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COMPOSITIONS OF TWO ADDITIVE ALMOST CONTINUOUS FUNCTIONS

Abstract

In the paper we prove that an additive Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed as a composition of two additive almost continuous (connectivity) functions if and only if either f is almost continuous (connectivity) function or $\dim(\ker(f)) \neq 1$. We also show that for every cardinal number $\lambda \leq 2^\omega$ there exists an additive almost continuous functions with $\dim(\ker(f)) = \lambda$. A question whether every Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ can be expressed as a composition of two almost continuous functions (see [**?**] or [**?**]) remains open.

1 Definitions and Notation

Our terminology and notation is standard. In particular, functions will be identified with their graphs, and for a subset A of $\mathbb{R} \times \mathbb{R}$ (possibly, but not necessarily, a graph of a function) we will write $\mathrm{dom}\,(A)$ and $\mathrm{rng}\,(A)$ to denote the x-projection (the domain) and the y-projection (the range) of A, respectively. The cardinality of a set A will be denoted by $\mathrm{card}\,(A)$. Cardinals will be identified with the initial ordinals. The cardinality of the set $\mathbb{R}$ of real numbers, the continuum, will be denoted by 2^ω.

Throughout the paper we will consider $\mathbb{R}$ as a linear space over the field $\mathbb{Q}$ of rational numbers. A linear basis of this space will be referred to as a *Hamel basis*. It is evident that the cardinality of every Hamel basis is equal to 2^ω.

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For an arbitrary set $A \subset \mathbb{R}$ the symbol $L(A)$ will denote the linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ spanned by A, i.e., the set of all finite linear combinations of elements of A with coefficients from $\mathbb{Q}$. Similarly for an arbitrary planar set $A \subset \mathbb{R}\times\mathbb{R}$ we define the set $L_2(A)$. Also, for $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ we write $x+A$ for $\{x+a\colon a \in A\}$.

Now, let $L \neq \emptyset$ be a linear subspace of $\mathbb{R}$ over $\mathbb{Q}$. A function $f\colon L \to \mathbb{R}$ is said to be *additive* if it satisfies Cauchy's equation $f(x+y) = f(x)+f(y)$ for every $x, y \in L$. (See [**?**] or [**?**, p. 120].) The class of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ will be denoted by $\mathcal{A}dd$. Recall that if $H \subset \mathbb{R}$ is a Hamel basis, then every function $f_0\colon H \to \mathbb{R}$ can be uniquely extended to the additive function $f\colon \mathbb{R} \to \mathbb{R}$. In fact, $f = L_2(f_0)$.

For $f \in \mathcal{A}dd$ its kernel $\ker(f)$ is defined as $f^{-1}(0)$. Clearly $\ker(f)$ is a linear subspace of $\mathbb{R}$. Thus, $\dim(\ker(f))$ denotes the (linear) dimension of $\ker(f)$ over $\mathbb{Q}$.

A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a *Darboux function* if it has the intermediate value property, i.e., whenever for every $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, and every point c between $f(x_1)$ and $f(x_2)$ there exists $x \in [x_1, x_2]$ such that $f(x) = c$. The family of all Darboux functions from $\mathbb{R}$ to $\mathbb{R}$ will be denoted by $\mathcal{D}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *almost continuous* in the sense of Stallings if each open set (in $\mathbb{R}^2$) containing f contains also a (graph of) continuous function $g : \mathbb{R} \to \mathbb{R}$ [**?**]. The class of all almost continuous functions from $\mathbb{R}$ into $\mathbb{R}$ will be denoted by $\mathcal{AC}$.

A closed set $K \subset \mathbb{R}\times\mathbb{R}$ is said to be a *blocking set* for a function $f\colon \mathbb{R} \to \mathbb{R}$ provided $f \cap K = \emptyset$ while $g \cap K \neq \emptyset$ for every continuous function $g\colon \mathbb{R} \to \mathbb{R}$. A blocking set $K \subset \mathbb{R}\times\mathbb{R}$ for f is *irreducible* if no proper subset of K is a blocking set for f [**?**].

It is known that f is almost continuous if and only if it has no blocking set. Moreover, if f is not almost continuous, then there is an irreducible blocking set K for f, and the x-projection of K is a non–degenerate connected set [**?**]. Thus, if $f\colon \mathbb{R} \to \mathbb{R}$ intersects all closed sets $K \subset \mathbb{R}^2$ with the domain being a non-degenerate interval, then it is almost continuous (cf. [**?**]).

A function $f\colon \mathbb{R} \to \mathbb{R}$ is a *connectivity function* if its graph is connected (in $\mathbb{R}^2$). We will use a symbol $\mathcal{C}onn$ to denote the class of all connectivity functions $f\colon \mathbb{R} \to \mathbb{R}$. The class of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ will be denoted by $\mathcal{C}$. We have the following chain of proper inclusions [**?**].

$$\mathcal{C} \subset \mathcal{AC} \subset \mathcal{C}onn \subset \mathcal{D}.$$

It is well–known that the composition of two Darboux functions is a Darboux function again. The problem of characterization of these Darboux functions which can be expressed as a composition of two almost continuous functions was considered in [?]. (See also [?].) In this paper we will consider the analogous problem in the class of additive functions.

2 Main Theorem

Let $\mathcal{B}$ be the family of all closed sets $B \subset \mathbb{R} \times \mathbb{R}$ such that $\mathrm{dom}\,(B)$ is a non-degenerate interval and either

(A) $B = \mathbb{R} \times \{y\}$; or,

(B) $B^y = \{x \in \mathbb{R} \colon \langle x, y\rangle \in B\}$ is nowhere dense for each $y \in \mathbb{R}$.

We will use this family throughout the paper.

In what follows we will use the following lemma repeatedly.

Lemma 1. *Let $f \in \mathcal{A}dd$ be such that* $\ker(f) \neq \{0\}$. *If $f \cap B \neq \emptyset$ for every $B \in \mathcal{B}$, then $f \in \mathcal{AC}$.*

Proof. Fix an arbitrary closed set $K \subset \mathbb{R}^2$ such that $\mathrm{dom}\,(K)$ is a non-degenerate interval. It is enough to show that $f \cap K \neq \emptyset$. If K^y is nowhere dense for each $y \in \mathbb{R}$, then $K \in \mathcal{B}$ and $f \cap K \neq \emptyset$. So, assume otherwise.

Then there is $y \in \mathbb{R}$ such that K^y contains a non-degenerate interval I. But $\mathbb{R} \times \{y\} \in \mathcal{B}$; so $f \cap (\mathbb{R} \times \{y\}) \neq \emptyset$. In particular, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Also, $\ker(f)$ is dense, since $\ker(f) \neq \{0\}$, and so $f^{-1}(y)$ contains a dense set $x + \ker(f)$. Thus $f^{-1}(y) \cap I \supset (x + \ker(f)) \cap I \neq \emptyset$ and $\emptyset \neq f \cap (I \times \{y\}) \subset f \cap K$. $\square$

The next theorem constitutes one direction of our main characterization theorem.

Theorem 1. *Let $f \in \mathcal{D} \cap \mathcal{A}dd$ be such that* $\dim(\ker(f)) \neq 1$. *Then f is a composition of two additive almost continuous functions.*

Proof. Fix $f \in \mathcal{D} \cap \mathcal{A}dd$ with $\dim(\ker(f)) \neq 1$. If $\dim(\ker(f)) = 0$, then f is continuous (see [?]) and $f = f \circ id$. Similarly, if $f \equiv 0$, then $f = f \circ id$. Hence we can assume that $\dim(\ker(f)) \geq 2$ and $f \not\equiv 0$.

Let $\{K_\alpha \colon \alpha < 2^\omega\}$ be an enumeration of the family $\mathcal{B}$ such that $K_0 = \mathbb{R} \times \{0\}$ and let $\{b_\alpha \colon \alpha < 2^\omega\}$ be an enumeration of a fixed Hamel basis with $b_0 \in \ker(f)$.

We construct, by induction on $\alpha < 2^\omega$, the sequences $\langle g_\alpha \colon \alpha < 2^\omega\rangle$ and $\langle h_\alpha \colon \alpha < 2^\omega\rangle$ of additive functions from subsets of $\mathbb{R}$ into $\mathbb{R}$ maintaining the following inductive properties for every $\alpha < 2^\omega$.

(i) $g_\beta \subset g_\alpha$ and $h_\beta \subset h_\alpha$ for every $\beta < \alpha$;

(ii) $\text{card}\,(\text{dom}\,(g_\alpha)) \le \max(\omega, \alpha)$, and $\text{card}\,(\text{dom}\,(h_\alpha)) \le \max(\omega, \alpha)$;

(iii) $\text{rng}\,(g_\alpha) = \text{dom}\,(h_\alpha)$ and $h_\alpha \circ g_\alpha = f|\text{dom}\,(g_\alpha)$;

(iv) $g_\alpha \cap K_\alpha \neq \emptyset$ and $h_\alpha \cap K_\alpha \neq \emptyset$;

(v) $b_\alpha \in \text{dom}\,(g_\alpha)$.

To make an inductive step assume that for some $\alpha < 2^\omega$ the functions g_β and h_β satisfying conditions (i)–(v) have already been constructed for every $\beta < \alpha$.

If $\alpha = 0$, choose $s_0 \in \ker(f) \setminus L(\{b_0\})$. Such a choice is possible, since $\dim(\ker(f)) \ge 2$. Put $g_0 = L_2(\{\langle b_0, 0\rangle, \langle s_0, s_0\rangle\})$ and $h_0 = L_2(\{\langle s_0, 0\rangle\})$. It is easy to see that g_0 and h_0 fulfill the conditions (i)–(v).

So, assume that $\alpha > 0$ and put $\overline{g}_\alpha = \bigcup_{\beta<\alpha} g_\beta$ and $\overline{h}_\alpha = \bigcup_{\beta<\alpha} h_\beta$. Clearly functions $\overline{g}_\alpha$ and $\overline{g}_\alpha$ satisfy the conditions (i)-(iii). We will find $x_\alpha, y_\alpha, s_\alpha, v_\alpha, c_\alpha \in \mathbb{R}$ such that

(a) $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$;

(b) $\langle v_\alpha, f(s_\alpha)\rangle \in K_\alpha$;

(c) $g_\alpha = L_2(\overline{g}_\alpha \cup \{\langle x_\alpha, y_\alpha\rangle, \langle b_\alpha, c_\alpha\rangle, \langle s_\alpha, v_\alpha\rangle\})$ and
$h_\alpha = L_2(\overline{h}_\alpha \cup \{\langle y_\alpha, f(x_\alpha)\rangle, \langle c_\alpha, f(b_\alpha)\rangle, \langle v_\alpha, f(s_\alpha)\rangle\})$ remain functions.

It is easy to see that such g_α and h_α will satisfy the conditions (i)–(v).

As a first step we will construct x_α and y_α. If $K_\alpha \cap \overline{g}_\alpha \neq \emptyset$, we simply choose $\langle x_\alpha, y_\alpha\rangle \in K_\alpha \cap \overline{g}_\alpha$. So, assume that $K_\alpha \cap \overline{g}_\alpha = \emptyset$. In this case we will find $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$ such that

$$x_\alpha \notin \text{dom}\,(\overline{g}_\alpha), \qquad \text{and} \quad y_\alpha \notin \text{dom}\,(\overline{h}_\alpha) = \text{rng}\,(\overline{g}_\alpha). \tag{1}$$

Such a restriction is necessary to guarantee condition (c).

Let $X_\alpha = \text{dom}\,(\overline{g}_\alpha)$, and $Y_\alpha = \text{dom}\,(\overline{h}_\alpha) = \text{rng}\,(\overline{g}_\alpha)$. Then $\text{card}\,(X_\alpha) < 2^\omega$ and $\text{card}\,(Y_\alpha) < 2^\omega$. If K_α was chosen according to the part (A) of the definition of $\mathcal{B}$, then $K_\alpha = \mathbb{R} \times \{y\}$ for some $y \in \mathbb{R}$. Hence $y \notin Y_\alpha$, since $K_\alpha \cap \overline{g}_\alpha = \emptyset$. Put $y_\alpha = y$ and choose $x_\alpha \notin X_\alpha$. Then $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$ and the condition (**??**) is satisfied. So, assume that K_α was chosen according to the part (B) of the definition of $\mathcal{B}$, i.e., that K_α^y is nowhere dense for every $y \in \mathbb{R}$. To deal with this case recall the following fact. (See [**?**, Th. 29.19, p. 231].)

For every closed set $K \subset \mathbb{R}^2$ the set

$$Z(K) = \{y \in \mathbb{R} \colon K^y \text{ contains a non-empty perfect set}\}$$

is either countable or is of power continuum.

This leads us to the two natural subcases.

- $\operatorname{card}(Z(K_\alpha)) = 2^\omega$. Then $\operatorname{card}(Z(K_\alpha) \setminus Y_\alpha) = 2^\omega$ and we can choose $y_\alpha \in Z(K_\alpha) \setminus Y_\alpha$. Moreover, $\operatorname{card}(K_\alpha^{y_\alpha}) = 2^\omega$, and so we can pick $x_\alpha \in K_\alpha^{y_\alpha} \setminus X_\alpha$. Then $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$ satisfies (**??**).

- $\operatorname{card}(Z(K_\alpha)) \le \omega$. Then the set $E_\alpha = \operatorname{dom}(K_\alpha) \setminus \bigcup\{K_\alpha^y \colon y \in Z(K_\alpha)\}$ is a residual subset of the interval $\operatorname{dom}(K_\alpha)$ since each set K_α^y is nowhere dense. In particular, $\operatorname{card}(E_\alpha) = 2^\omega$. Moreover, K_α^y is countable for every $y \in \mathbb{R}\setminus Z(K_\alpha)$. So the set $E_\alpha^1 = E_\alpha \setminus (X_\alpha \cup \bigcup\{K_\alpha^y \colon y \in Y_\alpha \setminus Z(K_\alpha)\})$ has cardinality 2^ω. Choose $x_\alpha \in E_\alpha^1 \subset \operatorname{dom}(K_\alpha) \setminus \left(X_\alpha \cup \bigcup_{y\in Y_\alpha} K_\alpha^y\right)$ and $y_\alpha \in \mathbb{R}$ with $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$. Then $y_\alpha \notin Y_\alpha$ and (**??**) is satisfied.

This finishes the construction of x_α and y_α.

To construct s_α and v_α first note that by (**??**),

$$\underline{g}_\alpha = L_2(\overline{g}_\alpha \cup \{\langle x_\alpha, y_\alpha\rangle\}), \quad \text{and} \quad \underline{h}_\alpha = L_2(\overline{h}_\alpha \cup \{(y_\alpha, f(x_\alpha))\})$$

are the additive functions. If $K_\alpha \cap \underline{h}_\alpha \neq \emptyset$, we choose $\langle v_\alpha, w_\alpha\rangle \in K_\alpha \cap \underline{h}_\alpha$ and take s_α such that $\underline{g}_\alpha(s_\alpha) = v_\alpha$. Such an s_α exists since $\operatorname{dom}(\underline{h}_\alpha) = \operatorname{rng}(\underline{g}_\alpha)$. Then $w_\alpha = \underline{h}_\alpha(v_\alpha) = \underline{h}_\alpha(\underline{g}_\alpha(s_\alpha)) = f(s_\alpha)$, so the condition (b) is satisfied. So, assume that $K_\alpha \cap \underline{h}_\alpha = \emptyset$. Then, as in the construction of x_α and y_α, we can find $\langle v_\alpha, w_\alpha\rangle \in K_\alpha$ such that

$$v_\alpha \notin \operatorname{dom}(\underline{h}_\alpha) = \operatorname{rng}(\underline{g}_\alpha), \quad \text{and} \quad w_\alpha \notin \operatorname{rng}(\underline{h}_\alpha). \tag{2}$$

Now, note that $\operatorname{rng}(f) = \mathbb{R}$, since f is a non-zero additive Darboux function. Choose $s_\alpha \in f^{-1}(w_\alpha)$ and notice that $s_\alpha \notin \operatorname{dom}(\underline{g}_\alpha)$ since otherwise $w_\alpha = f(s_\alpha) = \underline{h}_\alpha(\underline{g}_\alpha(s_\alpha)) = \underline{h}_\alpha(v_\alpha) \in \operatorname{rng}(\underline{h}_\alpha)$, contradicting (**??**). Thus, $\langle v_\alpha, f(s_\alpha)\rangle \in K_\alpha$, as required in (b).

Finally, to choose c_α note that

$$G_\alpha = L_2(\underline{g}_\alpha \cup \{\langle s_\alpha, v_\alpha\rangle\}), \quad \text{and} \quad H_\alpha = L_2(\underline{h}_\alpha \cup \{\langle v_\alpha, f(s_\alpha)\rangle\})$$

are the additive functions. If $b_\alpha \in \operatorname{dom}(G_\alpha)$, we put $c_\alpha = G_\alpha(b_\alpha)$. Otherwise we choose $c_\alpha \in \mathbb{R} \setminus \operatorname{dom}(H_\alpha)$. It is easy to see that x_α, y_α, s_α, v_α, and c_α chosen above satisfy (a)–(c). This finishes the inductive construction.

Having constructed functions g_α and h_α let

$$g = \bigcup_{\alpha<2^\omega} g_\alpha, \quad \text{and} \quad h^0 = \bigcup_{\alpha<2^\omega} h_\alpha.$$

It is easy to see that g and h^0 are additive functions such that $\operatorname{dom}(g) = \mathbb{R}$ (by (v)) and that $f = h^0 \circ g$. Now, if $h \colon \mathbb{R} \to \mathbb{R}$ is any additive extension of

h^0, then, by (iv), g and h are almost continuous, while we still have $f = h \circ g$. This finishes the proof. $\square$

Next we will prove the converse of Theorem **??**. For this we will need the following simple and well known fact.

Lemma 2. *If $g, h \in \mathcal{A}dd$ and g is a surjection, then*

$$\dim(\ker(h \circ g)) = \dim(\ker(h)) + \dim(\ker(g)).$$

PROOF. Let G, H be linearly independent sets such that $L(G) = \ker(g)$ and $L(H) = \ker(h)$. For every $w \in H$ choose $s_w \in g^{-1}(w)$ and notice that $F = G \cup \{s_w : w \in H\}$ is linearly independent. Indeed, suppose that

$$x = \sum_{i=1}^{n} q_i v_i + \sum_{j=1}^{k} p_j s_{w_j} = 0 \tag{3}$$

for some $n, k \in \mathbb{N}$, $q_i, p_j \in \mathbb{Q}$, $v_i \in G$, and $w_j \in H$, where $i = 1, \ldots, n$, and $j = 1, \ldots, k$. Then

$$g(x) = \sum_{i=1}^{n} q_i g(v_i) + \sum_{j=1}^{k} p_j g(s_{w_j}) = \sum_{j=1}^{k} p_j g(s_{w_j}) = \sum_{j=1}^{k} p_j w_j = 0$$

which shows that $p_j = 0$ for $j = 1, \ldots, k$. Hence, by (**??**), $\sum_{i=1}^{n} q_i v_i = 0$, which implies that $q_i = 0$ for $i = 1, \ldots, n$.

It is easy to see that $L(F) = \ker(h \circ g)$ and consequently,

$$\dim(\ker(g)) + \dim(\ker(h)) = \mathrm{card}\,(G) + \mathrm{card}\,(H) = \mathrm{card}\,(F) = \dim(\ker(h \circ g)).$$

This finishes the proof. $\square$

With this lemma in hand we are ready for the next theorem.

Theorem 2. *Assume $f \in \mathcal{A}dd$ and* $\dim(\ker(f)) = 1$.

(I) *If $f \notin \mathcal{AC}$, then $f = h \circ g$ for no $h, g \in \mathcal{A}dd \cap \mathcal{AC}$.*

(II) *If $f \notin \mathcal{C}onn$, then $f = h \circ g$ for no $h, g \in \mathcal{A}dd \cap \mathcal{C}onn$.*

PROOF. Fix $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $\dim(\ker(f)) = 1$ and suppose that there exist $g, h \in \mathcal{A}dd \cap \mathcal{D}$ with $f = h \circ g$. Then, g is surjection, since $g \not\equiv 0$. By Lemma **??**, either $\dim(\ker(g)) = 0$ or $\dim(\ker(h)) = 0$. Consequently, either g or h is a Darboux injection, so it is equal to a linear homeomorphism $L(x) = ax$. (Any other additive function has a dense graph, so it cannot be

Darboux and one-to-one at the same time.) Since the classes $\mathcal{AC}$ and $\mathcal{C}onn$ are closed under composition with homeomorphisms (cf, e.g., [?]), we conclude that $f \in \mathcal{AC}$ ($f \in \mathcal{C}onn$) if and only if $g, h \in \mathcal{AC}$ ($g, h \in \mathcal{C}onn$). $\square$

Theorems **??** and **??** give us as a corollary the main characterization. (Since $\mathcal{AC} \subset \mathcal{C}onn$.)

Corollary 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive Darboux function. Then*

(I) *f is a composition of two additive almost continuous functions if and only if either f is almost continuous or* $\dim(\ker(f)) \neq 1$*;*

(II) *f is a composition of two additive connectivity functions if and only if either f is a connectivity function or* $\dim(\ker(f)) \neq 1$*.*

3 Final Remarks

Although Corollary **??** gives a full characterization of additive Darboux functions which can be expressed as a composition of two additive almost continuous (or connectivity) functions it still does not exclude the possibility that every additive Darboux function can be expressed as a such composition. To conclude this, we need also the following example.

Example 1. *There exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that* $\dim(\ker(f)) = 1$ *and $f \in \mathcal{A}dd \cap \mathcal{D} \setminus \mathcal{C}onn$.*

PROOF. Let H be a Hamel basis and H_0 be a proper subset of H be with $\mathrm{card}\,(H_0) = 2^\omega$. Choose $h_0 \in H_0$, fix a bijection $\varphi\colon H_0 \setminus \{h_0\} \longrightarrow H_0$ and define $\overline{f}\colon H \to \mathbb{R}$ as follows.

$$\overline{f}(h) = \begin{cases} 0 & \text{for} \quad h = h_0 \\ \varphi(h) & \text{for} \quad h \in H_0 \setminus \{h_0\} \\ h & \text{for} \quad h \in H \setminus H_0. \end{cases}$$

Let f be the additive extension of $\overline{f}$. It is easy to observe that

$$\overline{f}(h) \in h + L(H_0) \ \text{ for } \ h \in H, \tag{4}$$

and therefore

$$f(x) \in x + L(H_0) \ \text{ for every } \ x \in \mathbb{R}. \tag{5}$$

It is obvious that $\ker(f) = L(\{h_0\})$. Also $\mathrm{rng}\,(f) = \mathbb{R}$, since $\mathrm{rng}\,(\overline{f}) = H$. Thus $f^{-1}(y) \neq \emptyset$ for every $y \in \mathbb{R}$. Moreover, since all level sets are congruent under translations and $\ker(f)$ is dense [?], $f^{-1}(y)$ is dense for every $y \in \mathbb{R}$.

Hence, the graph of f is dense in $\mathbb{R}^2$ and $f[J] = \mathbb{R}$ for every interval $J \subset \mathbb{R}$. In particular, $f \in \mathcal{D}$. Moreover, by (**??**),

$$f \subset \bigcup_{b \in L(H_0)} \{\langle x, x+b\rangle : x \in \mathbb{R}\}$$

and consequently, the line $y = x + c$ separates the graph of f for every number $c \in \mathbb{R} \setminus L(H_0)$. So, f is not a connectivity function. □

Corollary 2. *There exists an additive Darboux function $f : \mathbb{R} \to \mathbb{R}$ such that $f = h \circ g$ for no $f, g \in \mathcal{A}dd \cap \mathcal{C}onn$.*

Our last theorem is a variation of the example above. For its proof we will need one more easy lemma.

Lemma 3. *Let f be an additive function and $F = L_2(f \cup \{\langle u, v\rangle\})$ where $u \notin \mathrm{dom}\,(f)$ and $v \notin \mathrm{rng}\,(f)$. Then $\ker(F) = \ker(f)$.*

PROOF. Obviously $\ker(f) \subset \ker(F)$. To prove that $\ker(F) \subset \ker(f)$, fix an arbitrary $x \in \ker(F)$. Then

$$x = q_0 u + q_1 w \quad \text{where} \quad q_0, q_1 \in \mathbb{Q} \quad \text{and} \quad w \in \mathrm{dom}\,(f).$$

Since $F(x) = q_0 v + q_1 f(w) = 0$, $q_0 v = -q_1 f(w)$. Because $v \notin \mathrm{rng}\,(f)$, $q_0 = 0$ and consequently, $x \in \mathrm{dom}\,(f)$. Which shows that $x \in \ker(f)$. □

Theorem 3. *For every cardinal number $\lambda \le 2^\omega$ there exists an additive almost continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\dim(\ker(f)) = \lambda$.*

PROOF. Since the function $f \equiv 0$ is almost continuous and $\dim(\ker(f)) = 2^\omega$ for such f, we can assume that $\lambda < 2^\omega$. If $\lambda = 0$, then the identity function id has required properties and so we may also assume that $\lambda > 0$.

Now, let $H \subset \mathbb{R}$ be a Hamel basis and $H_0 \subset H$ be such that $\mathrm{card}\,(H_0) = \lambda$. Also, let $\{b_\alpha : \alpha < 2^\omega\} = H \setminus H_0$ and choose an enumeration $\{K_\alpha : \alpha < 2^\omega\}$ of the family $\mathcal{B}$ of blocking sets from Lemma **??**, with $K_0 = \mathbb{R} \times \{0\}$. The construction will be a slight modification of that in the proof of Theorem **??**.

By transfinite induction construct a sequence $\langle f_\alpha : \alpha < 2^\omega\rangle$ of additive partial functions from $\mathbb{R}$ into $\mathbb{R}$ such the that following inductive conditions are satisfied for every $\alpha < 2^\omega$.

(i) $f_\beta \subset f_\alpha$ for every $\beta < \alpha$;

(ii) $f_\alpha \cap K_\alpha \ne \emptyset$;

(iii) $b_\alpha \in \mathrm{dom}\,(f_\alpha)$ and $\mathrm{card}\,(f_\alpha) \le \max(\lambda, \omega, \alpha)$;

(iv) $\ker(f_\alpha) = L(H_0)$.

We start the induction by putting $f_0 = L_2((H_0 \times \{0\}) \cup \{\langle b_0, 1\rangle\})$. It is obvious that f_0 fulfills the conditions (i)–(iv).

To make an inductive step, fix $\alpha < 2^\omega$, $\alpha > 0$, and assume that we have already chosen functions f_β for $\beta < \alpha$ which satisfy conditions (i)–(iv). If $b_\alpha \in \operatorname{dom}(\bigcup_{\beta<\alpha} f_\beta)$, we put $\overline{f}_\alpha = \bigcup_{\beta<\alpha} f_\beta$. Otherwise, by (iii), $\operatorname{card}(\operatorname{rng}(\bigcup_{\beta<\alpha} f_\beta)) < 2^\omega$ and we can choose $c_\alpha \in \mathbb{R} \setminus \operatorname{rng}(\bigcup_{\beta<\alpha} f_\beta)$. Put

$$\overline{f}_\alpha = L_2\left(\{\langle b_\alpha, c_\alpha\rangle\} \cup \bigcup_{\beta<\alpha} f_\beta\right).$$

Clearly $\overline{f}_\alpha$ satisfies (i), (iii) and (iv). Also, if $K_\alpha \cap \overline{f}_\alpha \neq \emptyset$, then $f_\alpha = \overline{f}_\alpha$ satisfies (ii) as well and the construction is completed.

So, assume that $K_\alpha \cap \overline{f}_\alpha = \emptyset$ and let $X_\alpha = \operatorname{dom}(\overline{f}_\alpha)$, and $Y_\alpha = \operatorname{rng}(\overline{f}_\alpha)$. From (iii) we have that $\operatorname{card}(Y_\alpha) \le \operatorname{card}(X_\alpha) \le \max(\omega, \alpha, \lambda) < 2^\omega$. We will choose $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$ such that

$$x_\alpha \notin X_\alpha \quad \text{and} \quad y_\alpha \notin Y_\alpha \tag{6}$$

and define $f_\alpha = L_2(\overline{f}_\alpha \cup \{\langle x_\alpha, y_\alpha\rangle\})$. This will finish the construction since, by Lemma **??**, $\ker(f_\alpha) = \ker(\overline{f}_\alpha) = L(H_0)$.

Now, if $K_\alpha = \mathbb{R} \times \{y\}$ for some $y \in \mathbb{R}$, then $y_\alpha = y \notin Y_\alpha$, since $K_\alpha \cap \overline{f}_\alpha = \emptyset$. Choose an arbitrary $x_\alpha \in \mathbb{R} \setminus X_\alpha$. Then $\langle x_\alpha, y_\alpha\rangle$ satisfies (**??**).

So, assume that $K_\alpha = \mathbb{R} \times \{y\}$ for no $y \in \mathbb{R}$. Then K_α^y is nowhere dense for every $y \in \mathbb{R}$. Since $Z(K_\alpha) = \{y \in \mathbb{R} \colon K_\alpha^y \text{ contains non-empty perfect set}\}$ is either countable or has the cardinality of the continuum, we have the following two cases to consider.

- $\operatorname{card}(Z(K_\alpha)) = 2^\omega$. Choose $y_\alpha \in Z(K_\alpha) \setminus Y_\alpha$. Then $\operatorname{card}(K_\alpha^{y_\alpha}) = 2^\omega$ and we may choose $x_\alpha \in K_\alpha^{y_\alpha} \setminus X_\alpha$. Clearly $\langle x_\alpha, y_\alpha\rangle$ satisfies (**??**).
- $\operatorname{card}(Z(K_\alpha)) \le \omega$. Then the set

  $$E_\alpha = \operatorname{dom}(K_\alpha) \setminus \bigcup\{K_\alpha^y \colon y \in Z(K_\alpha)\}$$

  is residual in $\operatorname{dom}(K_\alpha)$ and the set

  $$E_\alpha^1 = E_\alpha \setminus \Big(X_\alpha \cup \bigcup\{K_\alpha^y \colon y \in Y_\alpha \setminus Z(K_\alpha)\}\Big)$$

  has the cardinality of the continuum. Choose $x_\alpha \in E_\alpha^1$ and $y_\alpha \in \mathbb{R}$ such that $\langle x_\alpha, y_\alpha\rangle \in K_\alpha$. Then $\langle x_\alpha, y_\alpha\rangle$ satisfies (**??**) as well.

This finishes the inductive construction.

Now, put

$$f = \bigcup_{\alpha<2^\omega} f_\alpha.$$

It is easy to see that f has the desired properties. □

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