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BEST PROXIMITY POINTS OF CYCLIC φ-CONTRACTIONS IN ORDERED METRIC SPACES

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Abstract. In this paper, we shall give some results about best proximity points of cyclic φ-contractions in ordered metric spaces. These results generalize some known results.

1. Introduction

Let (X,d) be a complete metric space. The well-known Banach contraction theorem assures us a unique fixed point of a contraction $T\colon X \to X$. As a generalization of the Banach contraction principle, W. A. Kirk et al. proved the following fixed point result in 2003 ([5]).

Theorem 1.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Suppose that $T\colon A \cup B \to A \cup B$ is a map satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and there exists $k \in (0,1)$ such that $d(Tx,Ty) \le kd(x,y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

Let A and B be nonempty subsets of a metric space (X,d). We say that a map $T\colon A \cup B \to A \cup B$ is cyclic whenever $T(A) \subseteq B$ and $T(B) \subseteq A$. The map T is called a cyclic contraction whenever T is a cyclic map and there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \le \alpha d(x,y) + (1-\alpha)d(A,B)$$

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for all $x \in A$ and $y \in B$ ([5]). If $\varphi\colon [0,\infty) \to [0,\infty)$ is a strictly increasing map, then we say that the map T is a cyclic φ-contraction map whenever T is a cyclic map and

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B))$$

for all $x \in A$ and $y \in B$ ([1]). Also, $x \in A \cup B$ is called a best proximity point if $d(x,Tx) = d(A,B)$. Note that, a best proximity point x is a fixed point of T whenever $A \cap B \neq \emptyset$. Thus, it generalizes the notion of fixed point in case when $A \cap B = \emptyset$. Recently, J. Anuradha and P. Veeramani provided the notion of proximal pointwise contraction maps ([2]). They gave a result about best proximity points of proximal pointwise contraction maps whenever (A,B) is a nonempty weakly compact convex pair in a Banach space.

In 2005, G. Petruşel proved some results about periodic points of cyclic contraction maps ([6]). Then, A. A. Eldered and P. Veeramani proved some results about best proximity points of cyclic contraction maps in 2006 ([3]). They raised a question about the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space. In 2009, M. A. Al-Thagafi and N. Shahzad gave a positive answer to the question ([1]).

In this paper, we shall give some results about best proximity points of cyclic φ-contractions in ordered metric spaces. Note that a mapping on an ordered (cone) metric space can be a contraction but it is not a contraction in classical sense ([4]).

Let X be a nonempty set and T a selfmap on X. We denote the set of all nonempty subsets of X by 2^X and the set of all invariant nonempty subsets of X by $I(T)$, that is

$$I(T) = \{Y \in 2^X : T(Y) \subseteq Y\}.$$

For each pair of sets X and Y and selfmaps $T\colon X \to X$ and $S\colon Y \to Y$, we define the selfmap $T \times S\colon X \times Y \to X \times Y$ by $T \times S(x,y) = (Tx, Sy)$. If $(X,\le)$ is a partially ordered set, then we define

$$X_{\le} = \{(x,y) \in X \times X : x \le y \text{ or } y \le x\}.$$

Let $(X,d,\le)$ be an ordered metric space and $T\colon X \to X$ a selfmap on X. For each nonempty subset C of X and $x^* \in X$, we define

$$E_{T,C}(x^*) = \Big\{ x \in C : \lim_{n\to\infty} T^{2n}x = x^* \Big\}.$$

The space X is called regular whenever every bounded monotone sequence in X is convergent. We say that a selfmap $T\colon X \to X$ is orbitally continuous whenever for each $x \in X$ and sequence $\{n(i)\}_{i\ge1}$ with $T^{n(i)}x \to a$ for some $a \in X$, we have $T^{n(i)+1}x \to Ta$. Here, $T^{m+1} = T(T^m)$.

2. Main results

Now, we are ready to state and prove our results.

Theorem 2.1. *Let $(X,d,\leq)$ be an ordered metric space, $A,B \in 2^X$ and T a decreasing selfmap on $A\cup B$ such that $T(A)\subseteq B$ and $T(B)\subseteq A$. Suppose that there exists $x_0\in A$ such that $x_0 \leq T^2x_0 \leq Tx_0$ and*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B))$$

for all $x\in A$ and $y\in B$ with $x\leq y$, where $\varphi\colon[0,\infty)\to[0,\infty)$ is a strictly increasing map. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1},x_n)$ for all $n\geq 0$, then $d_n \to d(A,B)$.

Proof. First note that we have

$$x_0 \leq x_2 \leq \ldots \leq x_{2n} \leq x_{2n+1} \leq \ldots \leq x_3 \leq x_1$$

for all $n\geq 1$. Thus, we obtain

$$0 \leq d_{n+1} \leq d_n - \varphi(d_n) + \varphi(d(A,B))$$

for all $n\geq 1$. Hence, the sequence $\{d_n\}$ is decreasing and bounded from below. If $d_{n_0} = 0$ for some n_0, then $d_n \to d(A,B) = 0$. Suppose that $d_n > 0$ for all $n \geq 1$ and $d_n \to t_0$ for some $t_0 \geq d(A,B)$. Since

$$\varphi(d(A,B)) \leq \varphi(d_n) \leq d_n - d_{n+1} + \varphi(d(A,B)),$$

we have $\varphi(d_n)\to\varphi(d(A,B))$. This implies that $\varphi(t_0)=\varphi(d(A,B))$. So, $t_0 = d(A,B)$ because φ is strictly increasing. □

Theorem 2.2. *Let $(X,d,\leq)$ be a regular ordered metric space, $B\in 2^X$, A a closed nonempty subset of X and T a decreasing selfmap on $A\cup B$ such that $T(A)\subseteq B$ and $T(B)\subseteq A$. Suppose that there exists $x_0\in A$ such that $x_0\leq T^2x_0\leq Tx_0$ and*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B))$$

for all $x\in A$ and $y\in B$ with $x\leq y$, where $\varphi\colon[0,\infty)\to[0,\infty)$ is a strictly increasing map. If T is orbitally continuous, then there exists $x\in A$ such that $d(x,Tx) = d(A,B)$.

Proof. Again, note that $x_0\leq x_2\leq\ldots\leq x_{2n}\leq x_1$ for all $n\geq 1$. Since X is regular and A is closed, there exists $x\in A$ such that $x_{2n}\to x$. Also, note that

$$d(A,B) \leq d(x_{2n},Tx) = d(Tx_{2n-1},Tx) \leq d(Tx_{2n-1},Tx_{2n}) + d(Tx_{2n},Tx)$$

for all $n\geq 1$. If T is orbitally continuous, then $d(Tx_{2n},Tx)\to 0$. Hence, $d(x,Tx) = d(A,B)$ because $d(Tx_{2n-1},Tx_{2n})\to d(A,B)$ by Theorem 2.1. □

We note that T is not a cyclic φ-contraction in [1, Example 3]. To see this, let $x = -1/2$ and $y = 1/2$. Then $2/3 = d(Tx, Ty) > d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) = 1/2$. For improvment it is sufficient that we change the function φ by $\varphi(t) = t^2/(2(1 + t))$. The following is another example for a cyclic φ-contraction.

Example 2.3. Consider the Euclidian ordered metric space $X = \mathbb{R}$ with the usual norm. Suppose that $A = [-1, 0]$, $B = [0, 1]$ and $T\colon A \cup B \to A \cup B$ is defined by $Tx = -x/3$ for all $x \in A \cup B$. If $\varphi\colon [0, \infty) \to [0, \infty)$ is defined by $\varphi(t) = t/2$, then φ is strictly increasing and T is a cyclic φ-contraction map.

The following example shows that Theorem 2.2 may be applied in situations where [1, Theorem 8] does not work.

Example 2.4. Consider the regular ordered metric space $X = L^1([0, 1])$ with the norm $\|\cdot\|_1$ and the order $f \leq g$ if and only if $f(t) \leq g(t)$ for almost all $t \in [0, 1]$. Suppose that $A = \{f \in X : -1 \leq f \leq 0\}$, $B = \{g \in X : 0 \leq g \leq 1\}$ and $T\colon A \cup B \to A \cup B$ is defined by $Tf = -f/3$ for all $f \in A \cup B$. If $\varphi\colon [0, \infty) \to [0, \infty)$ is defined by $\varphi(t) = t/2$, then φ is strictly increasing and T is a decreasing cyclic φ-contraction map. Note that A is closed and convex, T is orbitally continuous and $T0 = 0$. But, X is not a reflexive Banach space.

Theorem 2.5. *Let $(X, d, \leq)$ be an ordered metric space, $A, B \in 2^X$ and T a selfmap on $A \cup B$ such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $((A \times B) \cup (B \times A)) \cap X_{\leq} \in I(T \times T)$. Suppose that there exists $x_0 \in A$ such that $(x_0, Tx_0) \in X_{\leq}$ and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$ with $(x, y) \in X_{\leq}$, where $\varphi\colon [0, \infty) \to [0, \infty)$ is a strictly increasing map. If $x_{n+1} = Tx_n$ and $d_n = d(x_{n+1}, x_n)$ for all $n \geq 0$, then

$$d_n \to d(A, B).$$

Proof. First note that we have

$$d(T^{2n+1}x_0, T^{2n}x_0) \leq d(T^{2n}x_0, T^{2n-1}x_0) - \varphi(d(T^{2n}x_0, T^{2n-1}x_0)) + \varphi(d(A, B))$$

for all $n \geq 1$. Thus, we obtain

$$0 \leq d_{n+1} \leq d_n - \varphi(d_n) + \varphi(d(A, B))$$

for all $n \geq 1$. Hence, the sequence $\{d_n\}$ is decreasing and bounded from below. If $d_{n_0} = 0$ for some n_0, then $d_n \to d(A, B) = 0$. Suppose that $d_n > 0$ for all $n \geq 1$ and $d_n \to t_0$ for some $t_0 \geq d(A, B)$. Since

$$\varphi(d(A, B)) \leq \varphi(d_n) \leq d_n - d_{n+1} + \varphi(d(A, B)),$$

we have $\varphi(d_n) \to \varphi(d(A,B))$. This implies that $\varphi(t_0) = \varphi(d(A,B))$. So, $t_0 = d(A,B)$ because φ is strictly increasing. $\square$

Theorem 2.6. *Let $(X,d,\leq)$ be an ordered metric space, $A, B \in 2^X$ and T a selfmap on $A \cup B$ such that $T(A) = B$, $T(B) \subseteq A$ and $((A \times B) \cup (B \times A)) \cap X_{\leq} \in I(T \times T)$. Suppose that for each $x, y \in A$ there exists $z \in A$ such that $(x,z), (y,z) \in X_{\leq}$. Also, suppose that there exist $x_0, x^* \in A$ such that $x_0 \in E_{T,A}(x^*)$, $(x_0, Tx_0) \in X_{\leq}$ and*

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B))$$

for all $x \in A$ and $y \in B$ with $(x,y) \in X_{\leq}$, where $\varphi\colon [0,\infty) \to [0,\infty)$ is a strictly increasing map. Also, suppose that $y \in A$, $(x,y) \in X_{\leq}$ and $x \in E_{T,A}(x^*)$ imply that $y \in E_{T,A}(x^*)$. Then, $E_{T,A}(x^*) = A$ and the following statement holds:

$$E_{T,B}(Tx^*) = B \quad \text{and} \quad d(x^*,Tx^*) = d(A,B) \;\Leftrightarrow\; T \text{ is orbitally continuous.}$$

Proof. Let $x \in A$. If $(x_0, x) \in X_{\leq}$, then $x \in E_{T,A}(x^*)$. If $(x_0,x) \notin X_{\leq}$, then there exists $z \in A$ such that $(x_0,z) \in X_{\leq}$ and $(x,z) \in X_{\leq}$. Hence, $x \in E_{T,A}(x^*)$. Thus, $E_{T,A}(x^*) = A$.

Now, suppose that T is orbitally continuous and $y \in B$. Choose $x' \in A$ such that $Tx' = y$. Since $E_{T,A}(x^*) = A$, $T^{2n}x' \to x^*$ and so $T^{2n+1}x' \to Tx^*$. Hence, we have $T^{2n}y \to Tx^*$. Thus, $E_{T,B}(Tx^*) = B$. If $d(x^*,Tx^*) \neq d(A,B)$, then $\{d(T^{2n+1}x_0, T^{2n}x_0)\}$ is a decreasing sequence because $(x_0,Tx_0) \in X_{\leq}$. By Theorem 2.5, $d(T^{2n+1}x_0, T^{2n}x_0) \downarrow d(A,B)$. Choose a natural number n such that

$$d(A,B) \leq d(T^{2n+1}x_0, T^{2n}x_0) < d(x^*,Tx^*).$$

Put $x = T^{2n}x_0$ and $y = T^{2n+1}x_0$. Since $(x,y) \in X_{\leq}$, $(Tx,Ty) \in X_{\leq}$ and so $\{d(T^{2n}x, T^{2n}y)\}$ is a decreasing sequence and $d(T^{2n}x,T^{2n}y) \downarrow d(x^*,Tx^*)$. Hence, $d(x^*,Tx^*) \leq d(T^{2n+1}x_0,T^{2n}x_0) < d(x^*,Tx^*)$ which is a contradiction. Therefore, $d(x^*,Tx^*) = d(A,B)$.

Now, suppose that $d(x^*,Tx^*) = d(A,B)$, $E_{T,B}(Tx^*) = B$, $x \in A \cup B$ and $T^{n(i)}x \to a$ for some $a \in A \cup B$. We shall show that $T^{n(i)+1}x \to Ta$. Put $A' = A \cap \{T^{n(i)}x\}$ and $B' = B \cap \{T^{n(i)}x\}$.

Case 1. Let $d(A,B) = 0$. First suppose that $A' = \{T^{n_1(i)}x\}$ and $B' = \{T^{n_2(i)}x\}$ are subsequences of $\{T^{n(i)}x\}$. Since $\{T^{n_1(i)}x\}$ is a subsequence of $\{T^{2n}x\}$, $T^{n_1(i)}x \to x^*$. Also, we have $T^{n_1(i)+1}x \to Tx^*$ because $Tx \in B$ and $E_{T,B}(Tx^*) = B$. Since $\{T^{n_1(i)}x\}$ is a subsequence of $\{T^{n(i)}x\}$ and $T^{n(i)}x \to a$, $T^{n_1(i)}x \to a$. Thus, $a = x^*$ and so $a = x^* = Ta = Tx^*$. Since $\{T^{n_2(i)}x\}$ is a subsequence of $\{T^{2n+1}x\} = \{T^{2n}(Tx)\}$, $Tx \in B$ and $E_{T,B}(Tx^*) = B$, $T^{n_2(i)}x \to Tx^*$. Also, we have $T^{n_2(i)+1}x \to x^*$ because $T^2x \in A$, $E_{T,A}(x^*) = A$ and $\{T^{n_2(i)}x\}$ is a subsequence of $\{T^{2n+2}x\} = \{T^{2n}(T^2x)\}$. Hence, $T^{n(i)+1}x \to Ta$.

Now, suppose that $B' = \{t_1, \dots, t_k\}$ is finite. By using a similar argument, we have $T^{n_1(i)}x \to x^*$, $T^{n_1(i)+1}x \to Tx^*$ and $a = x^* = Ta = Tx^*$. Since $\{T^{n(i)+1}x\} = \{T^{n_1(i)+1}x\} \cup \{Tt_1, \dots, Tt_k\}$, $T^{n(i)+1}x \to Ta$. If $A' = \{s_1, \dots, s_m\}$ is finite, then $B' = \{T^{n_2(i)}x\}$ is a subsequence of $\{T^{n(i)}x\}$ and so $T^{n_2(i)}x \to a$. By using a similar argument, we have $T^{n_2(i)}x \to Tx^*$ and $T^{n_2(i)+1}x \to x^*$. Thus, $a = x^* = Ta = Tx^*$. Since $\{T^{n(i)+1}x\} = \{T^{n_2(i)+1}x\} \cup \{Ts_1, \dots, Ts_m\}$, we have $T^{n(i)+1}x \to Ta$.

Case 2. Let $d(A, B) > 0$. We claim that A' or B' is finite.

In fact, if A' and B' are infinite, then similar to the above case we have $T^{n_1(i)}x \to x^*$ and $T^{n_2(i)}x \to Tx^*$.

Since $\{T^{n_1(i)}x\}$ and $\{T^{n_2(i)}x\}$ are subsequences of $\{T^{n(i)}x\}$ and $T^{n(i)}x \to a$, we obtain $a = x^* = Tx^*$. So, $d(A, B) = d(x^*, Tx^*) = 0$ which is a contradiction.

Now, suppose that $B' = \{t_1, \dots, t_k\}$ is finite. By using a similar argument as in Case 1, we have $T^{n_1(i)}x \to x^*$, $T^{n_1(i)+1}x \to Tx^*$ and $a = x^*$. Since $\{T^{n(i)+1}x\} = \{T^{n_1(i)+1}x\} \cup \{Tt_1, \dots, Tt_k\}$, $T^{n(i)+1}x \to Ta$.

If $A' = \{s_1, \dots, s_m\}$ is finite, then $B' = \{T^{n_2(i)}x\}$ is a subsequence of $\{T^{n(i)}x\}$ and so $T^{n_2(i)}x \to a$. By using a similar argument as in Case 1, we have $T^{n_2(i)}x \to Tx^*$. Thus, $a = Tx^*$. Also, we have $T^{n_2(i)+1}x \to x^*$ because $T^2x \in A$, $E_{T,A}(x^*) = A$ and $\{T^{n_2(i)}x\}$ is a subsequence of $\{T^{2n+2}x\} = \{T^{2n}(T^2x)\}$.

Now, we show that $Ta = x^*$. In fact, $(x^*, x^*) \in X_{\leq}$ and

$$d(x^*, T^2x^*) \leq d(T^{2n}x^*, x^*) + d(T^{2n}x^*, T^2x^*).$$

Hence, by using the assumptions we have $d(T^{2n}x^*, T^2x^*) \leq d(T^{2n-2}x^*, x^*)$. Thus $d(x^*, T^2x^*) \leq d(T^{2n}x^*, x^*) + d(T^{2n-2}x^*, x^*)$.

Since $E_{T,A}(x^*) = A$ and $x^* \in A$, $T^{2n}x^* \to x^*$ and $T^{2n-2}x^* \to x^*$. Hence, $x^* = T^2x^*$. Since $a = Tx^*$, $Ta = x^*$. Thus, $T^{n_2(i)+1}x \to Ta$.

Since $\{T^{n(i)+1}x\} = \{T^{n_2(i)+1}x\} \cup \{Ts_1, \dots, Ts_m\}$, we have $T^{n(i)+1}x \to Ta$. $\square$

The following example shows that the assumption

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$ with $(x, y) \in X_{\leq}$, does not imply the following assumption:

$$y \in A, (x, y) \in X_{\leq}, x \in E_{T,A}(x^*) \Rightarrow y \in E_{T,A}(x^*).$$

Example 2.7. Consider the subsets

$$A = \{x_1 = (6, 3), x_2 = (1, 3)\} \quad \text{and} \quad B = \{y_1 = (2, 0), y_2 = (0, 4)\}$$

of $\mathbb{R}^2$ via the following order:

$$(a, b) \leq (c, d) \iff a \leq c \text{ and } b \leq d.$$

Define $T: A \cup B \to A \cup B$ by $Tx_1 = y_2$, $Tx_2 = y_1$, $Ty_1 = x_2$, $Ty_2 = x_1$. Note that, $x_2 \leq x_1$ and $y_1 \leq x_1$ and other elements are not comparable. Also, we have $d(Tx_1, Tx_2) = d(x_2, y_2) = d(A, B) = \sqrt{2}$ and $d(x_1, y_1) = \sqrt{25}$. Consider the map $\varphi: [0, \infty) \to [0, \infty)$ by $\varphi(x) = x/2$. Then, we have

$$d(Tx_1, Ty_1) \leq d(x_1, y_1) - \varphi(d(x_1, y_1)) + \varphi(d(A, B)),$$

while $T^{2n}x_1 \to x_1$ and $T^{2n}x_2 \to x_2$.

The following example shows that the assumptions of Theorem 2.6 do not imply orbital continuity of T.

Example 2.8. Define $S: \mathbb{R} \to \mathbb{R}$ by $Sx = -x/3$ for all $x \in \mathbb{R}$. Put $a_0 = -1$ and define the sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 1}$ by $b_n = Sa_{n-1}$ and $a_n = Sb_n$ for all $n \geq 1$. Now, define the sequences $\{c_n\}_{n \geq 0}$ and $\{d_n\}_{n \geq 1}$ as follows:

$$c_n = a_{2n+1} \quad \text{and} \quad d_n = a_{2n} \quad \text{for all } n \geq 0.$$

Now, consider the subsets

$$\begin{aligned} A &= \{(c_n, 0)\}_{n \geq 0} \cup \{(d_n, 0)\}_{n \geq 0} \cup \{(0, 0)\}, \\ B &= \{(b_{2n}, -1)\}_{n \geq 0} \cup \{(b_{2n+1}, -2)\}_{n \geq 1} \cup \{(0, -1)\} \end{aligned}$$

of $\mathbb{R}^2$ via the following order:

$$(a, b) \leq (c, d) \;\Leftrightarrow\; a \leq c \text{ and } b \leq d.$$

Define $T: A \cup B \to A \cup B$ by

$$\begin{aligned} T(c_n, 0) &= (b_{2n}, -1), & T(d_n, 0) &= (b_{2n+1}, -2), \\ T(b_{2n}, -1) &= (d_{n+1}, 0), & T(b_{2n+1}, -2) &= (c_{n+1}, 0), \\ T(0, 0) &= (0, -1), & T(0, -1) &= (0, 0). \end{aligned}$$

If we define $\varphi: [0, \infty) \to [0, \infty)$ by $\varphi(t) = t/2$, then it is easy to check that

$$T(A) = B, \quad T(B) \subseteq A, \quad ((A \times B) \cup (B \times A)) \cap X_{\leq} \in I(T \times T),$$
$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B))$$

for all $x \in A$ and $y \in B$ with $(x, y) \in X_{\leq}$ and for each $x, y \in A$ there exists $z \in A$ such that $(x, z), (y, z) \in X_{\leq}$. If we put $x_0 = x^* = (0, 0)$, then

$$(x_0, Tx_0) = ((0, 0), (0, -1)) \in X_{\leq},$$

and $y \in A$, $(x, y) \in X_{\leq}$ and $x \in E_{T,A}(x^*)$ imply that $y \in E_{T,A}(x^*)$. Finally, note that $T^{2n}x \to (0, 0)$ for all $x \in A$, $T^{2n}x_0 \to x^*$, $E_{T,B}(Tx^*) = B$ and $d(x^*, Tx^*) = d(A, B)$ while $\lim_{n \to \infty} T^{2n+1}(c_n, 0) = (0, -2) \neq Tx^* = (0, -1)$ for all $m \geq 1$. This implies that T is not orbitally continuous.

Theorem 2.9. *Let $(X, d, \leq)$ be an ordered metric space, $A, B \in 2^X$ and T a selfmap on $A \cup B$ such that $T(A) = B$, $T(B) \subseteq A$ and $((A \times B) \cup (B \times A)) \cap X_\leq \in I(T \times T)$. Suppose that for each $x, y \in A$ there exists $z \in A$ such that $(x, z), (y, z) \in X_\leq$. Also, suppose that there exist $x_0, x^* \in A$ such that $x_0 \in E_{T,A}(x^*)$ and*

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Also, suppose that $y \in A$, $(x, y) \in X_\leq$ and $x \in E_{T,A}(x^*)$ imply that $y \in E_{T,A}(x^*)$. Then, $E_{T,A}(x^*) = A$ and the following statement holds:

$$E_{T,B}(Tx^*) = B \text{ and } d(x^*, Tx^*) = d(A, B) \iff T \text{ is orbitally continuous.}$$

Proof. Similar as in the proof of Theorem 2.6 we can show that $E_{T,A}(x^*) = A$ and $E_{T,B}(Tx^*) = B$ whenever T is orbitally continuous. If $d(x^*, Tx^*) \neq d(A, B)$, then there exists $x \in A$ and $y \in B$ such that

$$d(A, B) \leq d(x, y) < d(x^*, Tx^*).$$

Note that $\{d(T^{2n}x, T^{2n}y)\}$ is a decreasing sequence and

$$d(T^{2n}x, T^{2n}y) \downarrow d(x^*, Tx^*).$$

Hence, $d(x^*, Tx^*) \leq d(x, y) < d(x^*, Tx^*)$ which is a contradiction. Thus,

$$d(x^*, Tx^*) = d(A, B).$$

Similar to the proof of Theorem 2.6 we can show that T is orbitally continuous whenever $E_{T,B}(Tx^*) = B$ and $d(x^*, Tx^*) = d(A, B)$. □

The following example shows that the assumption

$$d(Tx, Ty) \leq d(x, y)$$

for all $x \in A$ and $y \in B$, does not imply the following assumption in Theorem 2.9:

$$y \in A,\ (x, y) \in X_\leq,\ x \in E_{T,A}(x^*) \ \Rightarrow\ y \in E_{T,A}(x^*).$$

Example 2.10. Consider the subsets

$$A = \{x_1 = (0, 0), x_2 = (0, 1)\} \quad \text{and} \quad B = \{y_1 = (1, 0), y_2 = (1, 1)\}$$

of $\mathbb{R}^2$ via the following order:

$$(a, b) \leq (c, d) \iff a \leq c \text{ and } b \leq d.$$

Define $T\colon A \cup B \to A \cup B$ by $Tx_1 = y_1$, $Tx_2 = y_2$, $Ty_1 = x_1$, $Ty_2 = x_2$. Note that

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x \in A \text{ and } y \in B,$$

$T^{2n}x_1 \to x_1$ and $T^{2n}x_2 \to x_2$. Thus, the following assumption does npt hold:

$$y \in A,\ (x,y) \in X_{\leq},\ x \in E_{T,A}(x^*)\ \Rightarrow\ y \in E_{T,A}(x^*).$$

The following example shows that the following assumption is necessary in Theorem 2.9:

$$d(Tx,Ty) \leq d(x,y) \quad \text{for all } x \in A \text{ and } y \in B.$$

Example 2.11. Let $X = \mathbb{R}$, $A = [0,1]$ and $B = [2,3]$. Define $T\colon A \cup B \to A \cup B$ by $Tx = x + 2$ for all $x \in A$ and $Tx = \frac{x-2}{2}$ for all $x \in B$. Note that, T is orbitally continuous and we have $T^{2n}x_0 = x_0/2^n$ and $T^{2n+1}x_0 = x_0/2^n + 2$ for all $x_0 \in A$ and $n \geq 0$. Thus, $T^{2n}x_0 \to 0$ and $T^{2n+1}x_0 \to 2$ for all $x_0 \in A$. But, note that the assumption doesn't hold because $d(T1, T2) \nleq d(1,2)$.

The following example shows that the assumption

$$d(Tx,Ty) \leq d(x,y) \quad \text{for all } x \in A \text{ and } y \in B$$

can not be replaced by the following assumption in Theorem 2.9:

$$d(Tx,Ty) \leq d(x,y) \quad \text{for all } x \in A \text{ and } y \in B \text{ with } (x,y) \in X_{\leq}.$$

Example 2.12. Consider the subsets

$$A = \{x_1 = (1,2), x_2 = (2,2)\} \quad \text{and} \quad B = \{y_1 = (3,1), y_2 = (4,1)\}$$

of $\mathbb{R}^2$ via the following order:

$$(a,b) \leq (c,d) \ \Leftrightarrow\ a \leq c \text{ and } b \leq d.$$

Define $T\colon A \cup B \to A \cup B$ by $Tx_1 = y_1$, $Tx_2 = y_2$, $Ty_1 = Ty_2 = x_2$. Note that, $x_1 \leq x_2$ and $y_1 \leq y_2$ and other elements are not comparable. It is easy to check that $T(A) = B$, $T(B) \subseteq A$, $((A \times B) \cup (B \times A)) \cap X_{\leq} \in I(T \times T)$ and for each $x, y \in A$ there exists $z \in A$ such that $(x,z), (y,z) \in X_{\leq}$. Also, there exist $x_0, x^* \in A$ such that $x_0 \in E_{T,A}(x^*)$. Finally, $y \in A$, $(x,y) \in X_{\leq}$ and $x \in E_{T,A}(x^*)$ imply that $y \in E_{T,A}(x^*)$. Note that $T^{2n}x_i \to x_2$, $T^{2n+1}x_i \to y_2$, $T^{2n}y_i \to y_2$ and $T^{2n+1}y_i \to x_2$ for $i = 1, 2$. Thus, T is orbitally continuous while $d(x_2, Tx_2) \neq d(A,B)$.

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References

[1] M. A. Al-Thagafi and N. Shahzad, *Convergence and existence for best proximity points*, Nonlinear Anal. **70** (2009), 3665–3671.

[2] J. Anuradha and P. Veeramani, *Proximal pointwise contraction*, Topology Appl. **156** (2009), 2942–2948.

[3] A. A. Eldered and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), 1001–1006.

[4] Z. Kadelburg, M. Pavlović and S. Radenović, *Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces*, Computer Math. Appl. **59** (2010), 3148–3159.

[5] W. A. Kirk, P. S. Srinavasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

[6] G. Petruşel, *Cyclic representations and periodic points*, Studia Univ. Babes-Bolyai. Math. **50** (2005), 107–112.

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