0, & x \in \Omega,\ t > 0, \\ u^\varepsilon(0, x) = u_0(x), \quad u^\varepsilon_t(0, x) = v_0(x), & x \in \Omega, \\ u^\varepsilon(t, x) = 0, \quad \Delta u^\varepsilon(t, x) = 0, & x \in \partial\Omega,\ t \ge 0, \end{cases}$$

where $\varepsilon, \delta \in (0, 1]$, $n \le 3$, $f(0) = 0$ and the nonlinear term f satisfies the assumptions (1.7)–(1.9) and (1.11).

Since all the estimates of solutions to (2.1) obtained in this section are independent of ε, we use the simplified notation $u^\varepsilon = u$.

LEMMA 2.1. *Let $0 < \varepsilon \le 1$. If b is a non-negative real number satisfying*

$$b \le \min\left\{\frac{1}{8}; \frac{\lambda_1}{12}; \frac{\lambda_1^2}{24}\right\}, \tag{2.2}$$

then for each $(\phi, \psi) \in X := H_0^1 \times H^{-1}$ we have the following inequalities:

(a) $(1 - 2\varepsilon b)\|\psi\|_{H^{-1}}^2 + 2b\|\phi\|_{H_0^1}^2 + 2b\langle(-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\psi\rangle + \delta\|\psi\|^2 + 2b\delta\langle\phi, \psi\rangle \ge (5/3)b(\|\psi\|_{H^{-1}}^2 + \|\phi\|_{H_0^1}^2) + (\delta/2)\|\psi\|^2$,

(b) $|2b\varepsilon\langle(-\Delta)^{-1/2}\phi, (-\Delta)^{-1/2}\psi\rangle| \le (1/8)\|\phi\|_{H_0^1}^2 + (\varepsilon/8)\|\psi\|_{H^{-1}}^2$.

PROOF. For the proof it suffices to use the Cauchy inequality. □

THEOREM 2.2. *Let $0 < \varepsilon \le 1$ and $(u_0, v_0) \in X^{1/2}$. Then there exists a positive constant M_2 and, for any r_0, a positive constant $t(r_0)$ such that, for any solution u of (2.1) with $\varepsilon\|v_0\|_{H^{-1}}^2 + \|u_0\|_{H_0^1}^2 \le r_0^2$, the following estimate holds:*

$$\varepsilon\|u_t(t)\|_{H^{-1}}^2 + \|u(t)\|_{H_0^1}^2 \le M_2^2 \quad \text{for } t \ge t(r_0).$$

The constant M_2 is independent of ε.

PROOF. Fix a constant b as in (2.2). We will consider the equation obtained formally by applying $(-\Delta)^{-1}$ to (1.1), that is the equation

$$(-\Delta)^{-1}\varepsilon u_{tt} = -(-\Delta)^{-1}u_t - (-\Delta)u + f(u) - \delta u_t. \tag{2.3}$$

For any fixed $\varepsilon \in (0, 1]$ we introduce the following energy functional

$$\begin{aligned} V_\varepsilon(u, u_t) = \frac{\varepsilon}{2}\|u_t\|_{H^{-1}}^2 + \frac{1}{2}\|u\|_{H_0^1}^2 \\ - \int_\Omega \overline{F}(u)\,dx + 2\varepsilon b\langle(-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_t\rangle, \end{aligned} \tag{2.4}$$

which will now be estimated from below and from above. Using Lemma 2.1(b) and the assumption (1.9), we obtain

$$\begin{aligned} &\frac{3}{8}(\varepsilon\|u_t\|_{H^{-1}}^2 + \|u\|_{H_0^1}^2) - \overline{C}|\Omega| \\ &\qquad \le V_\varepsilon(u, u_t) \le \frac{5}{8}(\varepsilon\|u_t\|_{H^{-1}}^2 + \|u\|_{H_0^1}^2) - \int_\Omega \overline{F}(u)\,dx. \end{aligned} \tag{2.5}$$

Our next concern is to estimate $dV_\varepsilon(u, u_t)/dt$. Thanks to (2.3), we have

$$\begin{aligned} \frac{d}{dt}V_\varepsilon(u, u_t) &= -(1 - 2\varepsilon b)\|u_t\|_{H^{-1}}^2 - 2b\|u\|_{H_0^1} \\ &- 2b\langle(-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_t\rangle + 2b\langle u, f(u)\rangle - \delta\|u_t\|^2 - 2b\delta\langle u, u_t\rangle. \end{aligned} \tag{2.6}$$

Thus, from Lemma 2.1(a) it follows that

$$\frac{d}{dt}V_\varepsilon(u, u_t) \le -\frac{5}{2}\mu b(\|u_t\|_{H^{-1}}^2 + \|u\|_{H_0^1}^2) + 2b\langle u, f(u)\rangle, \tag{2.7}$$

where μ is such that condition (1.11) holds. Since the values of V_ε can be negative, we introduce another functional V_ε^* defined as

$$V_\varepsilon^*(u,u_t) = \frac{8}{5}(V_\varepsilon(u,u_t) + \overline{C}|\Omega|). \tag{2.8}$$

Note that $V_\varepsilon^*(u,u_t) \geq 0$. Furthermore, combining (1.11), (2.5), (2.7) and (2.8), we get

$$\frac{d}{dt}V_\varepsilon^*(u(t),u_t(t)) \leq -k_1 V_\varepsilon^*(u(t),u_t(t)) + k_2 \quad \text{for } t \geq 0. \tag{2.9}$$

Let us analyse the above inequality. Observe that if $V_\varepsilon^*(u_0,v_0) \geq k_2/k_1$ then there exists a constant $t(r_0)$ such that $V_\varepsilon^*(u(t),u_t(t)) \leq k_2/k_1$ for $t \geq t(r_0)$; otherwise $V_\varepsilon^*(u(t),u_t(t)) \leq k_2/k_1$ for $t \geq 0$. In particular,

$$V_\varepsilon^*(u(t),u_t(t)) \leq \max\left\{V_\varepsilon^*(u_0,v_0); \frac{k_2}{k_1}\right\} \quad \text{for } t \geq 0. \tag{2.10}$$

□

Remark 2.3. Under the assumptions (1.7)–(1.8), for any $s \in \mathbb{R}$, we have

$$|f(s)| \leq \widetilde{C}(1 + |s|^{q+1}), \tag{2.11}$$

where $q > 0$ can be arbitrarily large if $n = 1,2$ and $0 < q < 2$ if $n = 3$.

Sometimes, we use the weaker condition (2.11) instead of the condition (1.8).

Corollary 2.4. *Let $0 < \varepsilon \leq 1$ and $(u_0,v_0) \in X^{1/2}$. For arbitrary $r_0 > 0$ there exists a positive constant $C_0^*(r_0)$ (independent of ε) such that, for any solution u of (2.1) with*

$$\varepsilon\|v_0\|_{H^{-1}}^2 + \|u_0\|_{H_0^1}^2 \leq r_0^2, \tag{2.12}$$

we have the estimate

$$\varepsilon\|u_t(t)\|_{H^{-1}}^2 + \|u(t)\|_{H_0^1}^2 \leq C_0^*(r_0) \quad \textit{for } t \geq 0. \tag{2.13}$$

Proof. By (2.10) we know that

$$V_\varepsilon^*(u(t),u_t(t)) \leq V_\varepsilon^*(u_0,v_0) + \frac{k_2}{k_1},$$

and then, thanks to (2.5) and (2.8), we obtain

$$\frac{3}{5}(\varepsilon\|u_t(t)\|_{H^{-1}}^2 + \|u(t)\|_{H_0^1}^2) \leq \frac{8}{5}(V_\varepsilon(u_0,v_0) + \overline{C}|\Omega|) + \frac{k_2}{k_1}.$$

From the assumption (2.12) and estimate (2.5) it follows that

$$\varepsilon\|u_t(t)\|_{H^{-1}}^2 + \|u(t)\|_{H_0^1}^2 \leq -\frac{8}{3}\int_\Omega \overline{F}(u_0)\,dx + C(r_0).$$

Thus, as a consequence of (1.11), (2.11) and the embedding $H_0^1 \subset L^6$, $n \leq 3$, we get (2.13). □

PROPOSITION 2.5. *Let* $0 < \varepsilon \le 1$ *and* $(u_0, v_0) \in X^{1/2}$. *For any* $r_0 > 0$ *there exists a positive constant* $C_1^*(r_0)$ (*independent of* ε) *such that, for any solution* $u(t)$ *of* (2.1) *with* $\varepsilon\|v_0\|_{H^{-1}}^2 + \|u_0\|_{H_0^1}^2 \le r_0^2$, *we have the estimate*

$$\int_0^\infty \|u_t(s)\|^2 \, ds \le C_1^*(r_0). \tag{2.14}$$

PROOF. For any fixed $\varepsilon \in (0,1]$ we will consider again the functional given by (2.4), but with $b = 0$. From (2.6) and (2.8) it follows that

$$\frac{5}{8}\frac{d}{dt}V_\varepsilon^*(u,u_t) = \frac{d}{dt}V_\varepsilon(u,u_t) = -\|u_t\|_{H^{-1}}^2 - \delta\|u_t\|^2.$$

Integrating this equality over $[0,t]$, we obtain

$$V_\varepsilon^*(u(t),u_t(t)) - V_\varepsilon^*(u_0,v_0) = -\frac{8}{5}\int_0^t (\|u_t(s)\|_{H^{-1}}^2 + \delta\|u_t(s)\|^2)\, ds.$$

Note that $V_\varepsilon^*(u(t),u_t(t))$ is non-increasing in time. In particular,

$$0 \le V_\varepsilon^*(u(t),u_t(t)) \le V_\varepsilon^*(u_0,v_0), \quad t \ge 0.$$

Consequently, we have

$$\delta\int_0^t \|u_t(s)\|^2\, ds \le \int_0^t (\|u_t(s)\|_{H^{-1}}^2 + \delta\|u_t(s)\|^2)\, ds \le \frac{5}{8}V_\varepsilon^*(u_0,v_0),$$

hence

$$\int_0^\infty \|u_t(s)\|^2\, ds \le \frac{5}{8\delta}V_\varepsilon^*(u_0,v_0).$$

Estimating $V_\varepsilon^*(u_0,v_0)$, as in the proof of Corollary 2.4, we obtain (2.14). □

REMARK 2.6. The above calculations show that also the integral

$$\int_0^\infty \|u_t(s)\|_{H^{-1}}^2\, ds$$

is estimated by the same constant.

Our next goal will be to investigate the behavior of the Lyapunov type functional $\Phi_\varepsilon\colon X^{\frac{1}{2}} \to \mathbb{R}$ connected with (1.1)

$$\Phi_\varepsilon(\phi,\psi) = \varepsilon\langle\phi,\psi\rangle + \varepsilon\|\psi\|^2 + \|\phi\|_{H^2\cap H_0^1}^2 + \frac{1}{2}\|\phi\|^2 + \frac{\delta}{2}\|\phi\|_{H_0^1}^2. \tag{2.15}$$

Since $0 < \delta \le 1$, we have

$$\frac{1}{2}(\varepsilon\|\psi\|^2 + \|\phi\|_{H^2\cap H_0^1}^2) \le \Phi_\varepsilon(\phi,\psi) \le \frac{3}{2}\varepsilon\|\psi\|^2 + \|\phi\|^2 + \frac{1}{2}\|\phi\|_{H_0^1}^2 + \|\phi\|_{H^2\cap H_0^1}^2,$$

but

$$\|\phi\|_{H_0^1} \le K_3\|\phi\|_{H^2\cap H_0^1} \quad \text{and} \quad \|\phi\| \le K_4\|\phi\|_{H^2\cap H_0^1}, \tag{2.16}$$

hence

$$\frac{1}{2}(\varepsilon\|\psi\|^2 + \|\phi\|^2_{H^2\cap H^1_0}) \le \Phi_\varepsilon(\phi,\psi) \le \alpha(\varepsilon\|\psi\|^2 + \|\phi\|^2_{H^2\cap H^1_0}), \tag{2.17}$$

where $\alpha := \max\{3/2; K_4^2 + K_3^2/2 + 1\}$.

Theorem 2.7. *Let $0 < \varepsilon \le 1$ and $(u_0, v_0) \in X^{1/2}$. Then, for any $r_0 > 0$ and $r_1 > 0$ there exists a positive constant $C_2^*(r_0, r_1)$ (independent of ε) such that, for any solution u of* (2.1) *with*

$$\varepsilon\|v_0\|^2_{H^{-1}} + \|u_0\|^2_{H^1_0} \le r_0^2 \quad \textit{and} \quad \varepsilon\|v_0\|^2 + \|u_0\|^2_{H^2\cap H^1_0} \le r_1^2, \tag{2.18}$$

we have the estimate

$$\varepsilon\|u_t(t)\|^2 + \|u(t)\|^2_{H^2\cap H^1_0} \le C_2^*(r_0, r_1) \quad \textit{for } t \ge 0. \tag{2.19}$$

Remark 2.8. Note that the second condition in (2.18) is stronger than the first one, so that it suffices to assume only the second one. We keep here both conditions, since in some estimates the weaker one is sufficient.

Proof. Multiplying equation (1.1) in L^2 first by $2u_t$ then by u, we obtain

$$\frac{d}{dt}(\varepsilon\|u_t\|^2 + \|u\|^2_{H^2\cap H^1_0}) + 2\|u_t\|^2 + 2\langle\Delta f(u), u_t\rangle + 2\delta\|u_t\|^2_{H^1_0} = 0$$

and

$$\frac{d}{dt}\Big(\varepsilon\langle u, u_t\rangle + \frac{1}{2}\|u\|^2 + \frac{\delta}{2}\|u\|^2_{H^1_0}\Big) - \varepsilon\|u_t\|^2 + \|u\|^2_{H^2\cap H^1_0} + \langle\Delta f(u), u\rangle = 0.$$

Summing up these identities, recalling (2.15) and (2.17), we get

$$\begin{aligned} \frac{d}{dt}\Phi_\varepsilon(u, u_t) &+ \frac{1}{2\alpha}\Phi_\varepsilon(u, u_t) \\ &\le -2\delta\|u_t\|^2_{H^1_0} - \frac{1}{2}\|u\|^2_{H^2\cap H^1_0} + \langle-\Delta f(u), u\rangle + 2\langle-\Delta f(u), u_t\rangle. \end{aligned} \tag{2.20}$$

Our next task is to estimate the components $\langle-\Delta f(u), u\rangle$ and $2\langle-\Delta f(u), u_t\rangle$ from above. Note first that thanks to Sobolev embeddings for every $s \ge 1/(2q)$, $r \ge 1$ if $n = 1, 2$ and for every $s \in [1/(2q), 3/q]$, $r \in [1, 3)$ if $n = 3$,

$$\|u\|_{L^{2sq}} \le K\|u\|_{H^1_0} \quad \text{and} \quad \|u\|_{W^{1,2r}} \le C_1\|u\|^\eta_{H^2\cap H^1_0}\|u\|^{1-\eta}_{H^1_0}, \tag{2.21}$$

for certain $\eta \in [0, 1)$.

From (2.11) and (2.21) it follows that

$$|\langle-\Delta f(u), u\rangle| \le \frac{1}{4}\|u\|^2_{H^2\cap H^1_0} + C_1\|u\|^{2q+2}_{H^1_0} + C_2.$$

Thus Corollary 2.4 implies

$$|\langle-\Delta f(u), u\rangle| \le \frac{1}{4}\|u\|^2_{H^2\cap H^1_0} + c_1(r_0). \tag{2.22}$$

Observe that for arbitrary $\delta > 0$ condition (1.8) yields

$$|2\langle -\Delta f(u), u_t\rangle| \le 2\delta \|u_t\|^2_{H^1_0} + C_1\bigg(\int_\Omega |\nabla u|^2\,dx + \int_\Omega |u|^{2q}|\nabla u|^2\,dx\bigg).$$

Using Hölder's inequality with $s \ge \max\{1/(2q), 1\}$ if $n = 1, 2$ and $s = 3/q$ if $n = 3$, we obtain

$$|2\langle -\Delta f(u), u_t\rangle| \le 2\delta \|u_t\|^2_{H^1_0} + C_2(\|u\|^2_{H^1_0} + \|u\|^{2q}_{L^{2sq}}\|u\|^2_{W^{1,2r}}).$$

From (2.13), (2.21) and the last inequality we deduce that

$$|2\langle -\Delta f(u), u_t\rangle| \le 2\delta \|u_t\|^2_{H^1_0} + C_3(r_0) + C_4(r_0)\|u\|^{2\eta}_{H^2\cap H^1_0},$$

then using Young's inequality, we get

$$|2\langle -\Delta f(u), u_t\rangle| \le 2\delta \|u_t\|^2_{H^1_0} + \frac{1}{4}\|u\|^2_{H^2\cap H^1_0} + c_2(r_0). \tag{2.23}$$

Consequently, from (2.20), (2.22) and (2.23) we infer that

$$\frac{d}{dt}\Phi_\varepsilon(u(t), u_t(t)) + \frac{1}{2\alpha}\Phi_\varepsilon(u(t), u_t(t)) \le c_3(r_0) \quad \text{for } t \ge 0.$$

Integrating this estimate over $[0, t]$, we get

$$\Phi_\varepsilon(u(t), u_t(t)) \le \Phi_\varepsilon(u_0, v_0) + c_4(r_0) \quad \text{for } t \ge 0.$$

Thus, thanks to (2.17) and (2.18), we obtain (2.19). □

We show now that $\mathcal{A}_\varepsilon$ are bounded in "more regular" space X^1 independently of ε. Differentiating equation (1.1) with respect to t, we have

$$\varepsilon u_{ttt} + u_{tt} + (\Delta)^2 u_t + \Delta(f'(u)u_t) - \delta\Delta u_{tt} = 0.$$

Setting $v(t) = u_t(t)$ we will consider next the following problem:

$$\begin{cases} \varepsilon v_{tt} + v_t + (\Delta)^2 v - \delta\Delta v_t = -\Delta(f'(u)v) & x \in \Omega,\ t > 0, \\ v(0, x) = v_0(x), & x \in \Omega, \\ v_t(0, x) = -\dfrac{1}{\varepsilon}(v_0 + (\Delta)^2 u_0 + \Delta f(u_0) - \delta\Delta v_0) & x \in \Omega, \\ v(t, x) = 0, \quad \Delta v(t, x) = 0 & x \in \partial\Omega,\ t \ge 0. \end{cases} \tag{2.25}$$

THEOREM 2.9. *Let $0 < \varepsilon \le 1$. There exists a positive constant C^* and, for any $r_i > 0, i = 0, 1, 2$, two positive constants $C^*_3(r_0, r_2)$ and $C^*_4(r_0, r_1)$ such that, for any solution u of (2.1) with*

$$\begin{aligned} \varepsilon\|v_0\|^2_{H^{-1}} + \|u_0\|^2_{H^1_0} &\le r_0^2, \\ \varepsilon\|v_0\|^2 + \|u_0\|^2_{H^2\cap H^1_0} &\le r_1^2, \\ \varepsilon\|v_0\|^2_{H^1_0} + \|u_0\|^2_{H^3_B} &\le r_2^2, \end{aligned} \tag{2.26}$$

we have, for $t \geq 0$, the estimate

$$\varepsilon\|u_{tt}(t)\|^2_{H^{-1}} + \|u_t(t)\|^2_{H^1_0} + \|u(t)\|^2_{H^3_B} \leq \frac{C_3^*(r_0,r_2)}{\varepsilon^2}e^{-C^*t} + C_4^*(r_0,r_1).$$

The constants C^*, $C_3^*(r_0,r_2)$ and $C_4^*(r_0,r_1)$ are independent of ε.

PROOF. Fix b satisfying (2.1). For any $\varepsilon \in (0,1]$ we introduce the following energy functional

$$W_\varepsilon(\phi,\psi) = \frac{\varepsilon}{2}\|\psi\|^2_{H^{-1}} + \frac{1}{2}\|\phi\|^2_{H^1_0} + 2\varepsilon b\langle(-\Delta)^{-\frac{1}{2}}\phi, (-\Delta)^{-\frac{1}{2}}\psi\rangle.$$

Let us analyse its basic properties, which prove among other things that W_ε defines an equivalent norm of the product space $H^1_0 \times H^{-1}$. Note that estimating W_ε exactly as V_ε before, we obtain

$$\frac{3}{8}(\varepsilon\|\psi\|^2_{H^{-1}} + \|\phi\|^2_{H^1_0}) \leq W_\varepsilon(\phi,\psi) \leq \frac{5}{8}(\varepsilon\|\psi\|^2_{H^{-1}} + \|\phi\|^2_{H^1_0}). \tag{2.27}$$

Our main goal here is to estimate $dW_\varepsilon(v,v_t)/dt$. Since

$$\varepsilon(-\Delta)^{-1}v_{tt} = -(-\Delta)^{-1}v_t + \Delta v + f'(u)v - \delta v_t,$$

we can write

$$\begin{aligned}\frac{d}{dt}W_\varepsilon(v,v_t) = -(1-2\varepsilon b)\|v_t\|^2_{H^{-1}} - 2b\|v\|^2_{H^1_0} + 2b\langle v, f'(u)v\rangle + \langle v_t, f'(u)v\rangle \\ - \delta\|v_t\|^2 - 2b\delta\langle v, v_t\rangle - 2b\langle(-\Delta)^{-1/2}v, (-\Delta)^{-1/2}v_t\rangle.\end{aligned}$$

Thus, from Lemma 2.1(a) and (2.27) it follows that

$$\frac{d}{dt}W_\varepsilon(v,v_t) \leq -k_3W_\varepsilon(v,v_t) - \frac{\delta}{2}\|v_t\|^2 + 2b\langle v, f'(u)v\rangle + \langle v_t, f'(u)v\rangle.$$

Observe that Theorem 2.7 and the estimate

$$\|u\|_{L^\infty} \leq K\|u\|_{H^2\cap H^1_0}, \quad u \in H^2 \cap H^1_0,\ n \leq 3, \tag{2.28}$$

yield

$$\|f'(u)v\| \leq \sup_{|u|\leq C_2(r_0,r_1)} |f'(u)|\|v\| \leq C_1(r_0,r_1)\|v\|,$$

so that

$$\frac{d}{dt}W_\varepsilon(v(t),v_t(t)) \leq -k_3W_\varepsilon(v(t),v_t(t)) + c_1(r_0,r_1)\|v\|^2.$$

Multiplying the above inequality by e^{k_3t} and integrating over $[0,t]$, we obtain

$$W_\varepsilon(v(t),v_t(t))e^{k_3t} - W_\varepsilon(v(0),v_t(0)) \leq \int_0^t c_1(r_0,r_1)\|v(s)\|^2e^{k_3s}\,ds.$$

Then Proposition 2.5 implies that

$$W_\varepsilon(v(t),v_t(t)) \leq W_\varepsilon(v(0),v_t(0))e^{-k_3t} + c_2(r_0,r_1),$$

but from (2.25)–(2.27)

$$W_\varepsilon(v(0), v_t(0)) \le \frac{C_2(r_0, r_2)}{\varepsilon^2},$$

hence, recalling the estimate of W_ε from below, we get

$$\varepsilon \|v_t(t)\|^2_{H^{-1}} + \|v(t)\|^2_{H^1_0} \le \frac{c_3(r_0, r_2)}{\varepsilon^2} e^{-k_3 t} + c_4(r_0, r_1). \tag{2.29}$$

Note that the proof will be completed if we estimate the component $\|u(t)\|_{H^3_B}$. Rewriting (1.1) in the form of an "elliptic" equation

$$\Delta^2 u = -\varepsilon u_{tt} - u_t - \Delta f(u) + \delta \Delta u_t$$

and because $\varepsilon, \delta \in (0,1]$, we obtain

$$\|\Delta^2 u\|^2_{H^{-1}} \le C(\varepsilon \|u_{tt}\|^2_{H^{-1}} + \|u_t\|^2_{H^{-1}} + \|\Delta f(u)\|^2_{H^{-1}} + \|\Delta u_t\|^2_{H^{-1}}).$$

Since

$$\|\Delta f(u)\|^2_{H^{-1}} \le C_2(r_0, r_1)$$

from (2.26) and (2.29) we get

$$\|u(t)\|^2_{H^3_B} \le C_2\bigg(\frac{c_3(r_0, r_2)}{\varepsilon^2} e^{-k_3 t} + c_4(r_0, r_1)\bigg).$$

This completes the proof. □

As a direct consequence of the above theorem it follows that the attractors $\mathcal{A}_\varepsilon$ are bounded in X^1 uniformly with respect to ε.

COROLLARY 2.10. *If $0 < \varepsilon \le 1$, then there exists a positive constant M_3 such that, for $0 < \varepsilon \le 1$,*

$$\forall_{(\phi,\psi)\in\mathcal{A}_\varepsilon} \|\psi\|^2_{H^1_0} + \|\phi\|^2_{H^3_B} \le M_3^2. \tag{2.30}$$

Moreover, there exists a positive constant M_4, such that, for $0 < \varepsilon \le 1$ and for any orbit $U_\varepsilon(t) = (u^\varepsilon(t), u^\varepsilon_t(t))$ of (2.1) with $U_\varepsilon(\mathbb{R}) \subset \mathcal{A}_\varepsilon$,

$$\sqrt{\varepsilon}\|u^\varepsilon_{tt}(t)\|_{H^{-1}} \le M_4. \tag{2.31}$$

3. Viscous Cahn–Hilliard equation

Now we will study viscous Cahn–Hilliard equation

$$\begin{cases} u_t + \Delta(\Delta u + f(u) - \delta u_t) = 0, & x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = 0, \quad \Delta u(t,x) = 0, & x \in \partial\Omega,\ t \ge 0, \end{cases} \tag{3.1}$$

where $\delta \in (0,1]$, $n \le 3$, $f(0) = 0$ and $f \in C^2(\mathbb{R}, \mathbb{R})$.

3.1. Setting of the problem and local solvability of (3.2). Let A_δ and B_δ denote realizations in H_0^1 of the operators $(-\Delta)^2(I-\delta\Delta)^{-1}$ and $(-\Delta)(I-\delta\Delta)^{-1}$, with domains $D(A_\delta)=H_B^3$ and $D(B)=H_0^1$, respectively. We now discuss properties of A_δ. We start with writing it in an equivalent form

$$A_\delta = \frac{1}{\delta}(-\Delta) - \frac{1}{\delta^2}I - \frac{1}{\delta^3}\bigg(-\frac{1}{\delta}I+\Delta\bigg)^{-1}.$$

We know that the operator $(-\Delta)$ has several nice properties; it is closed, positive definite and self-adjoint operator on H_0^1, so that $-1/\delta \in \varrho(-\Delta)$. Thus, [7, Lemma 1.1.10] implies that $(-(1/\delta)I+\Delta)^{-1}\in\mathcal{L}(H_0^1)$. Finally, using [5, Proposition 1.3.2], we obtain that A_δ is sectorial. It is also easy to show that A_δ has compact resolvent.

With the use of operators A_δ and B_δ the problem (3.1) will be rewritten abstractly on H_0^1 as

$$\begin{cases} u_t = -A_\delta u + B_\delta f(u), & t>0,\\ u(0)=u_0. \end{cases} \tag{3.2}$$

Note that the function $B_\delta f\colon H^2\cap H_0^1 \to H_0^1$ is well defined. Indeed, taking $\phi\in H^2\cap H_0^1$, we have

$$\begin{aligned} \|B_\delta f(\phi)\|_{H_0^1} &\le \frac{1}{\delta}(\|f(\phi)\|_{H_0^1} + \|(I-\delta\Delta)^{-1}f(\phi)\|_{H_0^1})\\ &\le \frac{C_1}{\delta}(\|f(\phi)\|_{H_0^1} + \|f(\phi)\|_{H^{-1}}) \le C\|f'(\phi)\nabla\phi\|. \end{aligned}$$

Thus, from the assumption that $f\in C^2(\mathbb{R},\mathbb{R})$ and the estimate (2.28), we deduce that the right-hand side of the last inequality is finite.

Theorem 3.1. *Let $u_0\in H^2\cap H_0^1$. Then there exists a unique local solution u of the problem* (3.2) *in H_0^1, defined on its maximal interval of existence* $(0,\tau_{\max})$ *and satisfying*

$$u\in C([0,\tau_{\max}),H^2\cap H_0^1)\cap C^1((0,\tau_{\max}),H_0^1)\cap C((0,\tau_{\max}),D(A_\delta)).$$

Proof. Since A_δ is a sectorial operator, it suffices to show (see [5, Chapter 2], [13, Chapter 3]) that the function $B_\delta f$ is Lipschitz continuous on each bounded subset of $H^2\cap H_0^1$. Fix such a bounded set G and let $\phi,\psi\in G$. Then, due to (1.7) and (2.28), we obtain

$$\begin{aligned} \|B_\delta f(\phi)-B_\delta f(\psi)\|_{H_0^1} &\le C_1(\|f'(\phi)\nabla(\phi-\psi)\| + \|\nabla\psi(f'(\phi)-f'(\psi))\|)\\ &\le C_2(G)(\|\phi-\psi\|_{H_0^1} + \|\nabla\psi\|\|\phi-\psi\|_{L^\infty})\\ &\le C_3(G)\|\phi-\psi\|_{H^2\cap H_0^1}. \end{aligned}$$

□

Until the end of this section we assume that the conditions (1.9), (2.11) and also

$$\text{(3.3)}\qquad \mathop{\exists}_{\sigma>0}\ \mathop{\exists}_{0<\mu\le\min\{1;2\sigma\lambda_1/(\lambda_1+1)\}}\ \mathop{\exists}_{C_\sigma>0}\ \mathop{\forall}_{s\in\mathbb{R}}\quad sf(s)-2\mu\overline{F}(s)\le-\sigma s^2+C_\sigma,$$

hold.

Our next goal will be to investigate the behavior of the Lyapunov type functional $\Phi\colon H_0^1\to\mathbb{R}$ connected with (3.2)

$$\text{(3.4)}\qquad \Phi_0(u)=\frac{1}{2}\|u\|^2_{H^{-1}}+\|u\|^2_{H_0^1}-2\int_\Omega\overline{F}(u)\,dx+\frac{\delta}{2}\|u\|^2.$$

It follows from the above assumptions that Φ_0 is well defined.

Lemma 3.2. *Under the assumptions* (1.9), (2.11), (3.3) *and as long as a local solution u to* (3.2) *exists, we have*

$$\text{(3.5)}\qquad \Phi_0(u(t))\le(\Phi_0(u_0)-\frac{M_1}{\mu})e^{-\mu t}+\frac{M_1}{\mu},$$

where M_1 is a positive constant and μ is as in (3.3).

Proof. Let μ be such that the condition (3.3) holds. For the proof we shall consider the equation obtained formally by applying the operator $(-\Delta)^{-1}$ to (3.1), that is the equation

$$\text{(3.6)}\qquad (-\Delta)^{-1}u_t+(-\Delta)u-f(u)+\delta u_t=0.$$

Multiplying (3.6) in L^2 first by $2u_t$ then by u, we obtain

$$\frac{d}{dt}\left(\|u\|^2_{H_0^1}-2\int_\Omega\overline{F}(u)\,dx\right)+2\|u_t\|^2_{H^{-1}}+2\delta\|u_t\|^2=0$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\|u\|^2_{H^{-1}}+\frac{\delta}{2}\|u\|^2\right)+\|u\|^2_{H_0^1}-\int_\Omega f(u)u\,dx=0.$$

Adding these identities and recalling (3.4), we get

$$\frac{d}{dt}\Phi_0(u)\le-\|u\|^2_{H_0^1}+\int_\Omega f(u)u\,dx.$$

Thus, thanks to (3.3), we infer that

$$\text{(3.7)}\qquad \frac{d}{dt}\Phi_0(u(t))+\mu\Phi_0(u(t))\le C_\sigma|\Omega|=:M_1.$$

Integrating the last inequality over $[0,t]$, we obtain (3.5). □

Corollary 3.3. *Under the assumptions* (1.9), (2.11) *and* (3.3), *as long as a local solution u to* (3.2) *exists, we have*

$$\|u(t)\|_{H_0^1} \le c(\|u_0\|_{H_0^1}),$$

where $c\colon [0,\infty) \to [0,\infty)$ is some locally bounded function.

Proof. Since for $u \in H_0^1$ conditions (2.11) and (3.3) give

$$\Phi_0(u) \le C_1\|u\|_{H_0^1}^2 + \frac{1}{\mu}(\widetilde{C}\|u\|_{H_0^1} + \widetilde{C}\|u\|_{H_0^1}^{q+2} + M_1),$$

thanks to Lemma 3.2 and (1.9), we obtain

$$\|u(t)\|_{H_0^1}^2 \le C_2(\|u_0\|_{H_0^1}^2 + \|u_0\|_{H_0^1} + \|u_0\|_{H_0^1}^{q+2})e^{-\mu t} + \frac{M_1}{\mu} + 2\overline{C}|\Omega|. \tag{3.8}$$ □

Remark 3.4. In Lemma 3.2 and Corollary 3.3 it suffice to assume that (2.11) holds with $q \le 4$ for $n = 3$. Moreover, as a direct consequence of (3.8), when $u_0 \in H^2 \cap H_0^1$ we obtain

$$\|u(t)\|_{H_0^1} \le \overline{c}(\|u_0\|_{H^2\cap H_0^1}),$$

where $\overline{c}\colon [0,\infty) \to [0,\infty)$ is some locally bounded function. Note that the above estimate is true for arbitrarily large q in (2.11) for $n \le 3$.

3.2. Global solution. Under an additional growth restriction (1.8) on the derivative of f the local solution will be now extended to the global one.

Theorem 3.5. *Under the assumptions* (1.8), (1.9) *and* (3.3) *local solution to* (3.2) *exists globally in time.*

Proof. By (1.8), (3.3) we get

$$\|B_\delta f(u)\|_{H_0^1} \le C_1\bigg[\bigg(\int_\Omega |\nabla u|^2\,dx\bigg)^{1/2} + \bigg(\int_\Omega |u|^{2q}|\nabla u|^2\,dx\bigg)^{1/2}\bigg].$$

Using Hölder's inequality with $s > \max\{1/(2q), 1\}$ if $n = 1, 2$ and $s = 3/q$ if $n = 3$, $(r = s/(s-1))$ we obtain

$$\|B_\delta f(u)\|_{H_0^1} \le C_2(\|u\|_{H_0^1} + \|u\|_{L^{2sq}}^q\|u\|_{W^{1,2r}}).$$

Consequently, from (2.21),

$$\begin{aligned}\|B_\delta f(u)\|_{H_0^1} &\le C\max\{\|u\|_{H_0^1}, \|u\|_{H_0^1}^{q+1-\eta}\}(1 + \|u\|_{H^2\cap H_0^1}^\eta)\\ &\le g(\|u\|_{H_0^1})(1 + \|u\|_{H^2\cap H_0^1}^\eta),\end{aligned}$$

where $g\colon [0,\infty) \to [0,\infty)$ is some non-decreasing function, so that any local solution to (3.2) exists globally in time (see [5, Theorem 3.1.1]). □

3.3. Existence of the global attractor for (3.2). Denote by $\{S_0(t)\}$ the C^0-semigroup of global solutions to (3.2) defined on $H^2 \cap H_0^1$ via the relation

$$\mathcal{S}_0(t)u_0 = u(t), \quad t \geq 0.$$

Following [5, Section 1.1] and [6, Section 1.6] we now study existence and structure of the global attractor for the semigroup $\{\mathcal{S}_0(t)\}$. Since the resolvent of A_δ is compact, we know in advance (see [5, Theorem 3.3.1]) that the semigroup is compact. If we show that the semigroup $\{\mathcal{S}_0(t)\}$ is point dissipative and that there exists a "nice" Lyapunov type functional $\mathcal{L}$ for $\{\mathcal{S}_0(t)\}$, then $\{\mathcal{S}_0(t)\}$ will have a global attractor $\widetilde{\mathcal{A}}_0$ coinciding with the unstable manifold of $E_0 := \{\phi \in H^2 \cap H_0^1 : \mathcal{S}_0(t)\phi = \phi \text{ for } t \geq 0\}$ in $H^2 \cap H_0^1$ (see [5, Corollary 1.1.6] and [6, Theorem 6.1]).

We first show that the semigroup $\{\mathcal{S}_0(t)\}$ is point dissipative. To this end, it suffices to prove (see [5, Corollary 4.1.4] that for all $u_0 \in H^2 \cap H_0^1$

$$\limsup_{t\to\infty} \|u(t)\|_{H_0^1} \leq \sqrt{\frac{M_1}{\mu} + 2\overline{C}|\Omega|},$$

where M_1 is the constant in (3.7). Note that this inequality follows directly from (3.8).

We now analyse basic properties of the functional $\mathcal{L} : H^2 \cap H_0^1 \to \mathbb{R}$ defined as

$$\mathcal{L}(u) = \|u\|_{H_0^1}^2 - 2\int_\Omega \overline{F}(u)\,dx. \tag{3.9}$$

Proposition 3.5. *Under the assumptions* (1.9), (2.11) *and* (3.3) *the following properties of the functional $\mathcal{L}$ hold:*

(a) *$\mathcal{L}$ is bounded from below.*
(b) *$\mathcal{L}$ is continuous.*
(c) *For each $u_0 \in H^2 \cap H_0^1$ the function $0 < t \mapsto \mathcal{L}(\mathcal{S}_0(t)u_0)$ is nonincreasing.*
(d) *If $\mathcal{L}(\mathcal{S}_0(t)u_0) = \mathcal{L}(u_0)$ for all $t > 0$ and some $u_0 \in H^2 \cap H_0^1$, then $\mathcal{S}_0(t)u_0 = u_0$ for all $t > 0$.*

Proof. (a) From (1.9) and (3.9) we obtain $\mathcal{L}(u) \geq -2\overline{C}|\Omega|$.

(b) Let $u, u_k \in H^2 \cap H_0^1$ be such that $\|u_k - u\|_{H^2\cap H_0^1} \to 0$ as $k \to \infty$, hence we may assume that $\|u_k\|_{L^\infty}, \|u\|_{L^\infty} \leq M$. Since

$$|\mathcal{L}(u_k) - \mathcal{L}(u)| \leq |\|u_k\|_{H_0^1}^2 - \|u\|_{H_0^1}^2| + 2\int_\Omega \left|\overline{F}(u_k) - \overline{F}(u)\right| dx,$$

it suffices to show that $\int_\Omega |\overline{F}(u_k) - \overline{F}(u)|\,dx \to 0$ as $k \to \infty$. From (1.7) we have

$$\int_\Omega |\overline{F}(u_k) - \overline{F}(u)|\,dx \leq \int_\Omega \left|\int_u^{u_k} |f(s)|\,ds\right| dx \leq C \sup_{|s|\leq M} |f(s)| \|u_k - u\|_{L^\infty}.$$

(c) Multiplying the equation $(-\Delta)^{-1}u_t - \Delta u - f(u) + \delta u_t = 0$ in L^2 by $2u_t$ and recalling (3.9), we get

$$\frac{d}{dt}\mathcal{L}(u(t)) = -2\|u_t\|^2_{H^{-1}} - 2\delta\|u_t\|^2 \leq 0.$$

(d) Let $u_0 \in H^2 \cap H^1_0$ be such that $\mathcal{L}(\mathcal{S}_0(t)u_0) = \mathcal{L}(u_0)$ for $t > 0$. Then we have

$$\frac{d}{dt}\mathcal{L}(u_0) = \frac{d}{dt}\mathcal{L}(u(t)) = -2\|u_t\|^2_{H^{-1}} - 2\delta\|u_t\|^2,$$

but the left side is independent for t, hence $\|u_t\|_{H^{-1}} = \|u_t\| = 0$, so that $u_t(t,x) = 0$ almost everywhere for $t > 0$. $\square$

Remark 3.6. Note that the functional $\mathcal{L}$ is "almost" as good as the one in the definition of the gradient system in [11]. It is easy to see that the condition (ii$_2$) of [11, Definition 3.8.1.]; $\mathcal{L}(u) \to \infty$ as $\|u\|_{H^2\cap H^1_0} \to \infty$, is not satisfied. As was observed in [8] *this condition was used in* [11] *to show that orbits of bounded sets are bounded.* Notice that for the problem considered here this property is also satisfied.

We have thus verified all the conditions required in [5], [6] for the existence of the global attractor. We have

Theorem 3.7. *Under the assumptions* (1.8), (1.9) *and* (3.3) *the semigroup* $\{\mathcal{S}_0(t)\}$ *has a global attractor* $\widetilde{\mathcal{A}}_0$ *coinciding with the unstable manifold of* E_0 *in* $H^2 \cap H^1_0$.

4. Upper semicontinuity of the global attractors at zero

In this section we will prove that the family of attractors $\{\mathcal{A}_\varepsilon\}_{\varepsilon\in[0,1]}$ is upper semicontinuous at $\varepsilon = 0$.

Theorem 4.1. *Under the assumptions* (1.7)–(1.9) *and* (1.11) *we have*

$$\lim_{\varepsilon\to 0^+} d_{X^{1/2}}(\mathcal{A}_\varepsilon, \mathcal{A}_0) = 0.$$

Proof. According to [12, p. 211] it suffices to show that the following property holds:

> Let $\{\varepsilon_k\}$ be a sequence of positive numbers converging to 0 when k goes to infinity and $\{u^k\}$ be the corresponding sequence of solutions to $(2.1)_{\varepsilon_k}$ such that, for any $t \in \mathbb{R}$, $(u^k(t), u^k_t(t)) \in \mathcal{A}_{\varepsilon_k}$. There is a subsequence $\{\varepsilon_{j_k}\}$ of $\{\varepsilon_k\}$ such that $(u^{j_k}(0), u^{j_k}_t(0))$ converges to (u_0, v_0) in $X^{1/2}$ and (u_0, v_0) belongs to $\mathcal{A}_0$.

Note that thanks to (2.30) the set $V := \bigcup_{t\in\mathbb{R}}\bigcup_{k\in\mathbb{N}}\{u^k(t)\}$ is bounded in H^3_B. Since the embedding $H^3_B \subset H^2 \cap H^1_0$ is compact and $H^2 \cap H^1_0$ is a Banach space, from each sequence of elements of V we can choose a subsequence, which converges to some element of $H^2 \cap H_0^1$. Hence V is a precompact set in $H^2 \cap H_0^1$ and for every $t \in \mathbb{R}$ the set $V(t) := \bigcup_{k\in\mathbb{N}}\{u^k(t)\}$ is precompact in $H^2 \cap H_0^1$ as a subset of a precompact set. Let $\{[-m,m] : m \in \mathbb{N}\}$ be the sequence of compact intervals in $\mathbb{R}$. Thus, from [16, Chapter 3, Lemma 1.1] and estimate (2.30) we infer that for all $m \in \mathbb{N}$ the family of mappings $u^k \in C(\mathbb{R}; H^2 \cap H_0^1)$, $k \in \mathbb{N}$, is equicontinuous on $[-m,m]$. Using Ascoli–Arzelà's theorem (see [17, p. 10]) it is easy to show by induction that we can choose a subsequence $\{u^{k_{m+1}}\}$ of $\{u^{k_m}\}$ such that $\{u^{k_{m+1}}\}$ converges to u in $C([-m-1,m+1]; H^2 \cap H_0^1)$. Then using a classical diagonalization procedure, we choose a subsequence of positive numbers $\{\varepsilon_{j_k}\}$ of $\{\varepsilon_k\}$ and the corresponding subsequence $\{u^{j_k}\}$ of $\{u^k\}$, of solutions to $(2.1)_{\varepsilon_{j_k}}$, such that

$$u^{j_k} \to u \quad \text{in } C([-a,a]; H^2 \cap H_0^1) \text{ for any } a > 0. \tag{4.1}$$

In particular, $u^{j_k}(0)$ converges to $u(0) =: u_0$ in $H^2 \cap H_0^1$. Thanks to (2.30) and (4.1), we obtain

$$u \in C_b(\mathbb{R}; H^2 \cap H_0^1). \tag{4.2}$$

Fix $a > 0$. As a direct consequence of (4.1) we obtain

$$\begin{aligned} \Delta u^{j_k} &\to \Delta u \quad \text{in } C([-a,a]; L^2), \\ \Delta^2 u^{j_k} &\to \Delta^2 u \quad \text{in } C([-a,a]; H^{-2}). \end{aligned} \tag{4.3}$$

Furthermore, (see [10, Section 4.1]), from (4.1) and (4.3) it follows that

$$(u_t^{j_k} - \delta\Delta u_t^{j_k}) \to (u_t - \delta\Delta u_t) \quad \text{in } \mathcal{D}'(-a,a; L^2). \tag{4.4}$$

Our next objective is to show that the component $(u_t^{j_k} - \delta\Delta u_t^{j_k})$ converges in $C([-a,a]; H^{-2})$ to $(-\Delta^2 u - \Delta f(u))$. Since

$$u_t^{j_k} - \delta\Delta u_t^{j_k} = -\varepsilon u_{tt}^{j_k} - \Delta^2 u^{j_k} - \Delta f(u^{j_k}),$$

it suffices to study convergence of the right-hand side of the above equality. Thanks to (2.31), we obtain

$$\varepsilon_{j_k} u_{tt}^{j_k}(t) \to 0 \quad \text{in } C([-a,a]; H^{-1}). \tag{4.5}$$

Observe that since $f \in C^2(\mathbb{R},\mathbb{R})$, we can write

$$\begin{aligned} &\sup_{t\in[-a,a]} \|\Delta f(u^{j_k}) - \Delta f(u)\|_{H^{-1}} \\ &\qquad \le C_1 \sup_{t\in[-a,a]} \| f'(u^{j_k})|\nabla u^{j_k} - \nabla u| + |\nabla u|(f'(u^{j_k}) - f'(u))\|, \end{aligned}$$

but from (2.30) and (4.2) we have

$$\sup_{t\in\mathbb{R}} |u^{j_k}(t)| \le \sup_{t\in\mathbb{R}} \|u^{j_k}(t)\|_{H_B^3} \le M_3 \quad \text{and} \quad \sup_{t\in\mathbb{R}} |u(t)| \le \sup_{t\in\mathbb{R}} \|u(t)\|_{H^2\cap H_0^1} \le M_3,$$

so that

$$\sup_{t\in[-a,a]} \|\Delta f(u^{j_k}) - \Delta f(u)\|_{H^{-1}} \le C_2 \sup_{t\in[-a,a]} (\|\nabla u^{j_k} - \nabla u\| + \|u^{j_k} - u\|_{L^\infty}).$$

Consequently,

$$\Delta f(u^{j_n}) \to \Delta f(u) \quad \text{in } C([-a,a]; H^{-1}). \tag{4.6}$$

Finally, from (4.3), (4.5) and (4.6) we deduce

$$(u_t^{j_k} - \delta\Delta u_t^{j_k}) \to (-\Delta^2 u - \Delta f(u)) \quad \text{in } C([-a,a]; H^{-2}),$$

so that $u_t^{j_k}(0)$ converges to $-(I-\delta\Delta)^{-1}\Delta(\Delta u_0 + f(u_0)) =: v_0$ in L^2. It remains to show that $u_0 \in \widetilde{\mathcal{A}}_0$. By uniqueness of the limit in $\mathcal{D}'(-a,a;H^{-2})$, we have

$$u_t - \delta\Delta u_t = -\Delta^2 u - \Delta f(u),$$

but the right-hand side of the above equality belongs to $C([-a,a];H^{-2})$, hence $(I-\delta\Delta)u_t$ belongs to $C((-a,a);H^{-2})$. Consequently,

$$u_t \in C(\mathbb{R}; L^2) \tag{4.7}$$

and

$$\begin{cases} u_t = -A_\delta u + B_\delta f(u) & \text{in } L^2 \text{ for a.e. } t, \\ u(0) = u_0 \in H^2 \cap H_0^1. \end{cases} \tag{4.8}$$

From (4.2) and (4.7) it follows that u is a solution of (4.8) in L^2. Then [5, Corollary 2.3.1] implies that u_t belongs to $C(\mathbb{R}; H_0^1)$, so that the equality (4.8) holds in H_0^1 for almost every t . Finally, using regularity properties of the operator A_δ, we obtain that u is a solution of (4.8) in H_0^1. Since u is a bounded complete trajectory of the semigroup $\{\mathcal{S}_0(t)\}$, thanks to [4, Chapter 2, Theorem 3.2], we conclude that u_0 belongs to $\widetilde{\mathcal{A}}_0$, which completes the proof. □

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