more systematic, if at times relatively informal but still technically sound, presentation of its subject, and is much more concerned with helping the novice understand and appreciate the subject for its own sake, much less concerned, certainly far less concerned than the McNeill and Freiberger book, with whiz-bang gadgetry.

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Alan Ross Anderson, Nuel D. Belnap, Jr., and J. Michael Dunn. *Entailment*, Volume II. Princeton University Press, 1992. xxvii + 749$ pp.

Reviewed by

GREG RESTALL

Automated Reasoning Project
Australian National University
Canberra 0200, Australia

This is an enormous work, in many senses of that word. It is *long* at over 700 pages. It is blessed with an exhaustive bibliography (of 146 pages, provided by Robert G. Wolf). It has an extensive cast of contributors: the three main authors — Alan Anderson, Nuel Belnap and J. Michael Dunn, together with the contributors of smaller sections — Kit Fine, Alasdair Urquhart, Daniel Cohen, Glen Helman, Steve Giambrone, Errol Martin, Dorothy Grover, Michael McRobbie, Anil Gupta and Stuart Shapiro. It covers a wide range of topics, and it had an unusually large period of gestation, the acknowledgements state that the book had been in preparation since 1959. There were thirty-three years between the book's inception and its completion. But most importantly, the book contains a wealth of insights given only through the many collective years of hard work. In this review I will attempt to give the prospective reader an idea of the range of its contents, and then I will cast a (friendly) critical eye over the work as a whole and just some of the detail.

1. Scope

The volume commences with a simple summary of what the reader will need to know from Volume I. They write "we do not wish to penalize too severely the reader who is patiently working through this volume while floating on a raft in a swimming pool, having left Volume I up at the house" (p. xxiii). While I doubt that many will study *Entailment II* in this way, it is reassuring to know that you can. The authors assume that the readers will at least have some knowledge of relevant logics, so this section only covers the notation and particular axiom systems from Volume I which will be used in what follows. The reader familiar with relevant logics from some other source — like J. Michael Dunn's 1986 *Handbook* article, or Stephen Read's volume, will not need to have Volume I at hand to understand and appreciate this book.

The first chapter (Chapter 6) deals with the topic of propositional quantification. This is particularly interesting in a relevant context, and it is surprising what can be achieved once we allow it into our systems. The chief results are the embeddings of other systems into relevant logics. Particularly, we learn that both intuitionistic logic and the positive part of **S4** can be embedded into the system **E** of relevant entailment, by paying careful attention to enthymemes. We define $A \supset B$ as $\exists r(r \wedge (r \wedge A \rightarrow B))$. A intuitionistically implies B just when there is some true r (the enthymeme) such that A together with r entails B. So, the intuitionistic conditional can be 'irrelevant' because while A may not be relevant to B, the enthymeme may be. Specifically, we have $A \supset B$ true when B is true, because choosing r to be B suffices to make $(r \wedge A \rightarrow B)$ true. The result for **S4** is similar, except that we require that r be necessarily true. $A \Rightarrow B$ (A **S4**-entails B) is cashed out as $\exists r(\Box r \wedge (r \wedge A \rightarrow B))$. In this section we see how insights from relevant logics can live together with more 'traditional' pursuits in logic. We will return to this point later.

Chapter 7 brings us the discussion of individual quantification. This is a particularly difficult subject in relevant logics (perhaps more so than the thorny issues of quantification in other modal, intensional logics) because while the proof theory and purely algebraic study of quantification seems to pick out natural axioms for quantifiers, with all the usual properties, modelling this theory with the frame ('possible worlds') semantics is fiendishly difficult. In this chapter we have the proof theoretic and algebraic study of the quantified versions of relevant logics. A nice result is that monadic quantificational relevant logics are undecidable. The result follows from the undecidability of quantificational classical logic with binary relations. It is not difficult to show that we can mimic an arbitrary binary relation R holding of a and b by the relevant entailment (or implication) $Fa \rightarrow Gb$ for some appropriately chose F and G. Thus the undecidability of dyadic classical quantificational logic gives us the undecidability of monadic relevant quantificational logic.

The next chapter focuses on Ackermann's *strenge Implikation.* At 12 pages it is by far the shortest chapter, and it is chiefly of historical interest. The authors show that Ackermann's systems Σ', Π' and Π'' are all identical (in theorems) to **E**.

The next two chapters concern semantics and proof theory. They are the formally richest chapters of the book, and it is here that most of the target audience will spend most of their time. The chapter on semantics covers Urquhart's semilattice semantics (in a section written by Urquhart), the Routley-Meyer relational semantics, Dunn's binary relational semantics for **RM**, the coupled-trees semantics for first-degree entailments, and Fine's relational-operational semantics (written by Fine). It is in this section that we learn of the difficulties of modelling individual quantification in frame semantics. Fine shows us that the constant-domain semantics gets things wrong, and then he shows us just how difficult it is to get things 'right.' Fine's semantics for quantification is positively labyrinthine, but the central idea is not too difficult. A proposition $\forall xA$ is true if and only if A is true of an *arbitrary individual.* This is necessary because Fine shows that even the material equivalence of $\forall xA$ with the truth of all of its instances captures too much.

The chapter on proof theory and decidability is the most broad-ranging of all. We have McRobbie's work on relevant analytic tableaux, the Belnap-Dunn-Gupta work on a Gentzen-style consecution calculus for **R** with necessity, Belnap's display logic, Urquhart's undecidability results, Martin's proof of the **P** – **W** conjecture (that in the arrow fragment of the minimal system **P** – **W**, $A \leftrightarrow B$ is a theorem only if $A = B$, and Giambrone's work on the decidability of relevant systems without contraction. This is a very rich chapter, and each of its sections repays close reading. The reader might be confused about some of the subtleties in this chapter, not only with the details of the proofs — Urquhart's undecidability result takes detours through projective geometry in order to embed semigroups into the algebras of relevant logics to prove them undecidable, but also in their interpretation. The difficulty is the fact that Urquhart shows that the *deducibility* problem in almost all relevant logics is undecidable (even in their positive parts), whereas Giambrone shows that *theoremhood* in (the positive parts of) contractionless logics is decidable. What is the reader to make of this? The answer (and it is one that the authors sketch) is that in logics without contraction, you cannot reduce deducibility to theoremhood. Specifically, if we know that from A you can deduce B, in a contractionless system, you cannot infer that $A \wedge t \rightarrow B$ is a theorem (where t is the Ackermann constant for 'truth'), but only that either $A \wedge t \rightarrow B$ is, or that $A \wedge t \rightarrow (A \wedge t \rightarrow B)$ is, or that $A \wedge t \rightarrow (A \wedge t \rightarrow (A \wedge t \rightarrow B))$ is or . . . you get the idea. Without contraction (from $A \rightarrow (A \rightarrow B)$ to infer $A \rightarrow B$) these do not collapse. So, having a decision procedure for theorems in contractionless logics does not ensure that you can decide deducibility.

The next chapter is a grab-bag of sorts, as the title "Functions, Arithmetic, and Other Special Topics" attests. We start off with some programmatic reflections on functions, from a relevant standpoint. A *real* function is one which somehow makes use of its argument, just as in a *real* implication the consequent 'uses' the antecedent. The connections between functions on the one hand and conditionals on the other have been known for quite a while, especially among constructivists. It is reassuring to know that this intuition has some purchase in relevant logics as well. Just as the $\supset$ theorems in intuitionistic logic correspond to the types of closed λ terms, so the $\rightarrow$ theorems of the relevant logic **R** correspond to the types of closed λ terms *in which there are no vacuous* λ *abstractions*. That is, we are not allowed to make abstractions like $\lambda x \lambda y x$, where the λy doesn't bind any free variables at all. This corresponds to the relevant non-theoremhood of $A \rightarrow (B \rightarrow A)$. I take this correspondence to be one of the major points in favour of **R** as a stable system. Its condition that consequents depend on antecedents is straightforwardly tied to the condition that functions depend on their arguments. Glen Helman's section in this chapter extends that correspondence to deal with conjunction. The rest of the chapter chronicles work on relevant arithmetic (much of it recording the groundbreaking work of Robert Meyer), Dunn's work on relevant predication, in which he turns his attention to the question of what it is for an object to 'really' bear a property, and Cohen's work on conditional assertion.

The final chapter looks at a number of applications of relevant logic: the authors examine Belnap's "Useful Four-Valued Logic", the connections between relevant logic and Rescher's hypothetical reasoning, and Shapiro's section on relevant logic in computer science.

However, most readers' attention will rightly focus on the discussion of the question of disjunctive syllogism. In this section the authors discuss possible positions an aficionado of relevant logic might hold on the validity of the disjunctive syllogism: the argument from A and $\sim A \vee B$ to infer B. I will discuss this at greater length below.

2. Evaluation

The reader will probably by now have some idea of the massive breadth of the book. This has both positive and negative consequences. On the positive side, there is much here to learn and to digest. There are the distilled thoughts of many years' work by many very good researchers on what are difficult topics. On the negative side, the reader must be aware that not all of the book's sections are of equal worth. Some of them look quite dated when read with hindsight gained from living in the 1990's. It is fairly clear that had the book been planned and written in the last five years, it would have been quite a different book (and there is no doubt that many of the good things in it would have been lost had the authors done this).

A more surprising fact is that, given the breadth of scope of the book, it is not more general. For example, take the logic **B**. It is said to be the most *basic* relevant logic, and it is the weakest logic for which Fine gives his operational-relational semantics. However, one of its axioms is excluded middle, $A \vee \sim A$. There are many relevant logics in which excluded middle does not hold (like contractionless ones: it is not hard to see that $A \vee \sim A$ is equivalent to $A \wedge (A \rightarrow f) \rightarrow f$, which is an instance of $A \wedge (A \rightarrow B) \rightarrow B$, which fails in contractionless logics) so it ought not have a place in the most basic logic. Similar phenomena are discernible elsewhere, where results which are easily generalizable are restricted to a small family of systems, specifically **R** and **E** and perhaps **T**.

Another fact which might be a surprise to some is that Robert K. Meyer, who was promised to be a collaborator along with Dunn on Volume II, does not appear. Unfortunately, the Belnap-Meyer collaboration broke down, so Meyer is not one of the authors. There is no doubt that the results would have differed in many ways had he been a collaborator. However, his voice is still heard in the faithful transmission of many of his results.

To end the evaluation, I will discuss two particular issues in *Entailment II*, disjunctive syllogism, and the semantics of quantification. Neither are simply 'critical' remarks, but rather, indications of how research on these topics could perhaps be fruitfully pursued.

Disjunctive Syllogism and Boolean Negation. The discussion of disjunctive syllogism and boolean negation in *Entailment II* is both fascinating and troubling. Firstly, the authors discuss what the relevant logician ought to make of a connective like 'boolean negation', defined to satisfy $A \wedge \neg A \rightarrow B$ and $A \rightarrow B \wedge \neg B$. The discussion is fascinating because of the number of distinctions the authors draw and the fine intuitions they bring to bear on the issues. The discussion is troubling because it concentrates on the particularly simple model of propositions as taking four semantic values: *True*, *False*, *Both* and *Neither*. This is troubling, because there are other ways of examining the problem which might shed more light on the question. The evaluation in terms of 'four values' is particularly difficult because the question arises: why not define $\neg A$ as that proposition which takes the value *True* when A is *False*, *False* when A is *True*, *Neither* when A is *Both*, and *Both* when A is *Neither*? Then $\neg$ will be a 'Boolean Negation'.

Instead of pursuing this line, I wish to point out another way forward, and that is to take the frame semantics for relevant logics seriously. Consider the parallels with intuitionistic logic. In intuitionistic frames, points in the frame correspond to states of information, and these are partially ordered, so that if $x \leq y$ and $x \models A$ (x makes A true, or some such thing), then $y \models A$ too. The partial order is an order of *increasing information*. Now, the question arises: can an intuitionist make sense of boolean negation? Well, in one important sense, *no*. For if there were a connective $\neg$ such that $x \models \neg A$ if and only if $x \not\models A$, then the partial ordering on information states would collapse. We would have $x \leq y$ only if $x = y$. There would be no increase of information. The 'metatheoretic' fact, that x doesn't make A true cannot be forced down into an 'object-theoretic' fact (some fact that x makes true) without destroying the semantic structure. Of course, you can model boolean negation in frames for intuitionistic logic, but then the resulting logic is **S4** and intuitionistic logic is embedded within that new logic as the necessitated formulæ.

The situation is the same in the semantics for relevant logics. In frames for relevant logics, the points can also be taken to be information states (or theories, or possible situations, or something similar). They come naturally ordered in the same way. To define a notion of boolean negation on these frames is to destroy their structure. True, there is a fact *about* a point in the model that it doesn't support A for some A, but this need not be a fact {\em at\/} that point. So, boolean negation is not a meaningful connective in any system with this sort of interpretation. This kind of argument is one example of how insights from more 'traditional' areas in logic, like constructivism, can help enlighten problems in relevant logic. (Though it should be clear that this does not commit a relevant logician to the constructivist programme. It is only a similarity of interpretations of structures that is at work in this argument.)

Models for Quantification. Similar remarks can be made about the problems in modelling quantification. The problem with quantification is that constant domain semantics gets things wrong. You cannot assume that every point in the model has the same domain, because this captures too much, just as constant domain semantics on intuitionistic frames captures more than intuitionistic predicate logic. So, the natural question is: why not try 'increasing' domains? Why not say that if $x \leq y$, then $D(x) \subseteq D(y)$? Well, the reason is not too far away: the semantics of negation precludes it. For negation in relevant logics, we have $x \vDash {\sim}A$ just when $x^* \nvDash {\sim}A$, where x^* is the theory (piece of information, situation, whatever) which asserts all that x doesn't deny. The trouble is, we have $x^{**} = x$, and if $x \leq y$ then $y^* \geq x^*$. If we hold that the domain of x^* is the same as the domain of x (as seems sensible, given the interpretation of x^*) then the order twisting condition (if $x \leq y$ then $y^* \leq x^*$) means that we get constant domain semantics again.For if $x \leq y$ then $y^* \leq x^*$) and $D(y) = D(y^*) \subseteq D(x^*) = D(x)$.

So, what can the relevant logician do? Two things. First, investigate the positive part of relevant logics. Ignore negation, and see whether the 'increasing domain' semantics captures the quantified logic. This is sensitive to the choice of quantified logic, of course. In particular, the distribution axiom $\forall x(A \vee B) \rightarrow \forall xA \vee \exists xB$ will not be provable. But, that isn't provable in the natural algebraic or proof theoretical accounts of quantification without using negation, so that is no great loss. I conjecture that the increasing domain semantics will capture the natural quantified positive relevant logics.

Second, just consider the logic of conjunction, disjunction and (relevant, De Morgan) negation (perhaps paired with the intuitionistic conditional) and see how to model quantification in that. There is no need to worry about relevant implication, for we have seen that the problem arises at the level of interaction between negation and the containment relation on points in model structures.

The third part is to put the two together. This is not easy, but it gives another framework within which to examine what is a very hard problem.

Entailment II is an excellent book which provokes the reader to much thought, and offers many helpful insights into the intricacies and beauties of relevant logics. It is essential reading for any who wish to do work in this area, and that reading will be enjoyable and worthwhile (but not necessarily *easy*) for any who attempt it. More importantly, it will be the jumping off point for many more logical enterprises. There is nothing more that you can ask of such a book.

3. Note

I would like to thank Nuel Belnap, Mike Dunn, and Bob Meyer for their helpful comments on an earlier draft of this review.

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