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WEIGHTED INEQUALITIES OF HARDY-TYPE ON AMALGAMS

Abstract

Weighted Hardy-type inequalities between suitable amalgams $\ell^q(L^p, u)$ and $\ell^{\bar{q}}(L^{\bar{p}}, v)$ are characterized. The Hardy-type operator involved in the inequalities involves functions which are not necessarily non-negative.

1 Introduction.

In [1, Corollary 1.3], the following form of discrete Hardy inequality has been obtained:

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=-\infty}^{n} a_k\right)^q u_n\right)^{\frac{1}{q}} \leq C\left(\sum_{n\in\mathbb{Z}} a_n^p v_n\right)^{\frac{1}{p}} \tag{1.1}$$

which holds for $1 < p \leq q < \infty$ if and only if

$$\sup_n \left(\sum_{k=n}^{\infty} u_k\right)^{\frac{1}{q}} \left(\sum_{k=-\infty}^{n} v_k^{1-p'}\right)^{\frac{1}{p'}} < \infty \tag{1.2}$$

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and for $1 < q < p < \infty$ if and only if

$$\left\{\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}^{\infty}u_k\right)^{\frac{r}{q}}\left(\sum_{k=-\infty}^{n}v_k^{1-p'}\right)^{\frac{r}{q'}}v_n^{1-p'}\right\}^{\frac{1}{r}} < \infty \tag{1.3}$$

where $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$, $\{a_n\}$, $\{u_n\}$, $\{v_n\}$ are sequences on positive real numbers and $p'=\frac{p}{p-1}$. Also the complementary inequality

$$\left(\sum_{n\in\mathbb{Z}}\left(\sum_{k=n}^{\infty}a_k\right)^{q}u_n\right)^{\frac{1}{q}} \le C\left(\sum_{n\in\mathbb{Z}}a_n^p v_n\right)^{\frac{1}{p}}$$

is well known which holds for $1 < p \le q < \infty$ if and only if

$$\sup_n\left(\sum_{k=-\infty}^{n}u_k\right)^{\frac{1}{q}}\left(\sum_{k=n}^{\infty}v_k^{1-p'}\right)^{\frac{1}{p'}} < \infty\,.$$

The above inequalities can be regarded as the boundedness of the discrete Hardy operators

$$H(a_n)=\sum_{k=-\infty}^{n}a_k \quad \text{and} \quad H^*(a_n)=\sum_{k=n}^{\infty}a_k$$

between weighted sequence spaces $\ell^p(v_n)$ and $\ell^q(u_n)$. For a detailed description of such operators along with their continuous versions, one may see the recent monographs [2], [3]. For a sequence weight $\{w_n\}$ and $1 \le p < \infty$, the weighted sequence space $\ell^p(w_n)$ consists of all real sequences $\{a_n\}$ such that

$$\|a_n\|_{\ell^p(w_n)}=\left(\sum_{n\in\mathbb{Z}}|a_n|^p w_n\right)^{\frac{1}{p}} < \infty\,.$$

It is known that $\ell^p(w_n)$ is a Banach space. More general discrete Hardy operators can be considered, e.g.,

$$\begin{aligned} T_1(a_n) &= \phi_{1,n}\sum_{k=-\infty}^{n}\psi_{1,k}a_k\,,\\ T_2(a_n) &= \phi_{2,n}\sum_{k=n}^{\infty}\psi_{2,k}a_k\,,\end{aligned}$$

$\{\phi_{i,n}\}$, $\{\psi_{i,n}\}$, $i = 1, 2$ being non-negative sequences. By suitable transformations, the boundedness of $T_1(a_n)$ and $T_2(a_n)$ between sequence spaces can be obtained using the corresponding boundedness of $H(a_n)$ and $H^*(a_n)$.

In the present paper, first we deal with the operator $T = T_1 + T_2$, where $\{\phi_{i,n}\}$ and $\{\psi_{i,n}\}$, $i = 1, 2$ are general sequences, not necessarily non-negative. In fact, we characterize the boundedness of T between suitable weighted sequence spaces. For the continuous case, the operator T has been studied by Zharov [5] (see also [3]).

Next, we consider the weighted amalgam $\ell^q(L^p, w)$ which consists of functions f defined on $(-\infty, \infty)$ such that

$$||f||_{p,w,q} = \left\{ \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(x)|^p w(x)\, dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty$$

with $1 < p, q < \infty$ and w a weight function. We study between such suitable amalgams, the boundedness of the operator $S = S_1 + S_2$, where

$$\begin{aligned} (S_1 f)(x) &= \phi_1(x) \int_{-\infty}^{x} \psi_1(t) f(t)\, dt, \\ (S_2 f)(x) &= \phi_2(x) \int_{x}^{\infty} \psi_2(t) f(t)\, dt \end{aligned}$$

with ϕ_i, ψ_i , $i = 1, 2$ being general functions not necessarily non-negative. When $\phi_i \equiv \psi_i \equiv 1$, the corresponding $\ell^q(L^p, u)$ - $\ell^{\bar{q}}(L^{\bar{p}}, v)$ boundedness of the operators S_1 and S_2 has been studied by Carton-Lebrun, Heinig and Hofmann [1]. In the case when ϕ_i , ψ_i are general functions, which along with the functions f are defined on $(0, \infty)$, the L^p - L^q boundedness of S_1 and S_2 has been obtained by Stepanov [4] while the operator S has been studied by Zharov [5].

2 Discrete Hardy-type inequalities.

The following theorem characterizes the ℓ^p - ℓ^q boundedness of the operator T .

Theorem 2.1. *Let* $1 < p \le q < \infty$, $\{\phi_{1,n}\}$, $\{\psi_{1,n}\}$, $\{\phi_{2,n}\}$, $\{\psi_{2,n}\}$ *be the sequences of real numbers (not necessarily non-negative) and* $\{u_n\}$, $\{v_n\}$ *be the sequence weights.*

The inequality

$$\left(\sum_{n\in\mathbb{Z}} \left| \phi_{1,n} \sum_{k=-\infty}^{n} \psi_{1,k} a_k + \phi_{2,n} \sum_{k=n}^{\infty} \psi_{2,k} a_k \right|^q u_n \right)^{\frac{1}{q}} \le C \left(\sum_{n\in\mathbb{Z}} |a_n|^p v_n \right)^{\frac{1}{p}} \quad (2.1)$$

holds for all non-negative sequences $\{a_n\} \in \ell^p(v_n)$ *if and only if*

$$B_1 := \sup_n \left(\sum_{k=-\infty}^{n} |\psi_{1,k}|^{p'} v_k^{1-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=n}^{\infty} |\phi_{1,k}|^q u_k\right)^{\frac{1}{q}} < \infty \tag{2.2}$$

and

$$B_2 := \sup_n \left(\sum_{k=n}^{\infty} |\psi_{2,k}|^{p'} v_k^{1-p'}\right)^{\frac{1}{p'}} \left(\sum_{k=-\infty}^{n} |\phi_{2,k}|^q u_k\right)^{\frac{1}{q}} < \infty\,. \tag{2.3}$$

PROOF. According to the triangle inequality for the weighted sequence space $\ell^q(u_n)$, we have

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} \left|\phi_{1,n} \sum_{k=-\infty}^{n} \psi_{1,k} a_k + \phi_{2,n} \sum_{k=n}^{\infty} \psi_{2,k} a_k\right|^q u_n\right)^{\frac{1}{q}} \\ \le \left(\sum_{n\in\mathbb{Z}} \left|\phi_{1,n} \sum_{k=-\infty}^{n} \psi_{1,k} a_k\right|^q u_n\right)^{\frac{1}{q}} + \left(\sum_{n\in\mathbb{Z}} \left|\phi_{2,n} \sum_{k=n}^{\infty} \psi_{2,k} a_k\right|^q u_n\right)^{\frac{1}{q}}.\end{aligned}$$

Consequently, we find that the inequality (2.1) holds if both the following inequalities hold:

$$\left(\sum_{n\in\mathbb{Z}} \left|\phi_{1,n} \sum_{k=-\infty}^{n} \psi_{1,k} a_k\right|^q u_n\right)^{\frac{1}{q}} \le C \left(\sum_{n\in\mathbb{Z}} |a_n|^p v_n\right)^{\frac{1}{p}} \tag{2.4}$$

$$\left(\sum_{n\in\mathbb{Z}} \left|\phi_{2,n} \sum_{k=n}^{\infty} \psi_{2,k} a_k\right|^q u_n\right)^{\frac{1}{q}} \le C \left(\sum_{n\in\mathbb{Z}} |a_n|^p v_n\right)^{\frac{1}{p}}. \tag{2.5}$$

By variable transformations $|\psi_{1,k} a_k| = b_k$, the inequality (2.4) can be compared with (1.1) (with different weight sequences) which holds if (2.2) holds. Similarly, the sufficiency of (2.3) for (2.5) can be argued. Hence the sufficiency follows.

For the necessity, take integers ℓ and m such that $0 < \ell < m$.
Define a sequence $\{a_n\}$ as follows:

$$a_n = \begin{cases} \left(\dfrac{v_{n_\varepsilon}}{|\psi_{1,n}|}\right)^{1-p'} \operatorname{sgn} \psi_{1,n}\,, & \ell \le n < m \\ 0, & \text{otherwise,} \end{cases}$$

where v_{n_ε} is the modified sequence weight defined as

$$v_{n_\varepsilon} = \max\{v_n, |\psi_{1,n}|^p \varepsilon\}.$$

For this sequence, the RHS of (2.1) becomes

$$C\left(\sum_{n=\ell}^{m-1} v_{n_\varepsilon}^{1-p'} |\psi_{1,n}|^{p'}\right)^{\frac{1}{p}}$$

which can be shown to be finite.

Also, the LHS can be estimated as follows:

$$\begin{aligned}
&\left(\sum_{n\in\mathbb{Z}}\left|\phi_{1,n}\sum_{k=-\infty}^{n}\psi_{1,k}a_k + \phi_{2,n}\sum_{k=n}^{\infty}\psi_{2,k}a_k\right|^q u_n\right)^{\frac{1}{q}}\\
&\quad\geq \left(\sum_{n=m-1}^{\infty}\left|\sum_{k=-\infty}^{n}\psi_{1,k}a_k\right|^q (u_n|\phi_{1,n}|^q)\right)^{\frac{1}{q}}\\
&\quad= \left(\sum_{n=m-1}^{\infty}\left|\sum_{k=\ell}^{m-1}\psi_{1,k}\left(\frac{v_{k_\varepsilon}}{|\psi_{1,k}|}\right)^{1-p'}\operatorname{sgn}\psi_{1,k}\right|^q (u_n|\phi_{1,n}|^q)\right)^{\frac{1}{q}}\\
&\quad\geq \left(\sum_{n=m-1}^{\infty}\left|\sum_{k=\ell}^{m-1}(v_{k_\varepsilon}^{1-p'}|\psi_{1,k}|^{p'})\right|^q (u_n|\phi_{1,n}|^q)\right)^{\frac{1}{q}}\\
&\quad= \left(\sum_{n=m-1}^{\infty}(u_n|\phi_{1,n}|^q)\right)^{\frac{1}{q}}\left(\sum_{k=\ell}^{m-1}(v_{k_\varepsilon}^{1-p'}|\psi_{1,k}|^{p'})\right).
\end{aligned}$$

Therefore, the inequality (2.1) becomes

$$\left(\sum_{n=m-1}^{\infty}u_n|\phi_{1,n}|^q\right)^{\frac{1}{q}}\left(\sum_{k=\ell}^{m-1}v_{k_\varepsilon}^{1-p'}|\psi_{1,k}|^{p'}\right)^{\frac{1}{p'}} \leq C.$$

Writing m for $m-1$, taking $\ell\to-\infty$, and $\varepsilon\to 0$ (via a subsequence) and taking supremum over all m, we find that (2.2) is necessary for the inequality (2.1).

Finally, take ℓ and m such that $\ell > m$. Define a sequence $\{a_n^*\}$ by

$$a_n^* = \begin{cases}\left(\dfrac{\hat{v}_{n_\varepsilon}}{|\psi_{2,n}|}\right)^{1-p'}\operatorname{sgn}\psi_{2,n}, & \ell\geq n> m\\[2ex] 0, & \text{otherwise},\end{cases}$$

where $\hat{v}_{n_\varepsilon}$ is the modified weight sequence defined as

$$\hat{v}_{n_\varepsilon} = \max\{v_n, |\psi_{2,n}|^p \varepsilon\} .$$

The necessity of (2.3), for the inequality (2.1), now, can be obtained by using the sequence $\{a_n^*\}$ in (2.1) and making similar arguments as above. $\square$

3 Inequalities in Weighted Amalgams.

In this section, we shall characterized the boundedness of the operator S defined in Section 1 between suitable weighted amalgams $\ell^q(\ell^p, u)$ and $L^{\bar{q}}(\ell^{\bar{q}}, v)$. Let us first recall the integral form of standard Hardy inequality.

Theorem A. *Let $1 < p, q < \infty$, $-\infty \le a < b \le \infty$, u, v be weight functions defined on (a, b). Then the inequality*

$$\left(\int_a^b \left(\int_a^x f(t)\,dt\right)^q u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_a^b f^p(x)v(x)\,dx\right)^{\frac{1}{p}}$$

holds for all measurable functions $f \ge 0$ if and only if

(i) *in case $p \le q$*

$$B_1 = \sup_{x\in(a,b)} \left(\int_x^b u(t)\,dt\right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t)\,dt\right)^{\frac{1}{p'}} < \infty$$

(ii) *in case $q < p$*

$$B_2 = \left(\int_a^b \left(\int_x^b u(t)\,dt\right)^{\frac{r}{q}} \left(\int_a^x v^{1-p'}(t)\,dt\right)^{\frac{r}{q'}} v^{1-p'}(x)\,dx\right)^{\frac{1}{r}} < \infty,$$

with $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p}$.

Denote

$$C_1 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} \left(\int_n^{n+1} u|\phi_1|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \times \left(\sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} \left(v|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right)^{\frac{1}{\bar{q}'}}, \tag{3.1}$$

$$C_2 = \sup_{m\in\mathbb{Z}} \sup_{0<\alpha<1} \left(\int_{n+\alpha}^{n+1} u|\phi_1|^p \right)^{\frac{1}{p}} \left(\int_n^{n+\alpha} \left(v|\psi_1|^{-\bar{p}'}\right)^{1-\bar{p}} \right)^{\frac{1}{\bar{p}'}}, \tag{3.2}$$

$$C_3 = \sup_{n\in\mathbb{Z}} \left[\left\{ \int_n^{n+1} \left(\int_t^{n+1} u|\phi_1|^p \right)^{\frac{s}{p}} \right\} \times \left\{ \int_n^t \left(v|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'} \right\}^{\frac{s}{\bar{p}'}} \left(v|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'} dt \right]^{\frac{1}{s}}, \tag{3.3}$$

$$C_4 = \left[\sum_{k\in\mathbb{Z}} \left\{ \sum_{n=k}^{\infty} \left(\int_n^{n+1} u|\phi_1|^p \right)^{\frac{q}{p}} \right\}^{\frac{r}{q}} \left\{ \sum_{n=-\infty}^{k} \left(\int_{n-1}^{n} \left(v|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{r}{\bar{q}'}} \times \left(\int_{k-1}^{k} \left(v|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right]^{\frac{1}{r}}, \tag{3.4}$$

$$C_5 = \sup_{m\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} u|\phi_2|^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \times \left\{ \sum_{n=m}^{\infty} \left(\int_n^{n+1} (v|\psi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{1}{\bar{q}'}} < \infty, \tag{3.5}$$

$$C_6 = \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \left(\int_n^{n+\alpha} u|\phi_2|^p \right)^{\frac{1}{p}} \left(\int_{n+\alpha}^{n+1} (v|\psi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{1}{\bar{p}'}}, \tag{3.6}$$

$$C_7 = \sup_{n\in\mathbb{Z}} \left[\int_n^{n+1} \left(\int_n^t u|\phi_2|^p \right)^{\frac{s}{p}} \left(\int_t^{n+1} (v|\psi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{s}{\bar{p}'}} (u|\phi_2|^p) dt \right]^{\frac{1}{s}}, \tag{3.7}$$

$$C_8 = \left[\sum_{k\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{k} \left(\int_n^{n+1} u|\phi_2|^p \right)^{\frac{q}{p}} \right\}^{\frac{r}{q}} \left\{ \sum_{n=k}^{\infty} \left(\int_{n-1}^{n} (v|\psi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{r}{\bar{q}'}} \times \left(\int_k^{k+1} u|\phi_2|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{r}}, \tag{3.8}$$

where $\dfrac{1}{s} = \dfrac{1}{p} - \dfrac{1}{\bar{p}}$ and $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{\bar{q}}$.

Now, we prove the boundedness of the operator $S = S_1 + S_2$.

Theorem 3.1. *Let $1 < p, \bar{p}, q, \bar{q} < \infty$ and u, v be weight functions. Then the inequality*

$$\|Sf\|_{p,u,q} \leq C\|f\|_{\bar{p},v,\bar{q}} \tag{3.9}$$

holds for all functions $f \in \ell^{\bar{q}}(L^{\bar{p}}, v)$ if and only if

(a) *in case $\bar{p} \leq p$ and $\bar{q} \leq q$,*

$$C_1 < \infty, \quad C_2 < \infty, \quad C_5 < \infty, \quad C_6 < \infty,$$

(b) *in case $p < \bar{p}$ and $\bar{q} \leq q$,*

$$C_1 < \infty, \quad C_3 < \infty, \quad C_5 < \infty, \quad C_7 < \infty,$$

(c) *in case $\bar{p} \leq p$ and $q < \bar{q}$,*

$$C_2 < \infty, \quad C_4 < \infty, \quad C_6 < \infty, \quad C_8 < \infty,$$

(d) *in case $p < \bar{p}$ and $q < \bar{q}$*

$$C_3 < \infty, \quad C_4 < \infty, \quad C_7 < \infty, \quad C_8 < \infty.$$

PROOF. We first prove the necessity. In case, $\phi_i, \psi_i, i = 1, 2$ are non-negative, then for all $f \geq 0$

$$\|S_i f\|_{p,u,q} \leq \|Sf\|_{p,u,q}$$

and consequently, both the inequalities

$$\|S_1 f\|_{p,u,q} \leq C\|f\|_{\bar{p},v,\bar{q}} \tag{3.10}$$

and

$$\|S_2 f\|_{p,u,q} \leq C\|f\|_{\bar{p},v,\bar{q}} \tag{3.11}$$

hold.

By writing $g = f \cdot |\psi_1|$, $\widetilde{u} = u|\phi_1|^p$ and $\widetilde{v} = v|\psi_1|^{-\bar{p}}$, we find that the inequality (3.10) reduces to

$$\|Hg\|_{p,\widetilde{u},q} \le C\|g\|_{\bar{p},\widetilde{v},\bar{q}}\,, \tag{3.12}$$

where $(Hg)(x) = \int_{-\infty}^{x} g(t)\,dt$ is the Hardy operator. Similarly, the inequality (3.11) reduces to

$$\|H^*h\|_{p,\widehat{u},q} \le C\|h\|_{\bar{p},\widehat{h},\bar{q}} \tag{3.13}$$

if we write $h = f \cdot |\psi_2|$, $\widehat{u} = u|\phi_2|^p$, $\widehat{v} = v|\psi_2|^{-\bar{p}}$, where $(H^*h)(x) = \int_x^\infty h(t)\,dt$ is the adjoint Hardy operator.

But the inequality (3.12) and (3.13) are the ones considered by Carton-Lebrun, Heining and Hofmann [1]. Thus the necessity follows in the case when ϕ_i, ψ_i are non-negative.

Now assume that the functions ϕ_i, ψ_i are general functions. Suppose that the inequality (3.9) holds. For a given $\varepsilon > 0$ and a measurable function $g > 0$, define a new weight function v_ε as

$$v_\varepsilon(x) = \max\left\{v(x), \frac{|\psi_1(x)|^{\bar{p}}}{(g(x))^{\bar{p}}}\varepsilon\right\}. \tag{3.14}$$

Then $v(x) \le v_\varepsilon(x)$ and therefore

$$\|Sf\|_{p,u,q} \le C\|f\|_{\bar{p},v,\bar{q}} \le C\|f\|_{\bar{p},v_\varepsilon,\bar{q}}\,. \tag{3.15}$$

Now fix $m \in \mathbb{Z}$, take α, β such that $m < \alpha < \beta < m+1$ and define the function f by

$$f(x) = g(x)|\psi_1(x)|^{-1}\operatorname{sgn}(\psi_1(x))\chi_{[\alpha,\beta]}(x)\,. \tag{3.16}$$

We find that for f and v_ε defined above

$$\begin{aligned}
\|f\|_{\bar{p},v_\varepsilon,\bar{q}} &= \left(\int_m^{m+1} |f|^{\bar{p}} v_\varepsilon\right)^{\frac{1}{\bar{p}}} \\
&= \left(\int_\alpha^\beta (g(x))^{\bar{p}}|\psi_1(x)|^{-\bar{p}} v_\varepsilon(x)\,dx\right)^{\frac{1}{\bar{p}}} \\
&\le \left(\int_\alpha^\beta (g(x))^{\bar{p}}|\psi_1(x)|^{-\bar{p}}|\psi_2(x)|^{\bar{p}}(g(x))^{-\bar{p}}\varepsilon\,dx\right)^{\frac{1}{\bar{p}}} \\
&= (\varepsilon(\beta-\alpha))^{\frac{1}{\bar{p}}} < \infty
\end{aligned} \tag{3.17}$$

so that $f \in \ell^{\bar{q}}(L^{\bar{p}}, v_\varepsilon)$ and in view of (3.15) $f \in \ell^{\bar{q}}(L^{\bar{p}}, v)$ as well. Also

$$\begin{aligned}
\|Sf\|_{p,u,q} &= \left(\int_m^{m+1} |(Sf)(x)|^p u(x)\, dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_\beta^{m+1} \left|\phi_1(x)\int_{-\infty}^x \psi_1(t)f(t)\, dt + \phi_2(x)\int_x^\infty \psi_2(t)f(t)\, dt\right|^p u(x)\, dx\right)^{\frac{1}{p}} \\
&= \left(\int_\beta^{m+1} \left|\phi_1(x)\int_{-\infty}^x \psi_1(t)f(t)\, dt\right|^p u(x)\, dx\right)^{\frac{1}{p}} \\
&\geq \left(\int_\beta^{m+1} \left|\int_m^x \psi_1(t)f(t)\, dt\right|^p (u(x)|\phi_1(x)|^p)\, dx\right)^{\frac{1}{p}} \\
&= \left(\int_\beta^{m+1} \left(\int_m^x g(t)\, dt\right)^p (u(x)|\phi_1(x)|^p)\, dx\right)^{\frac{1}{p}}. \qquad (3.18)
\end{aligned}$$

Now, since the inequality (3.15) holds for all $f \in \ell^{\bar{q}}(L^{\bar{q}}, v)$, by putting the function defined by (3.16) in it and using (3.17) and (3.18), we obtain

$$\left(\int_\beta^{m+1} \left(\int_m^x g\right)^p (u(x)|\phi_1(x)|^p)\, dx\right)^{\frac{1}{p}} \leq C\left(\int_m^{m+1} (g(x))^{\bar{p}} v_\varepsilon(x)|\psi_1(x)|^{-\bar{p}}\, dx\right)^{\frac{1}{\bar{p}}}$$

or

$$\left(\int_\beta^{m+1} \left(\int_m^x g\right)^p (u|\phi_1|^p)\, dx\right)^{\frac{1}{p}} \left(\int_m^{m+1} g^{\bar{p}}(v_\varepsilon|\psi_1|^{-\bar{p}})\, dx\right)^{\frac{-1}{\bar{p}}} \leq C.$$

Since C is independent of β and ε, we have when $\beta \to m$ and $\varepsilon \to 0$ (via a subsequence) that

$$\left(\int_m^{m+1} \left(\int_m^x g\right)^p (u|\phi_1|^p)\, dx\right)^{\frac{1}{p}} \left(\int_m^{m+1} g^{\bar{p}}(v|\psi_1|^{-\bar{p}})\, dx\right)^{\frac{-1}{\bar{p}}} \leq C$$

or

$$\left(\int_m^{m+1} \left(\int_m^x g\right)^p (u|\phi_1|^p)\, dx\right)^{\frac{1}{p}} \leq C\left(\int_m^{m+1} g^{\bar{p}}(v|\psi_1|^{-\bar{p}})\, dx\right)^{\frac{1}{\bar{p}}}.$$

Thus, we find that if the inequality (3.9) holds, then the last inequality holds for each $m \in \mathbb{Z}$ with $C > 0$ independent of m and by Theorem A, this holds in case $\bar{p} \leq p$ if $C_2 < \infty$ and in case $\bar{p} > p$ if $C_3 < \infty$.

On the similar lines, the necessity of $C_6 < \infty$ and $C_7 < \infty$ can be obtained if v_ε and f defined in (3.14) and (3.16) are replaced, respectively, by

$$v_\varepsilon(x) = \max\left\{v(x), \frac{|\psi_2(x)|^{\bar{p}}}{(g(x))^{\bar{p}}}\varepsilon\right\}$$

and

$$f(x) = g(x)|\psi_2(x)|^{-1}\operatorname{sgn}(\psi_2(x))\chi_{[\alpha,\beta]}(x),$$

where $m \in \mathbb{Z}$ is fixed and $m < \alpha < \beta < m+1$.

Note that all the conditions obtained so far are necessary for all choices of $q, \bar{q} \in (1,\infty)$. We now proceed to obtain necessary conditions which depend upon the choice of q and $\bar{q}$.

For a given $\varepsilon > 0$, now define $\hat{v}_\varepsilon$ as

$$\hat{v}_\varepsilon(x) = \max\left\{v(x), |\psi_1(x)|^{\bar{p}}\varepsilon\right\} \tag{3.19}$$

and as a result, the inequality

$$\|Sf\|_{p,u,q} \le C\|f\|_{\bar{p},\hat{v}_\varepsilon,\bar{q}} \tag{3.20}$$

holds.

For a fixed $\ell \in \mathbb{Z}$ and any non-negative sequence $\{a_n\} \in \ell^{\bar{q}}$, we define

$$f(x) = \begin{cases} \sum\limits_{n\in\mathbb{Z}} a_n \left(\dfrac{\hat{v}_\varepsilon(x)}{|\psi_1(x)|}\right)^{1-\bar{p}'} \operatorname{sgn}(\psi_1(x))\chi_{[n,n+1]}(x), & x \le \ell \\ 0, & x > \ell\,. \end{cases} \tag{3.21}$$

With this function, we find that

$$\begin{aligned} \|f(x)\|_{\bar{p},\hat{v}_\varepsilon,\bar{q}} &= \left\{\sum_{n=-\infty}^{\ell}\left(\int_{n-1}^{n} a_{n-1}^{\bar{p}}\frac{\hat{v}_\varepsilon^{\bar{p}(1-\bar{p}')}}{|\psi_1|^{\bar{p}(1-\bar{p}')}}\hat{v}_\varepsilon\right)^{\bar{q}\left(1-\frac{1}{\bar{p}'}\right)}\right\}^{\frac{1}{\bar{q}}} \\ &= \left\{\sum_{n=-\infty}^{\ell} A_n^{\bar{q}} V_n\right\}^{\frac{1}{\bar{q}}}, \end{aligned} \tag{3.22}$$

where $A_n = a_{n-1}\int_{n-1}^{n}\left(\hat{v}_\varepsilon|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'}$ and $V_n = \left(\int_{n-1}^{n}\left(\hat{v}_\varepsilon|\psi_1|^{-\bar{p}}\right)^{1-\bar{p}'}\right)^{\frac{-\bar{q}}{\bar{p}'}}$.

Further, keeping in mind that f defined in (3.21) vanishes for $x > \ell$, we get for $\ell < x < \ell + 1$

$$\begin{aligned}
(Sf)(x) &= \phi_1(x)\int_{-\infty}^{x}\psi_1(t)f(t)\,dt + \phi_2(x)\int_{x}^{\infty}\psi_2(t)f(t)\,dt \\
&= \phi_1(x)\int_{-\infty}^{\ell}\psi(t)f(t)\,dt + \phi_1(x)\int_{\ell}^{x}\psi_1(t)f(t)\,dt \\
&\quad +\phi_2(x)\int_{x}^{\ell+1}\phi_2(t)f(t)\,dt + \phi_2(x)\int_{\ell+1}^{\infty}\psi_2(t)f(t)\,dt \\
&= \phi_1(x)\int_{-\infty}^{\ell}\psi_1(t)f(t)\,dt \\
&= \phi_1(x)\left(\sum_{n=-\infty}^{\ell}\int_{n-1}^{n}\psi_1(t)f(t)\,dt\right)
\end{aligned}$$

so that

$$\begin{aligned}
|(Sf)(x)|^p &= |\phi_1(x)|^p\left|\sum_{n=-\infty}^{\ell}a_{n-1}\int_{n-1}^{n}(\hat{v}_\varepsilon(t)|\psi_1(t)|^{-\bar{p}})^{1-\bar{p}'}\right|^p \\
&= |\phi_1(x)|^p\left|\sum_{n=-\infty}^{\ell}A_n\right|^p
\end{aligned}$$

which gives

$$\left(\int_{\ell}^{\ell+1}|(Sf)(x)|^p u(x)\,dx\right)^{\frac{1}{p}} = \left(\int_{\ell}^{\ell+1}u(x)|\phi_1(x)|^p\,dx\right)^{\frac{1}{p}}\left(\sum_{n=-\infty}^{\ell}A_n\right)$$

for each $\ell \in \mathbb{Z}$, and consequently

$$\|Sf\|_{p,u,q} = \left\{\sum_{\ell\in\mathbb{Z}}U_\ell^{\frac{q}{p}}\left(\sum_{n=-\infty}^{\ell}A_n\right)^q\right\}^{\frac{1}{q}} \tag{3.23}$$

holds, where $U_\ell = \int_\ell^{\ell+1}u(x)|\phi_1(x)|^p\,dx$.

Now, by substituting the function f from (3.21) in (3.20) and using (3.22) and (3.23), we have

$$\left\{\sum_{\ell\in\mathbb{Z}}U_\ell^{\frac{q}{p}}\left(\sum_{n=-\infty}^{\ell}A_n\right)^q\right\}^{\frac{1}{q}} \le C\left(\sum_{n=-\infty}^{\ell}A_n^{\bar{q}}V_n\right)^{\frac{1}{\bar{q}}}.$$

Note that

$$\left(\sum_{n=-\infty}^{\ell} A_n^{\bar q} V_n\right)^{\frac{1}{\bar q}} \le \left(\sum_{n\in\mathbb{Z}} A_n^{\bar q} V_n\right)^{\frac{1}{\bar q}} \tag{3.24}$$

and that the RHS of the above estimate is finite. Indeed

$$\begin{aligned}\left(\sum_{n\in\mathbb{Z}} A_n^{\bar q} V_n\right)^{\frac{1}{\bar q}} &\le \left\{\sum_{n\in\mathbb{Z}} a_{n-1}^{\bar q} (\varepsilon^{1-\bar p'})^{\frac{\bar q}{\bar p}}\right\}^{\frac{1}{\bar q}} \\ &= (\varepsilon^{1-\bar p'})^{\frac{1}{\bar p}} \left(\sum_{n\in\mathbb{Z}} a_{n-1}^{\bar q}\right)^{\frac{1}{\bar q}} \\ &< \infty\end{aligned}$$

since $\{a_n\} \in \ell^{\bar q}$. Consequently (3.20), (3.22), (3.23) and (3.24) gives

$$\left\{\sum_{\ell\in\mathbb{Z}} U_\ell^{\frac{q}{p}} \left(\sum_{n=-\infty}^{\ell} A_n\right)^q\right\}^{\frac{1}{q}} \le C \left\{\sum_{n\in\mathbb{Z}} A_n^{\bar q} V_n\right\}^{\frac{1}{\bar q}}$$

which can be compared with the inequality (1.1) which holds if and only if

$$\hat C_1 = \sup_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} U_n^{\frac{q}{p}}\right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^{m} V_n^{1-\bar q'}\right)^{\frac{1}{\bar q'}} < \infty$$

for all $\bar q \le q$ and

$$\hat C_4 = \left\{\sum_{m\in\mathbb{Z}} \left(\sum_{n=m}^{\infty} U_n^{\frac{q}{p}}\right)^{\frac{r}{q}} \left(\sum_{n=-\infty}^{m} V_n^{1-\bar q'}\right)^{\frac{r}{\bar q'}} V_m^{1-\bar q'}\right\}^{\frac{1}{r}} < \infty$$

for all $q < \bar q$ where $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{\bar q}$. Now, taking $\varepsilon \to 0$, we find that $\hat C_1 = C_1$ and $\hat C_4 = C_4$ and therefore the necessity of $C_1 < \infty$ and $C_4 < \infty$ for, respectively, $\bar q \le q$ and $q < \bar q$ follows.

On the similar lines, the necessity of $C_5 < \infty$ and $C_8 < \infty$ can be obtained if $\hat v_\varepsilon$ and f defined in (3.19) and (3.21) are replaced, respectively, by

$$\hat v_\varepsilon(x) = \max\left\{v(x), |\psi_2(x)|^{\bar p}\varepsilon\right\}$$

and

$$f(x) = \begin{cases} \sum_{n\in\mathbb{Z}} a_n \left(\frac{\hat{v}_\varepsilon(x)}{|\psi_2(x)|}\right)^{1-\bar{p}'} \operatorname{sgn}(\psi_2(x))\chi_{[n,n+1]}(x), & x \geq \ell \\ 0, & x < \ell \end{cases}$$

where $\ell \in \mathbb{Z}$ is fixed.

Finally, to prove the sufficiency, we note by an application of the triangle inequality that

$$\|Sf\|_{p,u,q} \leq \|S_1 f\|_{p,u,q} + \|S_2 f\|_{p,u,q}\,.$$

Therefore the inequality (3.9) holds if both the inequalities

$$\|S_1 f\|_{p,u,q} \leq C\|f\|_{\bar{p},v,\bar{q}} \tag{3.25}$$

and

$$\|S_2 f\|_{p,u,q} \leq C\|f\|_{\bar{p},v,\bar{q}} \tag{3.26}$$

hold. Now writing $g = f\cdot|\psi_1|$, $\tilde{u} = u|\phi_1|^p$ and $\tilde{v} = v|\psi_1|^{-\bar{p}}$ we find that (3.25) becomes

$$\|Hg\|_{p,\widetilde{u},q} \leq C\|g\|_{\bar{p},\widetilde{v},\bar{q}} \tag{3.27}$$

and with $h = f\cdot|\psi_2|$, $\widehat{u} = u|\phi_2|^p$, $\widehat{v} = v|\psi_2|^{-\bar{p}}$, (3.26) becomes

$$\|H^* h\|_{p,\widehat{u},q} \leq C\|h\|_{\bar{p},\widehat{v},\bar{q}}\,. \tag{3.28}$$

Consequently, the sufficiency of both (3.27) and (3.28) is the sufficiency for the inequality (3.9). But (3.27) and (3.28) are precisely the ones characterized by Carton-Lebrun, Heinig and Hofmann [1] and the assertion follows.

□

Remark. It is natural to characterize the boundedness of the conjugate operator $S^* = S_1^* + S_2^*$, where

$$\begin{aligned} (S_1^* f)(x) &= \psi_1(x)\int_x^\infty \phi_1(t)f(t)\,dt\,,\\ (S_2^* f)(x) &= \psi_2(x)\int_{-\infty}^x \phi_2(t)f(t)\,dt\,. \end{aligned}$$

Note that the dual spaces of $\ell^q(L^p,u)$ and $\ell^{\bar{q}}(L^{\bar{p}},v)$ are, respectively, $\ell^{q'}(L^{p'},u^{1-p'})$ and $\ell^{\bar{q}'}(L^{\bar{p}'},v^{1-\bar{p}'})$.

Let us use the following notations

$$C_1^* = \sup_{m\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} (u|\psi_1|^p) \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \left\{ \sum_{n=m}^{\infty} \left(\int_n^{n+1} (v|\phi_1|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{1}{\bar{q}'}},$$

$$C_2^* = \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \left(\int_n^{n+\alpha} u|\psi_1|^p \right)^{\frac{1}{p}} \left(\int_{n+\alpha}^{n+1} (v|\phi_1|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{1}{\bar{p}'}},$$

$$C_3^* = \sup_{n\in\mathbb{Z}} \left[\int_n^{n+1} \left(\int_n^t u|\psi_1|^p \right)^{\frac{s^*}{p}} \left(\int_t^{n+1} (v|\phi_1|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{s^*}{\bar{p}'}} |\psi_1|^p u dt \right]^{\frac{1}{s^*}},$$

$$C_4^* = \left[\sum_{k\in\mathbb{Z}} \left\{ \sum_{n=k}^{\infty} \left(\int_{n-1}^{n} (v|\phi_1|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{r^*}{\bar{q}'}} \right.$$
$$\left. \times \left\{ \sum_{n=-\infty}^{k} \left(\int_n^{n+1} u|\psi_1|^p \right)^{\frac{q}{p}} \right\}^{\frac{r^*}{q}} \left(\int_k^{k+1} u|\psi_1|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{r^*}},$$

$$C_5^* = \sup_{m\in\mathbb{Z}} \left\{ \sum_{n=m}^{\infty} \left(\int_n^{n+1} u|\psi_2|^p \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \left\{ \sum_{n=-\infty}^{m-1} \left(\int_n^{n+1} (v|\phi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{1}{\bar{q}'}},$$

$$C_6^* = \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \left(\int_{n+\alpha}^{n+1} u|\psi_2|^p \right)^{\frac{1}{p}} \left(\int_n^{n+\alpha} (v|\phi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{1}{\bar{p}'}},$$

$$C_7^* = \sup_{n\in\mathbb{Z}} \left[\int_n^{n+1} \left(\int_t^{n+1} u|\psi_2|^p \right)^{\frac{s^*}{p}} \left(\int_n^t (v|\phi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{s^*}{\bar{p}'}} (|\phi_2|^{-\bar{p}} v)^{1-\bar{p}'} dt \right]^{\frac{1}{s^*}},$$

and

$$C_8^* = \left[\sum_{k\in\mathbb{Z}} \left\{ \sum_{n=-\infty}^{k} \left(\int_{n-1}^{n} (v|\phi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right\}^{\frac{r^*}{\bar{q}'}} \right.$$
$$\left. \times \left\{ \sum_{n=k}^{\infty} \left(\int_n^{n+1} u|\psi_2|^p \right)^{\frac{q}{p}} \right\}^{\frac{r^*}{q}} \left(\int_{k-1}^{k} (v|\phi_2|^{-\bar{p}})^{1-\bar{p}'} \right)^{\frac{\bar{q}'}{\bar{p}'}} \right]^{\frac{1}{r^*}},$$

where $\dfrac{1}{s^*} = \dfrac{1}{\bar{p}'} - \dfrac{1}{p}$ and $\dfrac{1}{r^*} = \dfrac{1}{\bar{q}'} - \dfrac{1}{q'}$.

Now, we have the following

Theorem 3.2. *Let* $1 < p, \bar{p}, q\ ,\ \bar{q} < \infty$ *and* u, v *be weight functions. Then the inequality*

$$\|S^* f\|_{p,u,q} \leq C \|f\|_{\bar{p},v,\bar{q}}$$

holds for all measurable functions $f \in \ell^{\bar{q}}(L^{\bar{p}}, v)$ *if and only if*

(a) *in case* $\bar{p} \leq p$ *and* $\bar{q} \leq q$

$$C_1^* < \infty, \quad C_2^* < \infty, \quad C_5^* < \infty, \quad C_6^* < \infty,$$

(b) *in case* $p < \bar{p}$ *and* $\bar{q} \leq q$

$$C_1^* < \infty, \quad C_3^* < \infty, \quad C_5^* < \infty, \quad C_7^* < \infty,$$

(c) *in case* $\bar{p} \leq p$ *and* $q < \bar{q}$

$$C_2^* < \infty, \quad C_4^* < \infty, \quad C_6^* < \infty, \quad C_8^* < \infty,$$

(d) *in case* $p < \bar{p}$ *and* $q < \bar{q}$

$$C_3^* < \infty, \quad C_4^* < \infty, \quad C_7^* < \infty, \quad C_8^* < \infty.$$

PROOF. It is known that the operator

$$S : \ell^{\bar{q}}(L^{\bar{p}}, v) \to L^q(L^p, u)$$

is bounded if and only if the operator

$$S^* : \ell^{q'}(L^{p'}, u^{1-p'}) \to \ell^{\bar{q}'}(L^{\bar{p}'}, v^{1-\bar{p}'})$$

is bounded. The result now follows from Theorem 3.1 if we replace $p, \bar{p}, q, \bar{q}, u, v$ by, respectively, $\bar{p}', p', \bar{q}', q', v^{1-\bar{p}'}, u^{1-p'}$. $\square$

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