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MULTIPLIERS FOR SOME NON-ABSOLUTE INTEGRALS IN EUCLIDEAN SPACES

Abstract

In this paper we prove a uniform boundedness theorem and use it to show that if fg is non-absolutely integrable on an interval in Euclidean space for each non-absolute integrable function f, then g is almost everywhere a function of strongly bounded variation on E.

1 Introduction

A general form of a bounded linear functional on the space of all Henstock integrable function was first given in [**7**, Theorem 3.2] and was used to show that if fg is Henstock integrable on a compact interval E in Euclidean space for each Henstock integrable function f, then g is almost everywhere a function of strongly bounded variation on E. The integration by parts formula [**7**, Theorem 3.1, equation (4)] is a key tool used to prove [**7**, Theorem 3.2]. However it is not clear whether integration by parts holds for other non-absolute integrals, and so the method used in [**7**] does not seem to generalize easily to other non-absolute integrals. In this paper, we shall prove a uniform boundedness theorem and use it to extend [**7**, Theorem 5.1] to some other non-absolute integrals.

2 Preliminaries

By $\mathbb{R}$ and $\mathbb{R}^+$ we denote the real line and the positive real line respectively. Let m be a fixed positive integer. The m-dimensional Euclidean space is denoted

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by $\mathbb{R}^m$. Let $E = \prod_{i=1}^{m}[a_i, b_i]$ be a compact interval in $\mathbb{R}^m$ with $a_i < b_i$ for $1 \leq i \leq m$. By an interval E_0 we mean a compact set of the form $\prod_{i=1}^{m}[s_i, t_i]$ with $s_i < t_i$ for $1 \leq i \leq m$. If $X \subseteq \mathbb{R}^m$, then $\mathrm{int}(X)$, $\mathrm{diam}\, X$, $\overline{X}$ and χ_X denote the interior, the diameter, the closure, the characteristic function of X respectively. If Z is a subset of E and $\xi \in E$, then $\mathrm{dist}(\xi, Z)$ denotes the distance of ξ from Z. The m-dimensional Lebesgue measure of the set $X \subseteq \mathbb{R}^m$ is denoted by $|X|$. We shall say that the intervals E_1 and E_2 are non-overlapping if $\mathrm{int}(E_1) \cap \mathrm{int}(E_2) = \emptyset$. A *figure* is a finite union of non-overlapping intervals. Also, $B(\xi, r)$ denotes the open ball with center ξ and radius r in $\mathbb{R}^m$. If the intervals $I_i \subseteq E$, $i = 1, 2, \ldots, k$ are non-overlapping, then we say that the set $D = \{I_i : i = 1, 2, \ldots k\}$ is a partial division of E. If, in addition, $\cup_{i=1}^{k} I_i = E$, we say that D is a division of E. Given a function $\delta : E \to \mathbb{R}^+$ and a partial division D, we say that D is δ-fine if for each interval I from D we have $I \subseteq B(\xi, \delta(\xi))$ where ξ is a vertex of I, and we write $D = \{(I, \xi)\}$. In [**2**, p.42], it is shown that a δ-fine division of E exists for each $\delta : E \to \mathbb{R}^+$. Let G be an open set in E. A figure I_0 is called a non-absolute subset of G if there exits $\delta : E \to \mathbb{R}^+$ such that I_0 is the complement of a δ-fine cover of $E \setminus G$.

All functions considered in this paper will be real-valued. A function $f : E \to \mathbb{R}$ is Henstock integrable on E if there is a real number A with the following property: for every $\varepsilon > 0$, there exists $\delta : E \to \mathbb{R}^+$ such that $|(D) \sum f(\xi)\, |I| - A| < \varepsilon$ for each δ-fine division $D = \{(I, \xi)\}$ of E, and we write $A = \int_E f$. The family of all Henstock integrable functions on E will be denoted by $H(E)$. If I is a subinterval of E, we denote the Henstock integral of f on I by $\int_I f$. We denote by $L(E)$ the family of all Lebesgue integrable functions f on E, or equivalently, the family of all absolutely Henstock integrable functions on E. It is known that $L(E) \subset H(E)$ (see, for example,[**2**, p.37]).

For each measurable function$f : E \to \mathbb{R}$, the set of all non-absolute integrability points of f is defined to be

$$\mathrm{NA}(f) = \{x \in E : f \notin L(J) \text{ for each } J\ x \notin \mathrm{int}\, J\}.$$

In [**1**], it is shown that for each $f \in H(E)$, f is Lebesgue integrable on a portion of E, so we see that $\mathrm{NA}(f)$ is a proper closed subset of E. We say that f is Cauchy-Lebesgue integrable on E if $f \in CL(E)$, where

$$CL(E) = \{f \in H(E) : \mathrm{NA}(f) \text{ is a finite set}\}.$$

It is easy to see that $L(E) \subset CL(E) \subset H(E)$.

An interval function F is said to satisfy the (SL) condition if for each subset $W \subset E$ of measure zero and $\varepsilon > 0$, there exists $\delta : W \to \mathbb{R}^+$ such that for any δ-fine partial division $D = \{(I, \xi)\}$ of W, we have $(D) \sum |F(I)| < \varepsilon$.

For a definition of strongly bounded variation, see [**3**, Definition 1.1.4]. For the two dimensional definition of strongly bounded variation, see [**7**, Section 3].

3 Main Results

In this section, we prove a Banach-Steinhaus Theorem (Theorem 3.6), and an application will be given in section 4. We begin with a definition.

Definition 3.1. Let $\{c_k\} \subset E$, where $c_k = (c_k^{(1)}, c_k^{(2)}, c_k^{(3)}, \ldots, c_k^{(m)})$. For a nondegenerate subinterval $E_k \subset E$ with opposite vertices c_k, c_{k+1} , we write $E_k =< c_k, c_{k+1} >$. We say that $\{E_k\}$ is monotone if $\{c_k^{(j)}\}_{k=1}^{\infty}$ is strictly monotone for each $j = 1, 2, \ldots, m$. In this case, we say that $\{c_k\}$ is strictly monotone.

We remark that the two-dimensional version of Definition 3.1 is given in [**7**, Section 4].

Throughout this paper, $\langle u, v\rangle$ will denote a subinterval of E.

Let X be a linear space of integrable functions whose domain is E, and equip X with a norm $\|\cdot\|_X$. Suppose the following axioms hold for X.

(I) If $f \in X$, then $f\chi_I \in X$ for each subinterval I of E.

(II) Every $f \in X$ is measurable.

(III) G is a dense linear subspace of X, and if $f \in G$, then so is $f\chi_I$.

(IV) If $\{I_n\}$ is a monotone sequence of subintervals of E, and $\{f_n\} \subset G$, where $f_n(x) = 0$ for each $x \notin I_n$, and the series $\sum\limits_{k=1}^{\infty} \|f_k\|_X$ converges, then $f = \sum\limits_{k=1}^{\infty} f_k \in X$.

(V) For each $f \in X$, $\|f\chi_{\langle c,c_n\rangle} - f\chi_{\langle c,d\rangle}\|_X \to 0$ as $c_n \to d$.

(VI) For each $f \in X$, $\|f\chi_I\|_X \to 0$ as $|I| \to 0$, where I is a subinterval of E.

(VII) There exists $\gamma > 0$ such that $\|f\chi_I\|_X \le \gamma \|f\|_X$ for every subinterval I of E.

Two functions $f_1, f_2 \in X$ are regarded as identical if $f_1 = f_2$ almost everywhere in E. For each subinterval J of E, we let

$$X(J) = \{f \in X : f \text{ vanishes outside } J\}.$$

and

$$G(J) = \{f \in G : f \text{ vanishes outside } J\}.$$

A function g defined on E is said to be a multiplier for X if $fg \in X$ for each $f \in X$. Let Y be a normed space.

Definition 3.2. A linear operator $T : X \to Y$ is said to be nice if the following conditions are satisfied.

(N1) Given any subinterval J of E and $f \in X(J)$, there exists $\{f_n^*\} \subset G(J)$ such that $\|f_n^* - f\|_X \to 0$ and $\|T(f_n^*\chi_I) - T(f\chi_I)\|_Y \to 0$ as $n \to \infty$ for each subinterval I of J.

(N2) Let $c, d \in E$. If $\{c_n\} \subset \mathrm{int}(<\mathrm{c}, \mathrm{d}>)$ is strictly monotone with $c_n \to d$. Then
$$\|T(f\chi_{\langle c,c_n\rangle}) - T(f\chi_{\langle c,d\rangle})\|_Y \to 0$$
as $n \to \infty$.

(N3) If $f \in X$, then $\|T(f\chi_I)\|_Y \to 0$ as $|I| \to 0$.

If $T : X \to Y$ is a nice operator, we write $T \in \mathcal{N}(X, Y)$. We shall use the concept of nice operator to prove a uniform boundedness theorem for X. We need some lemmas and theorems.

Lemma 3.3. *Let $T : X \to Y$ be a nice operator. If p_0 is a positive integer and $f \in X(< c, d >)$ with $\|f\|_X < 1$ and $\|T(f)\|_Y > 4^{p_0}$, then there exist $c_p \in \mathrm{int}(< c, d >)$ and $f_p \in G(< c, c_p >)$ such that*

(i) $\|f_p\|_X < \frac{1}{2^{p_0}}$.

(ii) $\|T(f_p)\|_Y > 2^{p_0}$.

PROOF. Let $\{c_n\} \subset \mathrm{int}(< c, d >)$ be strictly monotone with $c_n \to d$. Since T is a nice operator, by (N2) condition,

$$\|T(f\chi_{<c,c_n>}) - T(f\chi_{<c,d>})\|_Y \to 0 \tag{1}$$

as $n \to \infty$, (1) implies that

$$\|T(f\chi_{<c,c_n>})\|_Y \to \|T(f\chi_{<c,d>})\|_Y \tag{2}$$

as $n \to \infty$, By Axiom (V), we have

$$\|f\chi_{<c,c_n>} - f\chi_{<c,d>}\|_X \to 0 \tag{3}$$

as $c_n \to d$. From (3), we have

$$\|f\chi_{\langle c,c_n\rangle}\|_X \to \|f\chi_{\langle c,d\rangle}\|_X \tag{4}$$

as $n \to \infty$.

By our hypothesis, $\|f\|_X < 1$ and $\|T(f\chi_{<c,d>})\|_Y > 4^{p_0}$, so by (4) and (2), we may choose a sufficiently large integer, say p, so that

$$\|f\chi_{\langle c,c_p\rangle}\|_X < 1 \tag{5}$$

and

$$\|T(f\chi_{\langle c,c_p\rangle})\|_Y > 4^{p_0}. \tag{6}$$

Then $c_p \in \operatorname{int}(\langle c,d\rangle)$. Since T is a nice operator, by (N1) condition, there exists $\{f_n^*\} \subset G(\langle c,c_p\rangle)$ such that

$$\|f_n^* - f\chi_{\langle c,c_p\rangle}\|_X \to 0 \tag{7}$$

and

$$\|T(f_n^*\chi_{<c,c_p>}) - T(f\chi_{<c,c_p>})\|_Y \to 0 \tag{8}$$

as $n \to \infty$. From (7), we have

$$\|f_n^*\|_X \to \|f\chi_{\langle c,c_p\rangle}\|_X \quad \text{as} \quad n \to \infty. \tag{9}$$

From (8), we have

$$\|T(f_n^*\chi_{<c,c_p>})\|_Y \to \|T(f\chi_{<c,c_p>})\|_Y \quad \text{as} \quad n \to \infty. \tag{10}$$

By (5), (9), (6) and (10), we may choose a sufficient large integer $N = N(p,p_0)$ so that $\|f_N^*\|_X < 1$ and $\|T(f_N^*\chi_{\langle c,c_p\rangle})\|_Y > 4^{p_0}$ Put $f_p = \frac{1}{2^{p_0}} f_N^*\chi_{<c,c_p>}$. $\square$

Lemma 3.4. *Let $\{I_i\}_{i=1}^k$ be a division of E and Y a normed space. If $\mathcal{T} \subset \mathcal{N}(X,Y)$ satisfies $\sup\{\|T\| : T \in \mathcal{T}\} = \infty$, then there exists I_p such that $\sup\{\|T\Big|_{X(I_p)}\| : T \in \mathcal{T}\} = \infty$.*

PROOF. Suppose not. Then for each $1 \le p \le k$, we have

$$\sup\{\|T\Big|_{X(I_p)}\| : T \in \mathcal{T}\} < \infty. \tag{11}$$

Recall that every two elements of X almost everywhere equal on E are regarded as identical, and $\{I_i\}_{i=1}^{k}$ be a division of E, we have

$$T(f) = \sum_{i=1}^{k} T(f\chi_{I_i}) \tag{12}$$

and thus by (12) $\|T(f)\|_Y \leq \sum_{i=1}^{k} \|T\Big|_{X(I_i)}\| \|f\|_X$ and so $\sup_{T\in\mathcal{T}} \|T\| \leq \sum_{i=1}^{k} \sup_{T\in\mathcal{T}} \|T\Big|_{X(I_i)}\| < \infty$ by (11) a contradiction to our original hypothesis. □

Lemma 3.5. *Let Y be a normed space. If $\mathcal{T} \subset \mathcal{N}(X,Y)$ satisfies*

$$\sup\{\|T\| : T \in \mathcal{T}\} < \infty,$$

then for each subinterval E_0 of E, we have $\sup\{\|T\Big|_{X(E_0)}\| : T \in \mathcal{T}\} < \infty$.

PROOF. Note that by (I), $f\chi_{E_0} \in X$ for each $f \in X$. Then we have for each $T \in \mathcal{T}$,

$$\|T\Big|_{X(E_0)}(f)\|_Y = \|T(f\chi_{E_0})\|_Y \leq \|T\|\|f\chi_{E_0})\|_X \leq \gamma\|T\|\|f\|_X \text{ by (VII).}$$

Consequently, $\sup_{T\in\mathcal{T}} \|T\Big|_{X(E_0)}\| \leq \gamma \sup_{T\in\mathcal{T}} \|T\| < \infty$. □

Theorem 3.6. *Let Y be a normed space. If $\mathcal{T} \subseteq \mathcal{N}(X,Y)$ such that for each $f \in X$, $M(f) = \sup\{\|T(f)\|_Y : T \in \mathcal{T}\} < \infty$, then* $\sup\{\|T\| : T \in \mathcal{T}\} < \infty$.

PROOF. Suppose the conclusion is false. By Lemma 3.4, there exists a subinterval J_1 of E such that $\sup\{\|T\Big|_{X(J_1)}\| : T \in \mathcal{T}\} = \infty$ By Lemma 3.4, there exists a subinterval J_2 of J_1 such that $\sup\{\|T\Big|_{X(J_2)}\| : T \in \mathcal{T}\} = \infty$. By induction, we can construct a decreasing sequence $\{J_i\}$ of subintervals of E with $\bigcap_{j=1}^{\infty} J_i = \{y\}$ and for each i,

$$\sup\{\|T\Big|_{X(J_i)}\| : T \in \mathcal{T}\} = \infty. \tag{13}$$

We may choose $x_0 \in E$ so that $|< x_0, y > \cap J_i| > 0$ and

$$\sup\{\|T\Big|_{X(\langle x_0,y\rangle\cap J_i)}\| : T \in \mathcal{T}\} = \infty \tag{14}$$

for infinitely many i. Put

$$l = \min\{i : |< x_0, y > \cap J_i| > 0 \text{ and } (14) \text{ holds for infinitely many i}\}.$$

Then for each $w \in \operatorname{int}(\langle x_0, y\rangle \cap J_l)$, we have $< w, y >\supset< x_0, y > \cap J_p$ for some p and (14) holds with $i = p$. By (14) and Lemma 3.5,

$$\sup\{\|T\Big|_{X(\langle w,y\rangle)}\| : T \in \mathcal{T}\} = \infty. \tag{15}$$

Choose $x_1 \in \operatorname{int}(\langle x_0, y\rangle)$. Then by (15) with $w = x_1$, there exists $f_1^* \in X(\langle x_1, y\rangle)$ and $T_1 \in \mathcal{T}$ such that $\|f_1^*\|_X < 1$ and $\|T_1(f_1^*)\|_Y > 4$. By Lemma 3.3 with $p_0 = 1$, there exists $x_2 \in \operatorname{int}(\langle x_1, y\rangle)$, $f_1 \in G(\langle x_1, x_2\rangle)$ such that $\|f_1\|_X < \frac{1}{2}$ and $\|T_1(f_1)\|_Y > 2$ By (15) with $w = x_2$, there exists $f_2^* \in X(\langle x_2, y\rangle)$ and $T_2 \in \mathcal{T}$ such that $\|f_2^*\|_X < 1$ and $\|T_2(f_2^*)\|_Y > 4^2$. By Lemma 3.3 with $p_0 = 2$, there exists $x_3 \in \operatorname{int}(\langle x_2, y\rangle)$, $f_2 \in G(\langle x_2, x_3\rangle)$ such that $\|f_2\|_X < \frac{1}{2^2}$ and $\|T_2(f_2)\|_Y > 2^2$. Proceeding in this way and and by Lemma 3.3, we can construct

(i) a strictly monotone sequence of points $\{x_n\}$ in $(< x_1, y >)$ converging to y with $x_{n+1} \in \operatorname{int}(< x_n, y >)$ and

(ii) a sequence $\{f_n\} \subset G$ and $\{T_n\} \subseteq \mathcal{T}$ such that $f_n \in G(< x_n, x_{n+1} >)$ with $\|f_n\|_X < \frac{1}{2^n}$ and $\|T_n(f_n)\|_Y > 2^n$.

Since $\sum\limits_{k=1}^{\infty} \|f_k\|_X < \infty$, by (i), (ii) and (IV), we have $f = \sum_{k=1}^{\infty} f_k \in X$ Claim. There is a subsequence $\{f_{n_k}\}$ such that for $k \geq 1$:

(a) $\|T_{n_{k+1}}(f_{n_{k+1}})\|_Y > 1 + k + \sum\limits_{j=1}^{k} M(f_{n_j})$ and

(b) $\sup\limits_{1\leq i\leq k} \|T_{n_i}(f_{n_{k+1}})\|_Y < 2^{-k-1}$.

The proof of the claim is done by induction. Choose $n_1 = 1$. Since $f \in X$, and $|< x_n, x_{n+1} >| \to 0$ as $n \to \infty$, by (VI), $\|f\chi_{\langle x_n,x_{n+1}\rangle}\|_X \to 0$ as $n \to \infty$. By (N3), $\|T(f\chi_{\langle x_n,x_{n+1}\rangle})\|_Y \to 0$ for each $T \in \mathcal{T}$. Note that we have $f_n =$

$f\chi_{\langle x_n,x_{n+1}\rangle}$ for all n. Recall also that $\|T_n(f_n)\|_Y > 2^n$ for all $n \geq 1$. We may choose $f_{n_2} \in \{f_n\}$ and T_{n_2} such that

$$\|T_{n_2}(f_{n_2})\|_Y > 1 + 1 + M(f_1) \text{ and } \|T_{n_1}(f_{n_2})\|_Y < 2^{-1-1};$$

so the claim holds when $k = 1$.

Now suppose that the claim is valid for some positive integer $k = q$ for some $\{f_1, f_{n_2}, \ldots, f_{n_q}, f_{n_{q+1}}\}$ and $\{T_1, T_{n_2}, \ldots, T_{n_{q+1}}\}$. Since $f \in X$, and $|< x_n, x_{n+1} >| \to 0$ as $n \to \infty$, by (VI), $\|f\chi_{\langle x_n,x_{n+1}\rangle}\|_X \to 0$ as $n \to \infty$. By (N3), $\|T(f\chi_{\langle x_n,x_{n+1}\rangle})\|_Y \to 0$ for each $T \in \mathcal{T}$. Note that $f_n = f\chi_{\langle x_n,x_{n+1}\rangle}$ and $\|T_n(f_n)\|_Y > 2^n$ for all $n \geq 1$. We may choose $f_{n_{q+2}} \in \{f_n\}$ and $T_{n_{q+2}}$ such that

$$\|T_{n_{q+2}}(f_{n_{q+2}})\|_Y > 1+q+1+\sum_{j=1}^{q+1} M(f_{n_j}) \text{ and } \sup_{1\leq i\leq q+1} \|T_{n_i}(f_{n_{q+2}})\|_Y < 2^{-(1+q)-1}$$

so the claim holds when $k = q + 1$. By induction, the claim is proved.

Since $\sum_{k=1}^{\infty} \|f_k\|_X < \infty$, the series $\sum_{k=1}^{\infty} \|f_{n_k}\|_X < \infty$, so by Axiom (IV), $f_0 = \sum_{k=1}^{\infty} f_{n_k} \in X$. Now, for $k \geq 1$,

$$\begin{aligned}
\|T_{n_{k+1}}(f_0)\|_Y &= \|\sum_{j=1}^{k} T_{n_{k+1}}(f_{n_j}) + T_{n_{k+1}}(f_{n_{k+1}}) + \sum_{j=k+2}^{\infty} T_{n_{k+1}}(f_{n_j})\|_Y \\
&= \|T_{n_{k+1}}(f_{n_{k+1}}) - \{-\sum_{j=1}^{k} T_{n_{k+1}}(f_{n_j}) - \sum_{j=k+2}^{\infty} T_{n_{k+1}}(f_{n_j})\}\|_Y \\
&\geq \|T_{n_{k+1}}(f_{n_{k+1}})\|_Y - \|\sum_{j=1}^{k} T_{n_{k+1}}(f_{n_j}) + \sum_{j=k+2}^{\infty} T_{n_{k+1}}(f_{n_j})\|_Y \\
&\geq 1 + k + \sum_{j=1}^{k} M(f_{n_j}) - \{\sum_{j=1}^{k} M(f_{n_j}) + \|\sum_{j=k+2}^{\infty} T_{n_{k+1}} f_{n_j}\|_Y\} \\
&\geq 1 + k - \sum_{j=k+2}^{\infty} 2^{-j-1} \geq k.
\end{aligned}$$

So we have $M(f_0) \geq \sup_{k\geq 1} \|T_{n_{k+1}}(f_0)\| = \infty$ a contradiction. $\square$

In Section 4, we shall give an application in which Theorem 3.6 holds but the classical Banach-Steinhaus Theorem (Corollary 3.7) does not seem to apply. Denoting the space of all bounded linear operators from X into Y by $\mathcal{B}(X,Y)$, we have the following assertion.

Corollary 3.7. *Let Y be a normed space. If $\mathcal{T} \subseteq \mathcal{B}(X,Y)$ such that for each $f \in X$, $M(f) = \sup\{\|T(f)\|_Y : T \in \mathcal{T}\} < \infty$, then $\sup\{\|T\| : T \in \mathcal{T}\} < \infty$.*

Proof. By (I), (III), (V), (VI) and (VII), every bounded linear operator is nice. □

4 An Application

Definition 4.1. An interval function F is said to be *continuous* if whenever $|I| \to 0$, we have $|F(I)| \to 0$.

By using [**8**, Theorem 6], it is easy to see that if $f \in H(E)$, then the interval function F defined by $F(I) = \int_I f$ is continuous on E. Hence the space $H(E)$ as well as its subspace $CL(E)$ can be equipped with the norm $\|\cdot\|$, where $\|f\| = \sup\limits_I \left|\int_I f\right|$ for each $f \in H(E)$, where the supremum is taken over all subinterval I of E.

The next result gives a characterization of all Cauchy-Lebesgue integrable functions on E.

Lemma 4.2. *$f \in CL(E) \iff$ there exists an additive continuous interval function F and a finite subset Q of E such that f is Lebesgue integrable on every subinterval J with $\int_J f = F(J)$, where $J \cap Q = \emptyset$.*

Proof. ($\Longrightarrow$) For each $f \in CL(E)$, we let $F(I) = (CL)\int_I f$ for every interval $I \subset E$, and $Q = \mathrm{NA(f)}$.

($\Longleftarrow$) Since Q is finite and since F is continuous, it is easy to verify that F satisfies (SL) condition on Q. Hence for $\varepsilon > 0$, there exists $\delta : Q \to \mathbb{R}^+$ such that for any δ-fine partial division $D = \{(I,\xi)\}$ of Q, we have $(D)\sum |F(I)| < \varepsilon$. We claim that [**6**, Theorem 3] applies here. Define $\delta_0 : E \to \mathbb{R}^+$ by $\delta_0(\xi) = \delta(\xi)$ if $\xi \in Q$ and $\delta_0(\xi) = \mathrm{dist}(\xi, Q)$ otherwise. Take $A = F(E)$ and $G = E - Q$. Then for any non-absolute subset I_0 of G involving δ_0, we have for some δ_0-fine cover $D_0 = \{(I,\xi)\}$ of Q,

$$|F(I_0) - F(E)| = \left|(D_0)\sum F(I)\right| \leq (D_0)\sum |F(I)| < \varepsilon$$

since F is an additive interval function satisfying (SL). Since f is Lebesgue integrable on every subinterval J with $\int_J f = F(J)$, where $J \cap Q = \emptyset$, [**6**, Theorem 3] holds. Thus f is Henstock integrable on E with $F(I) = \int_I f$ for each subinterval I of E. Hence $Q = \mathrm{NA}(f)$ and consequently $f \in CL(E)$.

Lemma 4.2. *If $f \in CL(E)$, then there exists a sequence $\{K_n\}$ of figures such that for all n, $K_n \subset K_{n+1} \subset E$ with $\overline{\bigcup_{n=1}^{\infty} K_n} = E$, $f \in L(K_n)$ and* $\lim_{n\to\infty} \|f\chi_{K_n} - f\| = 0$.

Proof. Let $F(I) = (CL)\int_I f$ and $Q = \mathrm{NA}(f) = \{x_1, x_2, \dots, x_l\}$. Since F satisfies (SL) on Q, for each n, there exists $\delta_n : Q \to \mathbb{R}^+$ such that for any δ_n-fine division D_n of Q, we have $(D_n)\sum |F(I)| < \frac{1}{n}$. We may assume that $\delta_1 > \delta_2 > \cdots > \delta_n > \dots$. For each n, we fix a δ_n-fine division D_n of Q. Put $P_n = \bigcup\{J : (J, x) \in D_n\}$. Then there are at most $l2^m$ interval-point pairs in each D_n, and $|\partial P_n| = 0$. Put $K_n = \overline{E - P_n}$. Then $\{K_n\}$ is a sequence of figures with $\overline{\bigcup_{n=1}^{\infty} K_n} = E$. Note that as $n \to \infty$, $|P_n| = |E - K_n| \to 0$. Consequently, by the continuity of F, we have $\|f\chi_{K_n} - f\| = \|f\chi_{P_n}\| \to 0$ as $n \to \infty$.

Lemma 4.3. *If g is a multiplier for $CL(E)$, and $\{K_n\}$ be given as in Lemma 4.3. Then* $\lim_{n\to\infty} \|fg\chi_{K_n} - fg\| = 0$.

Proof. Repeat the proof of Lemma 4.3 with the following modifications.

(a) f is replaced by fg and

(b) $F(I) = \int_I f$ is replaced by $F_1(I) = (CL)\int_I fg$.

Observing that F_1 satisfies (SL) on Q, we have the result. □

We shall next apply our uniform boundedness theorem (Theorem 3.6) to prove Theorem 4.5. Note that Corollary 3.7 does not seem to apply.

Theorem 4.4. *If g is a multiplier for $CL(E)$, then $T : CL(E) \to \mathbb{R}$ defined by $T(f) = \int_E fg$ is a bounded linear functional on $CL(E)$.*

Proof. We first show that $T : CL(E) \to \mathbb{R}$ is a nice operator (see Definition 3.2) with $G = L(E)$, $X = CL(E)$ and $Y = \mathbb{R}$. Since each $f \in CL(E) \subset H(E)$, we see that T satisfies conditions (N2), (N3) of definition 3.2. It remains to verify that condition (N1) holds for T. Now, let J be any subinterval of E and for any subinterval I of J, $\left|\int_I f\chi_{K_n} - \int_I f\right| \le \|f\chi_{K_n} - f\|$ which tends to zero as $n \to \infty$ by Lemma 4.3. So the first condition of (N1) is satisfied.

Next we will prove that the second condition of (N1) is satisfied. We observe that

$$\left|T(f\chi_{K_n}\chi_I) - T(f\chi_I)\right| = \left|\int_I f\chi_{K_n} g - \int_I fg\right| \le \|f\chi_{K_n} g - fg\|$$

which tends to zero by Lemma 4.4. Thus (N1) condition is satisfied. The theorem then follows from Theorem 3.6 with $X(E) = CL(E)$, $G(E) = L(E)$, Axioms (I) to (VII) hold and note that Axiom (IV) holds by [**7**, Lemma 4.1] and Lemma 4.2. $\square$

Corollary 4.5. *If $fg \in CL(E)$ for each $f \in CL(E)$, then g is almost everywhere a function of strongly bounded variation on E.*

Proof. By Theorem 4.5, the linear functional $T : CL(E) \to \mathbb{R}$ defined by $T(f) = \int_E fg$ is bounded on $CL(E)$. Although the spaces $(CL(E), \|\cdot\|)$ and $(H(E), \|\cdot\|)$ are not complete, we may still apply the Hahn-Banach Theorem to normed spaces. See, for example, [**9**, Theorem 3.3]. By Hahn-Banach Theorem [**9**, Theorem 3.3.], there exists a bounded linear functional T_1 on $H(E)$ such that $T(f) = T_1(f)$ for all $f \in CL(E)$. By [**7**, Theorem 3.2], there exists a function g_0 of strongly bounded variation on E such that

$$T_1(f) = \int_E fg_0 \text{ for all } f \in H(E).$$

As $T(f) = T_1(f)$ for all $f \in L(E) \subset CL(E)$, we have $\int_E fg = \int_E fg_0$ for all $f \in L(E)$. Hence $g = g_0$ almost everywhere on E and we are done. $\square$

From the proof of Corollary 4.6, we also have the following.

Corollary 4.6. *If $T \in CL(E)^*$, the conjugate space of $CL(E)$, then there exists a function of strongly bounded variation on E such that $T(f) = \int_E fg$ for all $f \in CL(E)$.*

We can now give the main result of this section.

Theorem 4.7. *Suppose $CL(E) \subset X(E)$. If g is a multiplier for X, then g is almost everywhere a function of strongly bounded variation on E.*

Proof. By (II) and repeating the proof of [**5**, Theorem 12.8], g is almost everywhere bounded on E. Since g is almost everywhere bounded on E, we can verify that $\mathrm{NA}(fg) \subset \mathrm{NA}(f)$. Note that $f \in CL(E)$, $\mathrm{NA}(f)$ is a finite set and so is $\mathrm{NA}(fg)$. By the continuity of the X-primitive of fg ((VI)) and Lemma 4.2, $fg \in CL(E)$. By Corollary 4.6, we have the result. $\square$

5 Remarks on One Dimensional Results

The first theorem is well known.

Theorem 5.1. *If $g \in BV([a,b])$ and F is continuous on $[a,b]$, then g is Riemann-Stieltjes integrable with respect to F on $[a,b]$ with*

$$(RS)\int_a^b g\ dF = F(b)g(b) - F(a)g(a) - (RS)\int_a^b F dg.$$

By using the integration by substitution theorem for non-absolute integral (see for example [**4**, page 186, Exercise 2], we have the next theorem.

Remark 5.2. Every function of bounded variation on $[a,b]$ is a multiplier for non-absolute integral (with a continuous primitive).

By Corollary 4.6 and Remark 5.2, we see that the multipliers for non-absolute integrals (with a continuous primitive) are essentially the space of all essentially bounded variation on $[a,b]$.

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