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ON ℓ^P-LIKE EQUIVALENCE RELATIONS

Abstract

For $f\colon [0,1] \to \mathbb{R}^+$, consider the relation $\mathbf{E}_f$ on $[0,1]^\omega$ defined by $(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n<\omega} f(|y_n - x_n|) < \infty$. We study the Borel reducibility of Borel equivalence relations of the form $\mathbf{E}_f$. Our results indicate that for every $1 \le p < q < \infty$, the order $\le_B$ of Borel reducibility on the set of equivalence relations $\{\mathbf{E}\colon \mathbf{E}_{\mathrm{Id}^p} \le_B \mathbf{E} \le_B \mathbf{E}_{\mathrm{Id}^q}\}$ is more complicated than expected, e.g. consistently every linear order of cardinality continuum embeds into it.

1 Introduction.

Let $f\colon [0,1] \to \mathbb{R}^+$ be an arbitrary function and consider the relation $\mathbf{E}_f$ on $[0,1]^\omega$ defined by setting, for every $(x_n)_{n<\omega}, (y_n)_{n<\omega} \in [0,1]^\omega$,

$$(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n<\omega} f(|y_n - x_n|) < \infty. \tag{1}$$

Several natural questions arise, e.g.

(i) when is $\mathbf{E}_f$ an equivalence relation?

(ii) which equivalence relations can be obtained in the form $\mathbf{E}_f$?

(iii) for what $f, g\colon [0,1] \to [0,1]$ is $\mathbf{E}_f$ Borel reducible to $\mathbf{E}_g$?

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In the present paper we answer (i), we initiate a study of (ii) and we obtain various conditions for (iii).

The prototypes of equivalence relations of the form $\mathbf{E}_f$ are induced by the Banach spaces ℓ^p $(1 \le p < \infty)$; i.e. they are defined by the functions $f = \mathrm{Id}^p$ for $1 \le p < \infty$, where $\mathrm{Id}\colon [0,1] \to [0,1]$ is the identity function. The Borel reducibility among these equivalence relations is fully described by a classical result of R. Dougherty and G. Hjorth [3, Theorem 1.1 p. 1836 and Theorem 2.2 p. 1840] stating that for every $1 \le p, q < \infty$,

$$\mathbf{E}_{\mathrm{Id}^p} \le_B \mathbf{E}_{\mathrm{Id}^q} \Leftrightarrow p \le q. \tag{2}$$

We note, however, that e.g. for the function $f(0) = 0$, $f(x) = 1$ $(0 < x \le 1)$ we have $\mathbf{E}_f$ is the equivalence relation of eventual equality on $[0,1]^\omega$, also denoted by E_1 in the literature; that is, the investigation of equivalence relations of the form $\mathbf{E}_f$ concerns equivalence relations which are not necessarily reducible to $\mathbf{E}_{\mathrm{Id}^p}$ for some $1 \le p < \infty$.

Our investigations were motivated by a question of S. Gao in [4] p. 74, asking whether for $1 \le p < \infty$, $\mathbf{E}_{\mathrm{Id}^p}$ is the greatest lower bound of $\{\mathbf{E}_{\mathrm{Id}^q} : p < q < \infty\}$; we note that formally the question in [4] p. 74 refers to equivalence relations on $\mathbb{R}^\omega$, but as we will see later in Lemma 2.3, the two formulations are equivalent. We answer this question in the negative by showing, for fixed $1 \le p < \infty$, that $\mathbf{E}_{\mathrm{Id}^p} <_B \mathbf{E}_f <_B \mathbf{E}_{\mathrm{Id}^q}$ for every $q > p$ whenever

$$\lim_{x \to +0} \frac{f(x)}{x^p} = 0 \text{ and } \lim_{x \to +0} \frac{f(x)}{x^q} = \infty \ (p < q < \infty), \tag{3}$$

and f satisfies some additional technical assumptions (see e.g. Corollary 5.4). However, toward this result we aim to carry out a general study of the relations $\mathbf{E}_f$ and their Borel reducibility. To this end, in Section 2 we characterize the functions for which $\mathbf{E}_f$ is an equivalence relation and, roughly speaking, we show that f is continuous if and only if $E_1 \not\le_B \mathbf{E}_f$. In Section 3 and in Section 4 we prove general reducibility and nonreducibility results for equivalence relations of the form $\mathbf{E}_f$. The results of these sections heavily build on techniques developed in [3]. Finally, in Section 5 we conclude our investigations by applying the technical results of the previous sections to concrete functions; in particular, we answer the above mentioned question of S. Gao, and we show that for $1 \le p < q < \infty$, every linear order which embeds into $(\mathcal{P}(\omega)/\mathrm{fin}, \subset)$ also embeds into the set of equivalence relations $\{\mathbf{E}_f : \mathbf{E}_{\mathrm{Id}^p} \le_B \mathbf{E}_f \le_B \mathbf{E}_{\mathrm{Id}^q}\}$ ordered by $<_B$.

Our results produce just examples. We are far from giving a full description of the Borel equivalence relations of the form $\mathbf{E}_f$ or a complete picture of the Borel reducibility relation among the $\mathbf{E}_f$'s. In particular, it remains open whether there are two functions f and g such that $\mathbf{E}_f$ and $\mathbf{E}_g$ are incomparable under $\le_B$. Nevertheless, we have one qualitative observation. Conditions (2) and (3) may suggest that reducibility among the $\mathbf{E}_f$'s is essentially governed by the growth order of the f's. However, this is far from being true. As we will see in Section 3, under mild additional assumptions on f we have e.g. $\mathbf{E}_{\mathrm{Id}^p} \le_B \mathbf{E}_f$ whenever $\lim_{x\to+0} f(x)/x^{p-\varepsilon} = 0$ for every $\varepsilon > 0$ (for the precise statement, see Theorem 3.3); this is in contrast with (3).

For basic terminology in descriptive set theory we refer to [5]. As above, if X and Y are Polish spaces, E and F are equivalence relations on X and Y, then we say *E is Borel reducible to F*, $E \le_B F$ in notation, if there exists a Borel function $\vartheta\colon X \to Y$ satisfying

$$xEx' \Leftrightarrow \vartheta(x)F\vartheta(x').$$

We say *E and F are Borel equivalent* if $E \le_B F$ and $F \le_B E$, while we write $E <_B F$ if $E \le_B F$ but $F \not\le_B E$.

Depending on the context, $|\cdot|$ denotes the absolute value of a real number, the length of a sequence or the cardinality of a set; $\lfloor\cdot\rfloor$ and $\{\cdot\}$ stand for lower integer part and fractional part. We denote by $\mathbb{Z}$ and $\mathbb{R}^+$ the set of integers and nonnegative reals.

2 Basic Properties.

Definition 2.1. Let $(G,+)$ be an Abelian group and let $H \subseteq G$ satisfy

(H_1) $0 \in H$;

(H_2) for every $x, y \in H$, $x - y \in H$ or $y - x \in H$;

(H_3) for every $x, y, z \in H$, $x - y \in H$ and $y - z \in H$ implies $x - z \in H$.

For every $x \in H \cup -H$, let $x^+ = x$ if $x \in H$ and $x^+ = -x$ if $x \in -H \setminus H$.

For every function $f\colon H \to \mathbb{R}^+$, we define the relation $\mathbf{E}_f$ on H^ω by setting, for every $(x_n)_{n<\omega}, (y_n)_{n<\omega} \in H^\omega$,

$$(x_n)\mathbf{E}_f(y_n) \Leftrightarrow \sum_{n<\omega} f((y_n - x_n)^+) < \infty; \tag{4}$$

the definition is valid by (H_2).

We say $f\colon H \to \mathbb{R}^+$ is *even* if for every $x \in H \cap -H$, $f(x) = f(-x)$.

Observe that for $\tilde{f}\colon H \to [0,1]$, $\tilde{f}(x) = \min\{f(x), 1\}$ $(x \in H)$ we have $\mathbf{E}_{\tilde{f}} = \mathbf{E}_f$. So in the sequel we only consider bounded functions.

We start this section by characterizing the bounded functions $f\colon H \to \mathbb{R}^+$ for which $\mathbf{E}_f$ is an equivalence relation. To avoid a meticulous bookkeeping of non-relevant constants, we will use the terminology "by $(\star)$, $A \lesssim B$" to abbreviate that "by property $(\star)$, there is a constant $C > 0$ depending on the parameters of $(\star)$ such that $A \leq CB$". The relations $\gtrsim$ and $\approx$ are defined analogously.

Proposition 2.2. *Let $f\colon H \to \mathbb{R}^+$ be a bounded even function. Let $\mathbf{E}_f$ be the relation on H^ω defined by (4). Then $\mathbf{E}_f$ is an equivalence relation if and only if the following conditions hold:*

(R_1) $f(0) = 0$;

(R_2) there is a $C \geq 1$ such that for every $x, y \in H$ with $x + y \in H$,

$$(a) \quad f(x+y) \leq C(f(x) + f(y)),$$

$$(b) \quad f(x) \leq C(f(x+y) + f(y)).$$

PROOF. Since f is even, $\mathbf{E}_f$ is symmetric. It is obvious that (R_1) is equivalent to $\mathbf{E}_f$ being reflexive, so it remains to show that (R_2) is equivalent to transitivity.

Suppose first (R_2) holds and let $(x_n)_{n<\omega}$, $(y_n)_{n<\omega}$, $(z_n)_{n<\omega} \in H^\omega$ such that $(x_n)\mathbf{E}_f(y_n)$ and $(y_n)\mathbf{E}_f(z_n)$. Let $n < \omega$ be fixed. Since the role of x_n and z_n is symmetric, by (H_2) we can assume $z_n - x_n \in H$. We distinguish several cases.

If $x_n - y_n \in H$, then by (H_3), $z_n - y_n \in H$ so by (R_2b) using $(z_n - y_n) = (z_n - x_n) + (x_n - y_n)$,

$$f(z_n - x_n) \lesssim f(z_n - y_n) + f((y_n - x_n)^+).$$

If $y_n - x_n \in H$, then either $z_n - y_n \in H$, hence by (R_2a), using $(z_n - x_n) = (z_n - y_n) + (y_n - x_n)$,

$$f(z_n - x_n) \lesssim f(z_n - y_n) + f(y_n - x_n);$$

or $y_n - z_n \in H$ hence by (R_2b) using $(y_n - z_n) + (z_n - x_n) = (y_n - x_n)$,

$$f(z_n - x_n) \lesssim f((z_n - y_n)^+) + f(y_n - x_n).$$

Thus

$$\sum_{n<\omega} f((z_n - x_n)^+) \lesssim \sum_{n<\omega} f((z_n - y_n)^+) + \sum_{n<\omega} f((y_n - x_n)^+) < \infty,$$

which gives $(x_n)\mathbf{E}_f(z_n)$; i.e. (R_2) implies transitivity.

To see the other direction, suppose first there is no $C \geq 1$ for which (R_2a) holds; i.e. for every $n < \omega$ there are $\xi_n, \eta_n \in H$ such that $\xi_n + \eta_n \in H$ and

$$f(\xi_n + \eta_n) > 2^n(f(\xi_n) + f(\eta_n)).$$

Set $k_n = \max\{1, \lfloor 1/f(\xi_n + \eta_n)\rfloor\}$; if $B \geq 1$ is an upper bound of f, we have

$$B \geq k_n f(\xi_n + \eta_n) \geq \frac{1}{2} \text{ and } B > 2^n k_n(f(\xi_n) + f(\eta_n)). \tag{5}$$

Let $(x_m)_{m<\omega} \in H^\omega$ be the sequence which, for every $n < \omega$, admits the value ξ_n with multiplicity k_n; and define the sequence $(y_m)_{m<\omega} \in H^\omega$ to admit η_n exactly there where $(x_m)_{m<\omega}$ admits ξ_n $(n < \omega)$. Then by (5),

$$\sum_{m<\omega} f(x_m) < 2B \text{ and } \sum_{m<\omega} f(y_m) < 2B;$$

i.e. if $\underline{0}$ denotes the constant zero sequence, we have $\underline{0}\mathbf{E}_f(x_m)$ and $(x_m)\mathbf{E}_f(x_m + y_m)$. Also by (5),

$$\sum_{m<\omega} f(x_m + y_m) = \infty;$$

i.e. $\underline{0}\not\mathbf{E}_f(x_m + y_m)$, which shows transitivity fails.

Finally suppose there is no $C \geq 1$ for which (R_2b) holds; i.e. for every $n < \omega$ there are $\xi_n, \eta_n \in H$ such that

$$f(\xi_n) > 2^n(f(\xi_n + \eta_n) + f(\eta_n)).$$

Set $k_n = \max\{1, \lfloor 1/f(\xi_n)\rfloor\}$ and let $(x_m)_{m<\omega}$, $(y_m)_{m<\omega}$ be as above. Then $(y_m)\mathbf{E}_f\underline{0}$ and $\underline{0}\mathbf{E}_f(x_m + y_m)$ but $(y_n)\not\mathbf{E}_f(x_m + y_m)$. □

If f is an arbitrary function, thus $\mathbf{E}_f$ is not necessarily an equivalence relation, then one could consider the equivalence relation generated by $\mathbf{E}_f$. However, it is very hard to control the properties of this generated equivalence relation by the properties of f, in particular we do not know how to ensure $\mathbf{E}_f$ is Borel. Therefore, from now on, we restrict our attention to such functions f for which $\mathbf{E}_f$ is an equivalence relation.

Despite the general setting of Definition 2.1 and Proposition 2, in the present paper we will work only with two special cases. At some point, we will set $G = H$ to be the circle group $\mathbb{S} = [0, 1)$ with mod 1 addition. Then (H_1)-(H_3) obviously hold, $x^+ = x$ $(x \in \mathbb{S})$, moreover (R_2a) and (R_2b) are equivalent. But mainly we will work with $G = \mathbb{R}$ and $H = [0, 1]$; then (H_1)-(H_3) hold and $x^+ = |x|$. Our reason for working with functions f defined on

$[0,1]$ instead of $\mathbb{R}$ is that on a smaller domain it is easier to define f such that it satisfies (R_1) and (R_2). Next we show that for $\mathbf{E}_{\mathrm{Id}^p}$, this change of domain makes no difference.

Lemma 2.3. *For $1 \leq p < \infty$, let ℓ^p denote the equivalence relation defined by (4) with $f\colon \mathbb{R} \to \mathbb{R}^+$, $f(x) = |x|^p$ $(x \in \mathbb{R})$. Then ℓ^p and $\ell^p|_{[0,1]^\omega \times [0,1]^\omega}$ are Borel equivalent.*

PROOF. It is obvious that $\ell^p|_{[0,1]^\omega \times [0,1]^\omega} \leq_B \ell^p$. To see the other direction, for every $k \in \mathbb{Z}$ let $\rho_k\colon \mathbb{R} \to [0,1]$,

$$\rho_k(x) = \begin{cases} 1, \text{ if } k < \lfloor x \rfloor; \\ \{x\}, \text{ if } k = \lfloor x \rfloor; \\ 0, \text{ if } k > \lfloor x \rfloor; \end{cases}$$

and set $\vartheta\colon \mathbb{R}^\omega \to [0,1]^{\mathbb{Z}\times\omega}$,

$$\vartheta((x_n)_{n<\omega}) = (\rho_k(x_n))_{k\in\mathbb{Z}, n<\omega}.$$

For every $x, y \in \mathbb{R}$ with $|y - x| \leq 1$, we have $\rho_k(x) \neq \rho_k(y)$ only if $k = \lfloor x \rfloor$ or $k = \lfloor y \rfloor$; moreover

$$|y - x| = \sum_{k\in\mathbb{Z}} |\rho_k(y) - \rho_k(x)| \ (x, y \in \mathbb{R}).$$

Thus

$$\sum_{k\in\mathbb{Z}} |\rho_k(y) - \rho_k(x)|^p \leq |y - x|^p \ (x, y \in \mathbb{R});$$

and for $x, y \in \mathbb{R}$ with $|y - x| \leq 1$,

$$|y - x|^p \leq 2^p \sum_{k\in\mathbb{Z}} |\rho_k(y) - \rho_k(x)|^p.$$

Since $(x_n)\ell_p(y_n)$ implies $\lim_{n<\omega} |y_n - x_n| = 0$, after reindexing the coordinates of its range, ϑ reduces ℓ^p to $\ell^p|_{[0,1]^\omega \times [0,1]^\omega}$, as required. $\square$

As we have seen already in the introduction, $\mathbf{E}_f$ may be an equivalence relation for a discontinuous f, e.g., for the function $f(0) = 0$, $f(x) = 1$ $(0 < x \leq 1)$ we have $\mathbf{E}_f$ is the equivalence relation of eventual equality on $[0,1]^\omega$. Following the literature, we denote this equivalence relation by E_1. In the remaining part of this section we show that f is continuous in zero if and only if $E_1 \not\leq_B \mathbf{E}_f$.

Theorem 2.4. *Let $f\colon [0,1] \to \mathbb{R}^+$ be a bounded Borel function such that $\mathbf{E}_f$ is an equivalence relation. Then f is continuous in zero if and only if $E_1 \not\leq_B \mathbf{E}_f$.*

Before proving Theorem 2.4 we show that up to Borel reducibility, requiring continuity in zero or continuity on the whole $[0,1]$ is the same condition for $\mathbf{E}_f$.

Proposition 2.5. *Let $f\colon [0,1] \to \mathbb{R}^+$ be a bounded function such that $\mathbf{E}_f$ is an equivalence relation. If f is continuous in zero, then there exists a continuous function $\tilde{f}\colon [0,1] \to \mathbb{R}^+$ such that $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$.*

As a corollary of Theorem 2.4 and Proposition 2.5, we obtain the following surprising result.

Corollary 2.6. *Let $f, g\colon [0,1] \to \mathbb{R}^+$ be bounded Borel functions such that $\mathbf{E}_f$ and $\mathbf{E}_g$ are equivalence relations. If g is continuous and $\mathbf{E}_f \leq_B \mathbf{E}_g$, then f is continuous in zero hence there is a continuous function $\tilde{f}\colon [0,1] \to \mathbb{R}^+$ such that $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$.*

We start with the proof of Proposition 2.5.

Proof of Proposition 2.5. Let $C \geq 1$ be the constant of (R_2). First we show that there exists an increasing function $\varepsilon\colon [0,1] \to [0,1]$ such that $\varepsilon(a) > 0$ for $a > 0$ and for every $x, y \in [0,1]$,

$$|y - x| \leq \varepsilon(f(x)) \Rightarrow \frac{f(x)}{2C} \leq f(y) \leq 2Cf(x). \tag{6}$$

Set

$$\varepsilon(a) = \frac{1}{2} \sup \left\{ y \in [0,1]\colon f(d) \leq \frac{a}{2C} \text{ for } 0 \leq d \leq y \right\};$$

then ε is increasing and since $f(0) = 0$ and f is continuous in zero, $\varepsilon(a) > 0$ for $a > 0$. We show (6). By (R_2a),

$$f(y) \leq C(f(x) + f(y - x)) \leq 2Cf(x) \ (0 \leq y - x \leq \varepsilon(f(x)))$$

and

$$\frac{f(x)}{2C} \leq \frac{f(x)}{C} - f(x - y) \leq f(y) \ (0 \leq x - y \leq \varepsilon(f(x)));$$

and by (R_2b),

$$\frac{f(x)}{2C} \leq \frac{f(x)}{C} - f(y - x) \leq f(y) \ (0 \leq y - x \leq \varepsilon(f(x)))$$

and

$$f(y) \le C(f(x)+f(x-y)) \le 2Cf(x)\ (0 \le x-y \le \varepsilon(f(x))),$$

as required.

As a corollary of (6), we get $U = \{x \in [0,1] : f(x) > 0\}$ is an open set. Moreover, for every $a > 0$ the $\varepsilon(a)$-neighborhood of $\{x \in [0,1] : f(x) > a\}$ is contained in U; i.e. f is continuous at every point of $[0,1] \setminus U = \{x \in [0,1] : f(x) = 0\}$. For every $x \in U$, set

$$I_x = (x - \varepsilon(f(x)), x + \varepsilon(f(x))) \cap [0,1].$$

Then $I_x \subseteq U$ and $\{I_x : x \in U\}$ is an open cover of U. Since the covering dimension of U is one, there is an open refinement $J_x \subseteq I_x$ $(x \in U)$ such that $\{J_x : x \in U\}$ is an open cover of U of order at most two; i.e. for every $x \in U$, $|\{y \in U : x \in J_y\}| \le 2$. So the function $\varphi : U \to 2^{\mathbb{R}}$,

$$\varphi(x) = \bigcup_{\substack{y \in U \\ x \in J_y}} \left[\frac{f(y)}{2C}, 2Cf(y)\right]$$

is closed convex valued and lower semicontinuous, hence Michael's Selection Theorem [9, Theorem 3.2 p. 364] can be applied to have a continuous function $\tilde{f} : U \to \mathbb{R}$ satisfying $\tilde{f}(x) \in \varphi(x)$ $(x \in U)$. Since f is continuous at every point of $[0,1] \setminus U$, $\tilde{f}$ extends continuously to $[0,1]$ with $\tilde{f}(x) = 0$ for $x \in [0,1] \setminus U$.

For fixed $x \in [0,1]$, $x \in J_y$ implies $x \in I_y$. So by (6),

$$f(x) \in \left[\frac{f(y)}{2C}, 2Cf(y)\right] \text{ hence } f(y) \in \left[\frac{f(x)}{2C}, 2Cf(x)\right].$$

Thus

$$\bigcup_{\substack{y \in U \\ x \in J_y}} \left[\frac{f(y)}{2C}, 2Cf(y)\right] \subseteq \left[\frac{f(x)}{4C^2}, 4C^2 f(x)\right]$$

and so

$$\frac{f(x)}{4C^2} \le \tilde{f}(x) \le 4C^2 f(x)\ (x \in [0,1]).$$

Therefore $\mathbf{E}_f = \mathbf{E}_{\tilde{f}}$, as required. □

We close this section with the proof of Theorem 2.4. We obtain the nonreducibility of E_1 to $\mathbf{E}_f$ for a continuous f via [7, Theorem 4.1 p. 238], which says that E_1 is not reducible to any equivalence relation induced by a Polish group action. To this end, first we show that for continuous f, $\mathbf{E}_f$ is essentially induced by a Polish group action. Recall that $\mathbb{S}$ denotes the circle group $[0,1)$ with mod 1 addition.

Lemma 2.7. *Let $f\colon [0,1] \to \mathbb{R}^+$ be a continuous function such that $\mathbf{E}_f$ is an equivalence relation. Then either f is identically zero or there is a continuous even function $\tilde{f}\colon \mathbb{S} \to \mathbb{R}^+$ such that $\tilde{f}(x) > 0$ for $x \neq 0$, $\mathbf{E}_{\tilde{f}}$ is an equivalence relation and $\mathbf{E}_f \leq_B \mathbf{E}_{\tilde{f}}$.*

Proof. Suppose f is not identically zero. We distinguish two cases. Suppose first $f(x) > 0$ for $x > 0$. Then set

$$\tilde{f}(x) = \begin{cases} f(2x), & \text{if } 0 \leq x < 1/2; \\ f(2-2x), & \text{if } 1/2 \leq x < 1. \end{cases}$$

It is obvious that $\tilde{f}$ is continuous, even and $\tilde{f}(x) > 0$ for $x \neq 0$. We show that $\mathbf{E}_{\tilde{f}}$ is an equivalence relation by verifying the conditions of Proposition 2.2. We have (R_1); since (R_2a) implies (R_2b), we prove only (R_2a). Let C be the constant of (R_2) for f. If $x \in [1/4, 3/4]$ or $y \in [1/4, 3/4]$, then

$$\tilde{f}(x+y) \leq \frac{\max \tilde{f}}{\min \tilde{f}|_{[1/4,3/4]}} (\tilde{f}(x) + \tilde{f}(y)).$$

If $x, y \in [0, 1/4]$ or $x, y \in [3/4, 1)$, then by (R_2a) for f, (R_2a) holds for $\tilde{f}$ with C. Finally if exactly one of x and y is in $[0, 1/4]$ and $[3/4, 1)$, then by (R_2b) for f, (R_2a) holds for $\tilde{f}$ with C.

Also, $\vartheta\colon [0,1]^\omega \to \mathbb{S}^\omega$, $\vartheta((x_n)_{n<\omega}) = (x_n/2)_{n<\omega}$ is a reduction of $\mathbf{E}_f$ to $\mathbf{E}_{\tilde{f}}$, so the proof of the first case is complete.

In the second case, suppose $f(x) = 0$ for some $x \in (0,1]$. By (R_2), the nonempty set $\{x \in [0,1]\colon f(x) = 0\}$ is closed under additions that are in $[0,1]$. Hence by the continuity of f, $x^\star = \inf\{x \in (0,1]\colon f(x) = 0\}$ satisfies $x^\star > 0$ and $f(x^\star) = 0$.

Set

$$\tilde{f}(x) = \begin{cases} f(xx^\star), & \text{if } 0 \leq x < 1/2; \\ f((1-x)x^\star), & \text{if } 1/2 \leq x < 1. \end{cases}$$

It is obvious that $\tilde{f}$ is continuous, even and $\tilde{f}(x) > 0$ for $x \neq 0$. Similarly to the previous case, we get $\mathbf{E}_{\tilde{f}}$ is an equivalence relation by distinguishing several cases. If $x \in [1/4, 3/4]$ or $y \in [1/4, 3/4]$, then

$$\tilde{f}(x+y) \leq \frac{\max \tilde{f}}{\min \tilde{f}|_{[1/4,3/4]}} (\tilde{f}(x) + \tilde{f}(y)).$$

If $x, y \in [0, 1/4]$, then by (R_2a) for f, (R_2a) holds for $\tilde{f}$ with C. If $x, y \in [3/4, 1)$, then again by (R_2a) for f,

$$\tilde{f}(x+y) = f(2x^\star - (x+y)x^\star) \lesssim f(x^\star - xx^\star) + f(x^\star - yx^\star) = \tilde{f}(x) + \tilde{f}(y).$$

Finally if exactly one of x and y is in $[0,1/4]$ and $[3/4,1)$, then by (R_2b) for f, (R_2a) holds for $\tilde{f}$ with C.

For every $x \in [0,1]$, let $\langle x\rangle = x/x^\star - \lfloor x/x^\star \rfloor$. We show that $\vartheta\colon [0,1]^\omega \to \mathbb{S}^\omega$, $\vartheta((x_n)_{n<\omega}) = (\langle x_n\rangle)_{n<\omega}$ is a reduction of $\mathbf{E}_f$ to $\mathbf{E}_{\tilde{f}}$. For every $0 \le x \le y \le 1$, with $k = \lfloor y/x^\star \rfloor - \lfloor x/x^\star \rfloor$ we have $\langle y\rangle - \langle x\rangle = y/x^\star - x/x^\star - k$, so

$$\tilde{f}\left(\langle y\rangle - \langle x\rangle\right) = \begin{cases} f\left(y - x - kx^\star\right), & \text{if } 0 \le y - x - kx^\star < x^\star/2;\\ f(-y + x + kx^\star), & \text{if } 0 \le -y + x + kx^\star < x^\star/2;\\ f\left(x^\star - y + x + kx^\star\right), & \text{if } x^\star/2 \le y - x - kx^\star < x^\star;\\ f(x^\star + y - x - kx^\star), & \text{if } x^\star/2 \le -y + x + kx^\star < x^\star. \end{cases}$$

For $l = k$ or $l = k \pm 1$, in any of the cases where applicable, by (R_2) we have

$$f(y-x) \lesssim f(y - x - lx^\star) + f(lx^\star),\ f(y-x) \lesssim f(lx^\star) + f(lx^\star - y + x),$$

$$f(y - x - lx^\star) \lesssim f(y-x) + f(lx^\star),\ f(lx^\star - y + x) \lesssim f(lx^\star) + f(y-x).$$

So $f(y-x) \approx \tilde{f}(\langle y\rangle - \langle x\rangle)$ follows from $f(lx^\star) = 0$. This implies that ϑ is a reduction, so the proof is complete. $\square$

In the next lemma, for an $\tilde{f}$ as in Lemma 2.7, we find a Polish group action inducing $\mathbf{E}_{\tilde{f}}$.

Definition 2.8. Let $f\colon H \to \mathbb{R}^+$ be an arbitrary function. For every $x = (x_n)_{n<\omega} \in H^\omega$ and $I \subseteq \omega$ we set

$$\|x\|_f = \sum_{n<\omega} f(x_n),\ \|x|_I\|_f = \sum_{n\in I} f(x_n).$$

We define $\mathcal{N}_f = \{x \in H^\omega \colon \|x\|_f < \infty\}$.

Lemma 2.9. *Let $f\colon \mathbb{S} \to \mathbb{R}^+$ be a continuous, even function such that $f(x) > 0$ for $x \neq 0$ and $\mathbf{E}_f$ is an equivalence relation.*

1. *There is a unique topology τ_f on $\mathcal{N}_f$ such that for every $x \in \mathcal{N}_f$, the sets*

$$B(x,\varepsilon) = \{y \in \mathcal{N}_f \colon \|y - x\|_f < \varepsilon\}\ (\varepsilon > 0)$$

   form a neighborhood base at x. This topology is regular, second countable and refines the topology inherited from $\mathbb{S}^\omega$.

2. *With τ_f, $(\mathcal{N}_f, +)$ is a Polish group. The natural action of $\mathcal{N}_f$ on $\mathbb{S}^\omega$ is continuous, and the equivalence relation induced by this action is $\mathbf{E}_f$.*

Proof. For 1, we show that for every $x \in \mathcal{N}_f$, $\varepsilon > 0$ and $y \in B(x,\varepsilon)$ there is a $\delta > 0$ such that $B(y,\delta) \subseteq B(x,\varepsilon)$; once this done, the first part of the statement follows from elementary topology (see e.g. [2]). Let $C \geq 1$ be the constant of (R_2), fix $x \in \mathcal{N}_f$, $\varepsilon > 0$ and $y \in B(x,\varepsilon)$. Let $n < \omega$ be such that

$$\|(y-x)|_{\omega\setminus n}\|_f < \frac{\varepsilon - \|y-x\|_f}{3C}.$$

Let $\delta > 0$ satisfy $\delta < (\varepsilon - \|y-x\|_f)/(3C)$, and such that for every $i < n$ and $z_i \in [0,1]$, $f(z_i - y_i) < \delta$ implies

$$|f(z_i - x_i) - f(y_i - x_i)| < \frac{\varepsilon - \|y-x\|_f}{3n};$$

such a δ exists by the continuity of f and by $f(x) > 0$ for $x \neq 0$.

Let $z \in B(y,\delta)$; then by (R_2),

$$\begin{aligned}
&\|z-x\|_f = \|(z-x)|_n\|_f + \|(z-x)|_{\omega\setminus n}\|_f < \\
&\quad \|(y-x)|_n\|_f + n\frac{\varepsilon - \|y-x\|_f}{3n} + C(\|(z-y)|_{\omega\setminus n}\|_f + \|(y-x)|_{\omega\setminus n}\|_f) < \\
&\qquad\qquad \|y-x\|_f + \frac{\varepsilon - \|y-x\|_f}{3} + \frac{\varepsilon - \|y-x\|_f}{3} + \frac{\varepsilon - \|y-x\|_f}{3} = \varepsilon,
\end{aligned}$$

as required.

Since $f(x) > 0$ for $x \neq 0$, τ_f refines the topology inherited from $\mathbb{S}^\omega$. The countable set of eventually zero rational sequences shows separability and hence second countability. To see regularity, let $F \subseteq (\mathcal{N}_f, \tau_f)$ be a closed set and take $x \notin F$. Then $\gamma = \inf\{\|y-x\|_f \colon y \in F\} > 0$. By (R_2), $B(x,\gamma/(2C)) \cap B(F,\gamma/(2C)) = \emptyset$, as required.

For 2, first we show $(\mathcal{N}_f, +)$ is a topological group. Let $x, y \in \mathcal{N}_f$ and $\gamma > 0$. By (R_2), $B(x,\gamma/2C) + B(y,\gamma/2C) \subseteq B(x+y,\gamma)$, so addition is continuous. The continuity of the inverse operation is obvious, so the statement follows.

Next we show $(\mathcal{N}_f, \tau_f)$ is strong Choquet (for the definition and notation see [5, Section 8.D p. 44]). The closed balls $\overline{B}(x,\varepsilon) = \{y \in \mathcal{N}_f \colon \|y-x\|_f \leq \varepsilon\}$ are closed in $\mathbb{S}^\omega$, thus every $\|\cdot\|_f$-Cauchy sequence is convergent in $\mathcal{N}_f$. If player I plays $(x_n, U_n)_{n<\omega}$, a winning strategy for player II is to choose $V_n = B(x_n,\gamma_n)$ such that $\overline{B}(x_n,\gamma_n) \subseteq U_n$ and $\gamma_n \leq 1/2^n$ $(n < \omega)$. So $(\mathcal{N}_f, \tau_f)$ is strong Choquet, hence Polish by Choquet's Theorem (see e.g. [5, (8.18) Theorem p. 45]).

The continuity of the action of $\mathcal{N}_f$ on $\mathbb{S}^\omega$ follows from the fact that τ_f refines the topology inherited from $\mathbb{S}^\omega$. It is obvious that the equivalence relation induced by this action is $\mathbf{E}_f$, so the proof is complete. $\square$

Definition 2.10. For a topological space X and $G \subseteq X$, we set

$$V(G) = \bigcup\{U \subseteq X : U \text{ is open, } G \cap U \text{ is comeager in } U\}.$$

The next lemma is a folklore result on the existence of a perfect set with special distance set.

Lemma 2.11. *Let $G \subseteq [0,1]$ be a Borel set such that zero is adherent to $V(G)$. Then there exists a nonempty perfect set $P \subseteq [0,1]$ such that*

$$\{|y - x| : x, y \in P\} \subseteq G \cup \{0\}. \tag{7}$$

Proof. By passing to a subset, we can assume that G is a comeager G_δ subset of $V(G)$. Set $\tilde{G} = G \cup (-G) \cup \{0\}$ and let $d_{\tilde{G}}$ be the metric on $\tilde{G}$ for which $(\tilde{G}, d_{\tilde{G}})$ is a Polish space with the topology inherited from $[-1,1]$ (see e.g. [5, (3.11) Theorem p. 17]). We construct inductively a sequence $(x_n)_{n<\omega} \subseteq [0,1]$ with the following properties:

1. for every $n < \omega$, $x_{n+1} < x_n/2$;
2. for every $s \in \{-1, 0, +1\}^{<\omega}$, $\sum_{i<|s|} s(i)x_i \in \tilde{G}$;
3. for every $s \in \{-1, 0, +1\}^{<\omega} \setminus \{\emptyset\}$, $d_{\tilde{G}}(\sum_{i<|s|-1} s(i)x_i, \sum_{i<|s|} s(i)x_i) \leq 1/2^{|s|}$.

Let $x_0 \in G$ be arbitrary. Let $0 < n < \omega$ and suppose x_i $(i < n)$ are defined such that 2 and 3 hold for every $s \in \{-1, 0, +1\}^{\leq n}$. By 2, if $s \in \{-1, 0, +1\}^n$ and $\sum_{i<n} s(i)x_i \neq 0$, then $\tilde{G}$ is comeager in a neighborhood of $\sum_{i<n} s(i)x_i$. Since zero is adherent to $V(G)$, by the Baire Category Theorem we can pick $x_n \in G$ sufficiently close to zero such that 1 holds; and for every $s \in \{-1, 0, +1\}^{n+1}$ with $\sum_{i<n} s(i)x_i \neq 0$ we have $\sum_{i<n+1} s(i)x_i \in \tilde{G}$, hence by $x_n \in G$, $\sum_{i<n+1} s(i)x_i \in \tilde{G}$ for every $s \in \{-1, 0, +1\}^{n+1}$; and in addition 3 holds. This completes the inductive step.

We show

$$P = \left\{ \sum_{n<\omega} \sigma(n)x_n : \sigma \in 2^\omega \right\}$$

fulfills the requirements. By 3, for every $\sigma \in \{-1, 0, +1\}^\omega$, $(\sum_{i<n} \sigma(i)x_i)_{n<\omega}$ is a Cauchy sequence in $\tilde{G}$, so $\sum_{n<\omega} \sigma(n)x_n \in \tilde{G}$. In particular $P \subseteq G \cup \{0\}$.

Let $x, x' \in P$, $x = \sum_{n<\omega} \sigma(n)x_n$ and $x' = \sum_{n<\omega} \sigma'(n)x_n$ with $\sigma, \sigma' \in 2^\omega$, $\sigma \neq \sigma'$; say for the first $n < \omega$ with $\sigma(n) \neq \sigma'(n)$ we have $\sigma(n) = 0$, $\sigma'(n) = 1$. Then for $\delta \in \{-1, 0, +1\}^\omega$, $\delta(n) = \sigma'(n) - \sigma(n)$ $(n < \omega)$ we have

$$|x' - x| = x' - x = \sum_{n<\omega} \delta(n)x_n \in \tilde{G},$$

moreover by 1, $x' - x > 0$; i.e. $x' - x \in G$. Thus P is a nonempty perfect set and satisfies (7), which completes the proof. $\square$

The last lemma points out a property of an f discontinuous in zero.

Lemma 2.12. *Let $f\colon [0,1] \to \mathbb{R}^+$ be a bounded Borel function such that $\mathbf{E}_f$ is an equivalence relation. If f is not continuous in zero, then there exists an $a > 0$ such that $G = \{x \in [0,1]\colon f(x) > a\}$ satisfies the condition of Lemma 2.11; i.e. zero is adherent to $V(G)$.*

PROOF. Let C be the constant of (R_2). Since f is not continuous in zero, there exists an $a > 0$ such that zero is adherent to $\{x \in [0,1]\colon f(x) > 2Ca\}$. If for every $x \in [0,1]$ with $f(x) > 2Ca$, x is adherent to

$$\bigcup \{U \subseteq (x,1)\colon U \text{ is open, } \{y \in U\colon f(y) > a\} \text{ is comeager in } U\}$$

then the statement follows. If not, by f being Borel, there is an $x \in [0,1]$ with $f(x) > 2Ca$ and a $\delta > 0$ such that

$$Y = \{y \in (x, x+\delta)\colon f(y) \leq a\}$$

is comeager in $(x, x+\delta)$. Since $f(x) > 2Ca$, by $(R_2 b)$ we have $f(y - x) > a$ whenever $y \in Y$. Hence $\{x \in [0,1]\colon f(x) > a\}$ is comeager in $(0,\delta)$, which finishes the proof. $\square$

PROOF OF THEOREM 2.4. Suppose first f is not continuous in zero. By Lemma 2.11 and Lemma 2.12, there is an $a > 0$ and a nonempty perfect set $P \subseteq [0,1]$ such that $f(|y - x|) > a$ for every $x, y \in P$, $x \neq y$. Thus $\mathbf{E}_f$ restricted to P^ω is E_1.

Suppose now that f is continuous in zero. By Proposition 2.5, we can assume f is continuous on $[0,1]$. If $f \equiv 0$, then $E_1 \not\leq_B \mathbf{E}_f$ is obvious. Else by Lemma 2.7 and Lemma 2.9, $\mathbf{E}_f \leq_B \mathbf{E}_{\tilde{f}}$ where $\mathbf{E}_{\tilde{f}}$ is induced by a Polish group action. Hence $E_1 \not\leq_B \mathbf{E}_{\tilde{f}}$ by [7, Thorem 4.1 p. 238] and [6]; in particular $E_1 \not\leq_B \mathbf{E}_f$. This completes the proof. $\square$

3 Reducibility Results.

In the remaining part of the paper, in most cases, we restrict our attention to equivalence relations $\mathbf{E}_f$ where $f\colon [0,1] \to \mathbb{R}^+$ is a *continuous* function. As we have seen in Proposition 2.5, requiring continuity on $[0,1]$ and continuity in zero for f are equivalent, and by Theorem 2.4, for Borel f it is the necessary and sufficient condition to have $E_1 \not\leq_B \mathbf{E}_f$. This assumption is acceptable to us since we aim to study equivalence relations $\mathbf{E}_f$ for which $\mathbf{E}_f \leq_B \mathbf{E}_{\mathrm{Id}^q}$ for some $1 \leq q < \infty$.

The main restriction, in addition to $(R_1)+(R_2)$, we impose in the sequel on the function f is formulated in the following definition.

Definition 3.1. Let $(R, \leq)$ be an ordered set and $f \colon R \to \mathbb{R}^+$ be a function. We say f is *essentially increasing* if for some $C \geq 1$, $\forall x, y \in R$ $(x \leq y \Rightarrow f(x) \leq Cf(y))$. Similarly, f is *essentially decreasing* if for some $C \geq 1$, $\forall x, y \in R$ $(x \leq y \Rightarrow Cf(x) \geq f(y))$.

Lemma 3.2. *With the notation of Definition 3.1, f is essentially increasing (resp. essentially decreasing) if and only if there is an increasing (resp. decreasing) function $\tilde{f}$ such that $\tilde{f} \approx f$.*

PROOF. If f is essentially increasing, set $\tilde{f} \colon R \to \mathbb{R}^+$, $\tilde{f}(x) = \sup\{f(y) \colon y \leq x\}$. Then $\tilde{f}$ is increasing and $f(x) \leq \tilde{f}(x) \leq Cf(x)$ $(x \in R)$. If f is essentially decreasing, let $\tilde{f}(x) = \inf\{f(y) \colon y \leq x\}$; then, as above, $\tilde{f} \approx f$. The other directions are obvious, so the proof is complete. □

We remark that for $R = [0,1]$ or $R = (0,1]$, by its definition above, $\tilde{f}$ is continuous if f is so. By $\tilde{f} \approx f$, $\tilde{f}$ and f have the same asymptotic behavior in 0.

In this section we prove the following two theorems.

Theorem 3.3. *Let $1 \leq \alpha < \infty$ and let $\psi \colon (0,1] \to (0,+\infty)$ be an essentially decreasing continuous function such that $\mathrm{Id}^\alpha \psi$ is bounded and for every $\delta > 0$, $\liminf_{x \to +0} x^\delta \psi(x) = 0$. Set $g(x) = x^\alpha \psi(x)$ for $0 < x \leq 1$ and $g(0) = 0$. Suppose $\mathbf{E}_g$ is an equivalence relation. Then $\mathbf{E}_{\mathrm{Id}^\alpha} \leq_B \mathbf{E}_g$.*

Theorem 3.4. *Let $f, g \colon [0,1] \to \mathbb{R}^+$ be continuous, essentially increasing functions such that $\mathbf{E}_f$ and $\mathbf{E}_g$ are equivalence relations. Suppose there exists a function $\kappa \colon \{1/2^i \colon i < \omega\} \to [0,1]$ satisfying the recursion*

$$f(1) = g(\kappa(1)),\ f(1/2^n) = \sum_{i=0}^{n} g(\kappa(1/2^i)/2^{n-i})\ (0 < n < \omega) \tag{8}$$

such that for some $L \geq 1$,

$$\sum_{i=n}^{\infty} g(\kappa(1/2^i)) \leq L \sum_{i=0}^{n} g(\kappa(1/2^i)/2^{n-i})\ (n < \omega). \tag{9}$$

Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Theorem 3.3 illustrates, e.g. by choosing $\psi(x) = 1 - \log(x)$ $(0 < x \leq 1)$, that reducibility among the $\mathbf{E}_f$'s is *not* characterized by the growth order of the f's. Theorem 3.4 is a stronger version of [3, Theorem 1.1 p. 1836], but we admit that our improvement is of technical nature. However, in Section 5 it will allow us to show the reducibility among $\mathbf{E}_f$'s for new families of f's.

These results neither give a complete description of the reducibility between the equivalence relations $\mathbf{E}_f$ nor are optimal. Nevertheless, we note that in Theorem 3.3, Id^α cannot be replaced by an arbitrary "nice" function: as we will see, e.g. $\mathbf{E}_{\mathrm{Id}^\alpha} <_B \mathbf{E}_{\mathrm{Id}^\alpha/(1-\log)}$. Also, the condition ψ is decreasing cannot be left out: e.g. we need the techniques of Theorem 3.4 in order to treat the $\psi(x) = x$ case; i.e. to show $\mathbf{E}_{\mathrm{Id}^\alpha} \leq_B \mathbf{E}_{\mathrm{Id}^{\alpha+1}}$. We comment on the optimality of Theorem 3.4 after its proof.

We start with a technical lemma.

Lemma 3.5. *Let $f, g\colon [0,1] \to \mathbb{R}^+$ be continuous functions such that $\mathbf{E}_f$, $\mathbf{E}_g$ are equivalence relations. Suppose there exists $K > 0$ and $I \in [\omega]^\omega$ such that for every $n \in I$ there is a mapping $\vartheta_n\colon \{i/n\colon 0 \leq i \leq n\} \to [0,1]^\omega$ satisfying*

$$\frac{1}{K} f((j-i)/n) \leq \|\vartheta_n(j/n) - \vartheta_n(i/n)\|_g \leq K f((j-i)/n) \ (0 \leq i < j \leq n). \quad (10)$$

Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Proof. For $x \in [0,1]$ and $0 < n < \omega$ set $[x]_n = \max\{i/n\colon i/n \leq x,\ 0 \leq i \leq n\}$. Since f is uniformly continuous on $[0,1]$, for every $k < \omega$ there is an $n_k \in I$ such that $|f(x) - f([x]_{n_k})| \leq 1/2^k$ $(x \in [0,1])$. We show that $\vartheta\colon [0,1]^\omega \to [0,1]^{\omega\cdot\omega}$,

$$\vartheta((x_k)_{k<\omega}) = (\vartheta_{n_k}([x_k]_{n_k}))_{k<\omega},$$

after reindexing the coordinates of the range, is a Borel reduction of $\mathbf{E}_f$ to $\mathbf{E}_g$.

Let $(x_k)_{k<\omega}, (y_k)_{k<\omega} \in [0,1]^\omega$. We have

$$[|y_k - x_k|]_{n_k} \leq |[y_k]_{n_k} - [x_k]_{n_k}| \leq [|y_k - x_k|]_{n_k} + 1/n_k \ (k < \omega).$$

So by the choice of n_k, $|f(|y_k - x_k|) - f([|y_k - x_k|]_{n_k})| \leq 1/2^k$ and $|f([|y_k - x_k|]_{n_k}) - f(|[y_k]_{n_k} - [x_k]_{n_k}|)| \leq 1/2^k$, thus

$$|f(|y_k - x_k|) - f(|[y_k]_{n_k} - [x_k]_{n_k}|)| \leq 2/2^k \ (k < \omega).$$

By (10), $\|\vartheta_{n_k}([y_k]_{n_k}) - \vartheta_{n_k}([x_k]_{n_k})\|_g \approx f(|[y_k]_{n_k} - [x_k]_{n_k}|)$ $(k < \omega)$, so the statement follows. □

PROOF OF THEOREM 3.3. For some $B \geq 1$, let $x^\alpha \psi(x) \leq B$ $(0 < x \leq 1)$. We find a $K > 0$ such that for every $0 < n < \omega$ there exist $M < \omega$ and $0 < \mu \leq 1$ such that for every $0 \leq i < j \leq n$,

$$\frac{1}{K}\left(\frac{j-i}{n}\right)^\alpha \leq M\left(\frac{j-i}{n}\mu\right)^\alpha \psi\left(\frac{j-i}{n}\mu\right) \leq K\left(\frac{j-i}{n}\right)^\alpha. \tag{11}$$

Once this done, the conditions of Lemma 3.5 are satisfied by the mapping $\vartheta_n \colon \{i/n \colon 0 \leq i \leq n\} \to [0,1]^\omega$,

$$\vartheta_n(i/n) = (\underbrace{i\mu/n, \ldots, i\mu/n}_{M}, 0, \ldots).$$

Observe that (11) is equivalent to

$$1/K \leq M\mu^\alpha \psi((j-i)\mu/n) \leq K \ (0 \leq i < j \leq n).$$

Since ψ is essentially decreasing, it is enough to have $1/2 \leq M\mu^\alpha\psi(\mu)$ and $M\mu^\alpha\psi(\mu/n) \leq 2B$. We will find a $0 < \mu \leq 1$ satisfying $\psi(\mu/n) \leq 2\psi(\mu)$. Then by choosing M to be minimal such that $1/2 \leq M\mu^\alpha\psi(\mu)$, by $\mu^\alpha\psi(\mu) \leq B$ and $B \geq 1$ we have $M\mu^\alpha\psi(\mu/n) \leq 2M\mu^\alpha\psi(\mu) \leq 2B$, so we fulfilled the requirements.

Suppose such a μ does not exist; i.e. $\psi(\mu/n) > 2\psi(\mu)$ $(0 < \mu \leq 1)$. Then for every $k < \omega$ and $\mu \in [1/n, 1]$, $\psi(n^{-k}\mu) \geq 2^k\psi(\mu)$. We have $x = n^{-k}\mu$ runs over $(0,1]$ as (k,μ) runs over $\omega \times [1/n, 1]$. So since ψ is essentially decreasing, with $\delta = \log(2)/\log(n)$ we have $\psi(x)x^\delta \gtrsim 1/n^\delta\psi(1) > 0$ $(0 < x \leq 1)$. This contradicts $\liminf_{x\to+0} x^\delta\psi(x) = 0$, so the proof is complete. □

PROOF OF THEOREM 3.4. Let $n < \omega$ be fixed. For $0 < l \leq 2^n$ let $r(l) \leq n$, $s(l) < \omega$ be such that $l/2^n = s(l)/2^{r(l)}$ and $s(l)$ is odd. With $\mathrm{Pr}_l x$ standing for the l^{th} coordinate of $x \in [0,1]^{2^n}$, for every $0 \leq i \leq 2^n$ we define $\vartheta(i/2^n)$ by

$$\mathrm{Pr}_l\vartheta(i/2^n) = \left(1 - 2^{r(l)}\left|\frac{i}{2^n} - \frac{l}{2^n}\right|\right)\kappa(1/2^{r(l)}) \tag{12}$$

if $l > 0$ and $\left|\frac{i}{2^n} - \frac{l}{2^n}\right| \leq 1/2^{r(l)}$, else let $\mathrm{Pr}_l\vartheta(i/2^n) = 0$. We show (10) holds for $\vartheta_{2^n} = \vartheta$.

Let $0 \leq i < j \leq 2^n$ be arbitrary. Let $m \leq n$ be minimal such that for some $e < 2^m$ we have

$$\frac{i}{2^n} \leq \frac{e}{2^m} < \frac{(e+1)}{2^m} \leq \frac{j}{2^n}.$$

We distinguish several cases.

Suppose first $i/2^n = e/2^m$ and $j/2^n = (e+1)/2^m$. For every $k \leq m$ there is exactly one l with $r(l) = k$ such that

$$|e/2^m - l/2^n| \leq 1/2^k \text{ and } |(e+1)/2^m - l/2^n| \leq 1/2^k;$$

and for this l, by (12),

$$|\mathrm{Pr}_l(\vartheta((e+1)/2^m) - \vartheta(e/2^m))| = \kappa(1/2^k)/2^{m-k}.$$

All the other coordinates of $\vartheta((e+1)/2^m)$ and $\vartheta(e/2^m)$ are zero so by (8),

$$\|\vartheta((e+1)/2^m) - \vartheta(e/2^m))\|_g = \sum_{k=0}^{m} g(\kappa(1/2^k)/2^{m-k}) = f(1/2^m); \qquad (13)$$

i.e. (10) holds with $K = 1$.

Next suppose e is even, $i/2^n = e/2^m$ and $(e+1)/2^m < j/2^n$; then we have $m \geq 1$. Observe that by the choice of m we have $j/2^n < (e+2)/2^m$. For every $k < m$ there is exactly one l with $r(l) = k$ such that

$$|e/2^m - l/2^n| \leq 1/2^k \textit{ and } |j/2^n - l/2^n| \leq 1/2^k;$$

and for this l, $l/2^n \notin (e/2^m, j/2^n)$. So by (12),

$$\frac{\kappa(1/2^k)}{2^{m-k}} \leq |\mathrm{Pr}_l(\vartheta(j/2^n) - \vartheta(e/2^m))| \leq 2\frac{\kappa(1/2^k)}{2^{m-k}} \ (0 \leq k < m).$$

Since e is even, $\vartheta(e/2^m)$ has no other nonzero coordinates. For every $m \leq k \leq n$ there is exactly one l with $r(l) = k$ such that $|j/2^n - l/2^n| \leq 1/2^k$, and for this l, $\mathrm{Pr}_l(\vartheta(j/2^n)) \leq \kappa(1/2^k)$. Since g is essentially increasing, we have

$$\begin{aligned} \sum_{k=0}^{m-1} g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) &\lesssim \|\vartheta(j/2^n) - \vartheta(e/2^m))\|_g \\ &\lesssim \sum_{k=0}^{m-1} g\left(2\frac{\kappa(1/2^k)}{2^{m-k}}\right) + \sum_{k=m}^{n} g(\kappa(1/2^k)). \qquad (14) \end{aligned}$$

By (R_2),

$$g\left(2\frac{\kappa(1/2^k)}{2^{m-k}}\right) \lesssim g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right) \ (0 \leq k < m). \qquad (15)$$

By (8) and since f is essentially increasing,

$$f\left(\frac{j}{2^n} - \frac{e}{2^m}\right) \lesssim f\left(\frac{1}{2^{m-1}}\right) = \sum_{k=0}^{m-1} g\left(\frac{\kappa(1/2^k)}{2^{m-1-k}}\right),$$

so by (14),

$$f\left(\frac{j}{2^n}-\frac{e}{2^m}\right)\lesssim\|\vartheta(j/2^n)-\vartheta(e/2^m))\|_g. \tag{16}$$

By (9) and (15), the right hand side of (14) is $\lesssim\sum_{k=0}^{m}g\left(\kappa(1/2^k)/2^{m-k}\right)$, so since f is essentially increasing,

$$\begin{aligned}\|\vartheta(j/2^n)-\vartheta(e/2^m))\|_g&\lesssim\sum_{k=0}^{m}g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right)\\&=f(1/2^m)\lesssim f\left(\frac{j}{2^n}-\frac{e}{2^m}\right).\end{aligned} \tag{17}$$

The case $e+1$ is even, $i/2^n<e/2^m$ and $j/2^n=(e+1)/2^m$ can be treated by an analogous argument.

Suppose now e is even, $i/2^n<e/2^m$ and $(e+1)/2^m\leq j/2^n$; then we have $m\geq 2$. By (R_2), (17), and also by (13) if $j/2^n=(e+1)/2^m$,

$$\begin{aligned}\|\vartheta(j/2^n)-\vartheta(i/2^n))\|_g&\lesssim\|\vartheta(j/2^n)-\vartheta(e/2^m))\|_g+\|\vartheta(e/2^m)-\vartheta(i/2^n))\|_g\\&\lesssim\left(f\left(\frac{j}{2^n}-\frac{e}{2^m}\right)+f\left(\frac{e}{2^m}-\frac{i}{2^n}\right)\right)\\&\lesssim f\left(\frac{j-i}{2^n}\right).\end{aligned}$$

To have a lower bound, observe that for every $k<m-1$ there is exactly one l with $r(l)=k$ such that

$$|i/2^n-l/2^n|\leq 1/2^k \textit{ and } |j/2^n-l/2^n|\leq 1/2^k,$$

and for this l, $l/2^n\notin(i/2^n,j/2^n)$. So by (12),

$$\frac{\kappa(1/2^k)}{2^{m-k}}\leq\left|\mathrm{Pr}_l(\vartheta(j/2^n)-\vartheta(i/2^n))\right|.$$

By (R_2) and since g is essentially increasing,

$$f(1/2^{m-2})=\sum_{k=0}^{m-2}g\left(\frac{\kappa(1/2^k)}{2^{m-2-k}}\right)\lesssim\sum_{k=0}^{m-2}g\left(\frac{\kappa(1/2^k)}{2^{m-k}}\right)\lesssim\|\vartheta(j/2^n)-\vartheta(i/2^n))\|_g.$$

By the choice of m we have $(j-i)/2^n<3/2^m$. So since f is essentially increasing, $f((j-i)/2^n)\lesssim f(4/2^m)=f(1/2^{m-2})$; thus $f\left((j-i)/2^n\right)\lesssim\|\vartheta(j/2^n)-\vartheta(i/2^n)\|_g$.

The case $e+1$ is even, $i/2^n\leq e/2^m$ and $(e+1)/2^m<j/2^n$ follows similarly, so the proof is complete. □

The assumptions of Theorem 3.4 are not necessary; they merely make it possible to imitate the construction in the proof of [3, Theorem 1.1 p. 1836]. We note, however, that the problem of characterizing whether $\{i/2^n : 0 \leq i \leq 2^n\}$ endowed with the $\|\cdot\|_f$-distance Lipschitz embeds into $[0,1]^\omega$ endowed with the $\|\cdot\|_g$-distance is very hard even if the distances $\|\cdot\|_f$ and $\|\cdot\|_g$ can be related to norms (see e.g. [8] and the references therein). So it is unlikely that there is a simple characterization of reducibility among $\mathbf{E}_f$'s using the approach of Lemma 3.5.

4 Nonreducibility Results.

In this section we improve [3, Theorem 2.2 p. 1840] in order to obtain nonreducibility results for a wider class of $\mathbf{E}_f$'s, as follows.

Theorem 4.1. *Let $1 \leq \alpha < \infty$ and let $\varphi, \psi\colon [0,1] \to [0,+\infty)$ be continuous functions. Set $f = \mathrm{Id}^\alpha \varphi$, $g(x) = \mathrm{Id}^\alpha \psi$ and suppose that f, g are bounded and $\mathbf{E}_f$ and $\mathbf{E}_g$ are equivalence relations. Suppose $\psi(x) > 0$ $(x > 0)$, and*

(A_1) *there exist $\varepsilon > 0$, $M < \omega$ such that for every $n > M$ and $x, y \in [0,1]$,*

$$\varphi(x) \leq \varepsilon\varphi(y)\varphi(1/2^n) \Rightarrow x \leq \frac{y}{2^{n+1}};$$

(A_2) $\lim_{n\to\infty} \psi(1/2^n)/\varphi(1/2^n) = 0$.

Then $\mathbf{E}_g \not\leq_B \mathbf{E}_f$.

Observe that $\varphi \equiv 1$, $\psi = \mathrm{Id}^\beta$ $(0 < \beta < \infty)$ satisfy the assumptions of Theorem 4.1, so it generalizes [3, Theorem 2.2 p. 1840].

The proof of [3, Theorem 2.2 p. 1840] has two fundamental constituents. The first idea is to pass to a subspace $X \subseteq [0,1]^\omega$ where a hypothetic Borel reduction ϑ of $\mathbf{E}_g$ to $\mathbf{E}_f$ is modular; i.e. for $x \in X$, $\vartheta(x)$ consists of finite blocks, each of which depends only on a single coordinate of x. This technique can be adopted without any difficulty. The second tool is an excessive use of the fact that for $f = \mathrm{Id}^p$, $f^{-1}(\|\cdot\|_f)$ is a norm, which does not follow from the assumptions of Theorem 4.1. We get around this difficulty by exploiting that φ is a perturbation when compared to Id^α.

Proof of Theorem 4.1. Suppose $\vartheta\colon [0,1]^\omega \to [0,1]^\omega$ is a Borel reduction of $\mathbf{E}_g$ to $\mathbf{E}_f$. With $Z_k = \{i/2^k : 0 \leq i \leq 2^k\}$, set $Z = \prod_{k<\omega} Z_k$; then ϑ is a Borel reduction of $\mathbf{E}_g|_{Z\times Z}$ to $\mathbf{E}_f$. For every finite sequence $t \in \prod_{i<|t|} Z_i$, let $N_t = \{z \in Z : z(i) = t(i)\ (i < |t|)\}$. We import several lemmas from [3].

Lemma 4.2. ([3, Claim (i) p. 1840]) *For any $j,k<\omega$ there exist $l<\omega$, a finite sequence $s^\star \in \prod_{i<|s^\star|} Z_{k+i}$, and a comeager set $D \subseteq Z$ such that for all $x,\hat{x} \in D$, if we have $x = r^\frown s^{\star\frown} y$ and $\hat{x} = \hat{r}^\frown s^{\star\frown} y$ for some $r,\hat{r} \in [0,1]^k$ and $y \in [0,1]^\omega$, then $\|(\vartheta(x)-\vartheta(\hat{x}))|_{\omega\setminus l}\|_f < 2^{-j}$.*

PROOF. For every $l<\omega$, we define $F_l \colon Z \to \mathbb{R}$ by $F_l(x) = \max\{\|(\vartheta(z)-\vartheta(\hat{z}))|_{\omega\setminus l}\|_f \colon z,\hat{z} \in Z,\ z(i)=\hat{z}(i)=x(i)\ (k \le i < \omega)\}$. For fixed $x \in Z$, there are only finitely many $z,\hat{z} \in Z$ satisfying $z(i)=\hat{z}(i)=x(i)$ $(k \le i < \omega)$. For each such pair we have $\|z-\hat{z}\|_g < \infty$, hence $\|\vartheta(z)-\vartheta(\hat{z})\|_f < \infty$, in particular $\lim_{l\to\infty} \|(\vartheta(z)-\vartheta(\hat{z}))|_{\omega\setminus l}\|_f = 0$. So $F_l(x) < \infty$ for all $l<\omega$ and $\lim_{l\to\infty} F_l(x) = 0$ $(x \in Z)$. Therefore, by the Baire Category Theorem, there exists an $l<\omega$ such that $\{x \in Z \colon F_l(x) < 2^{-j}\}$ is not meager. By f being Borel, this set has the property of Baire, so there is a nonempty open set O on which it is relatively comeager.

We can assume $O = N_t$ for some finite sequence $t \in \prod_{i<|t|} Z_i$, and we can also assume $|t| \ge k$. Let $t = r^{\star\frown} s^\star$ where $|r^\star| = k$. But $F_l(x)$ does not depend on the first k coordinates of x, so $\{x \in Z \colon F_l(x) < 2^{-j}\}$ is also relatively comeager in $N_{r^\frown s^\star}$ for all $r \in \prod_{i<k} Z_i$. Let D be a comeager set such that $F_l(x) < 2^{-j}$ whenever $x \in D \cap N_{r^\frown s^\star}$ for any r of length k. Now the conclusion of the claim follows from the definition of F_l. $\square$

By [5, (8.38) Theorem p. 52] there is a dense G_δ set $C \subseteq Z$ such that $\vartheta|_C$ is continuous.

Lemma 4.3. ([3, Claim (ii) p. 1841]) *For any $j,k,l<\omega$ there is a finite sequence $s^{\star\star} \in \prod_{i<|s^{\star\star}|} Z_{k+i}$ such that for all $x,\hat{x} \in C$, if we have $x = r^\frown s^{\star\star\frown} y$ and $\hat{x} = r^\frown s^{\star\star\frown} \hat{y}$ for some $r \in [0,1]^k$ and $y,\hat{y} \in [0,1]^\omega$, then $\|(\vartheta(x)-\vartheta(\hat{x}))|_l\|_f < 2^{-j}$.*

Furthermore, if G is a given dense open subset of Z, then $s^{\star\star}$ can be chosen such that $N_{r^\frown s^{\star\star}} \subseteq G$ for all $r \in \prod_{i<k} Z_i$.

PROOF. There are only finitely many $r \in \prod_{i<k} Z_i$; enumerate them as $r_0, r_1, \ldots, r_{M-1}$. We construct $s^{\star\star}$ by successive extensions.

Let $t_0 = \emptyset$. Let $m<M$ and suppose that we have the finite sequence $t_m \in \prod_{i<|t_m|} Z_{k+i}$. The basic open set $N_{r_m^\frown t_m}$ meets the comeager set C, so we can pick $w \in C \cap N_{r_m^\frown t_m}$. Since ϑ is continuous on C and f is continuous, we can pass to a smaller open neighborhood O of w such that for all $x,\hat{x} \in C\cap O$, $\|(\vartheta(x)-\vartheta(\hat{x}))|_l\|_f < 2^{-j}$. We can assume $O = N_{r_m^\frown t'_m}$ for some extension t'_m of t_m. Since G is dense open, we can further extend t'_m to get t_{m+1} such that $N_{r_m^\frown t_{m+1}} \subseteq G$. Once the sequences t_m $(m \le M)$ are constructed, $s^{\star\star} = t_M$ fulfills the requirements. $\square$

Lemma 4.4. [3, Claim (iii) p. 1842] *There exist strictly increasing sequences* $(b_i)_{i<\omega}, (l_i)_{i<\omega} \subseteq \omega$ *and functions* $f_i \colon Z_{b_i} \to [0,1]^{l_{j+1}-l_j}$ *such that* $b_0 = l_0 = 0$, *for* $Z' = \prod_{i<\omega} Z_{b_i}$ *and* $\vartheta' \colon Z' \to [0,1]^\omega$, $\vartheta'(x) = f_0(x_0)^\frown \dots ^\frown f_i(x_i)^\frown \dots$ *we have*

$$\|x - \hat{x}\|_g < \infty \Leftrightarrow \|\vartheta'(x) - \vartheta'(\hat{x})\|_f < \infty. \tag{18}$$

Proof. We construct the sequences $(b_i)_{i<\omega}, (l_i)_{i<\omega} \subseteq \omega$, finite sequences s_i $(i < \omega)$ and dense open sets D_i^j $(i, j < \omega)$ by induction, as follows.

We have $b_0 = l_0 = 0$. Let $j < \omega$ and suppose that we have b_j, l_j and $D_i^{j'}$ for every $i < \omega$ and $j' < j$. We apply Lemma 4.2 for j and $k = b_j + 1$ to get $l_{j+1} = l < \omega$, a finite sequence $s_j^\star \in \prod_{i<|s_j^\star|} Z_{b_j+1+i}$ and a comeager set $D^j \subseteq Z$ satisfying the conclusions of Lemma 4.2. We can assume $l_{j+1} > l_j$ and $D^j \subseteq C$. Let $(D_i^j)_{i<\omega}$ be a decreasing sequence of dense open subsets of Z such that $\bigcap_{i<\omega} D_i^j \subseteq D^j$. We apply Lemma 4.3 for j, $k = b_j + 1 + |s_j^\star|$, $l = l_{j+1}$, and $G = \bigcap_{j'<j} D_j^{j'}$ to get $s_j^{\star\star}$ as in Lemma 4.3. We set $s_j = s_j^{\star\frown} s_j^{\star\star}$ and $b_{j+1} = b_j + 1 + |s_j|$.

Let $Z' = \prod_{i<\omega} Z_{b_i}$ and set $h \colon Z' \to Z$,

$$h(x) = x_0^\frown s_0^\frown x_1^\frown s_1^\frown \dots ^\frown x_i^\frown s_i^\frown \dots .$$

For every $i < \omega$, we define $f_i \colon Z_{b_i} \to [0,1]^{l_{j+1}-l_j}$ by

$$f_i(a) = \vartheta(h(\underbrace{0^\frown \dots ^\frown 0}_{i}{}^\frown a^\frown 0^\frown 0^\frown \dots))|_{l_{j+1} \setminus l_j}; \tag{19}$$

and we set $\vartheta' \colon Z' \to [0,1]^\omega$, $\vartheta'(x) = f_0(x_0)^\frown \dots ^\frown f_i(x_i)^\frown \dots$

It remains to prove (18). To see this, it is enough to prove $\|\vartheta'(x) - \vartheta(h(x))\|_f < \infty$ for every $x \in Z'$ since then for every $x, \hat{x} \in Z'$, by (R_2),

$$\|\vartheta'(x) - \vartheta'(\hat{x})\|_f < \infty \iff \|\vartheta(h(x)) - \vartheta(h(\hat{x}))\|_f < \infty \iff \\ \|h(x) - h(\hat{x})\|_g < \infty \iff \|x - \hat{x}\|_g < \infty.$$

Let $x \in Z'$ be arbitrary; for every $j < \omega$ we define $e_j, e_j' \in Z'$ by setting

$$\mathrm{Pr}_i e_j = \begin{cases} x_i, & \text{if } i = j; \\ 0, & \text{if } i \in \omega \setminus \{j\}; \end{cases}, \ \mathrm{Pr}_i e_j' = \begin{cases} x_i, & \text{if } i \leq j; \\ 0, & \text{if } j < i < \omega. \end{cases}$$

Since $h(x)$ and $h(e_j')$ agree on all coordinates below b_{j+1}, by the definition of $s_j^{\star\star}$,

$$\|(\vartheta(h(x)) - \vartheta(h(e_j')))|_{l_{j+1}}\|_f < 2^{-j} \ (j < \omega).$$

On the other hand, for $j > 0$, $h(e'_j)$ and $h(e_j)$ agree on all coordinates above b_{j-1}, so by the definition of $s^\star_{j-1}$,

$$\|(\vartheta(h(e'_j)) - \vartheta(h(e_j)))|_{\omega\setminus l_j}\|_f < 2^{-j+1} \ (0 < j < \omega). \tag{20}$$

Moreover, (20) holds for $j = 0$, as well. Then by (R_2),

$$\|(\vartheta'(x) - \vartheta(h(x)))|_{l_{j+1}\setminus l_j}\|_f = \|(\vartheta(h(e_j)) - \vartheta(h(x)))|_{l_{j+1}\setminus l_j}\|_f \lesssim \\ \|(\vartheta(h(e_j)) - \vartheta(h(e'_j)))|_{\omega\setminus l_j}\|_f + \|(\vartheta(h(e'_j)) - \vartheta(h(x)))|_{l_{j+1}}\|_f \leq 3 \cdot 2^{-j}.$$

Therefore

$$\|(\vartheta'(x) - \vartheta(h(x)))\|_f = \sum_{j<\omega} \|(\vartheta'(x) - \vartheta(h(x)))|_{l_{j+1}\setminus l_j}\|_f \leq \sum_{j<\omega} 3 \cdot 2^{-j} < \infty,$$

as required. □

Lemma 4.5. [3, Claim (iv) p. 1843] *There exist $c > 0$ and $N < \omega$ such that with the notation of (19), for every $i > N$, $\|f_i(1) - f_i(0)\|_f > c$.*

Proof. If not, then we can find a strictly increasing sequence $(j_m)_{m<\omega} \subseteq \omega$ such that $\|f_{j_m}(1) - f_{j_m}(0)\|_f \leq 2^{-m}$ $(m < \omega)$. Let $\hat{x}$ be the constant 0 sequence, and let x be the sequence which is 1 at each coordinate j_m $(m < \omega)$ and 0 at all other coordinates. Then $\|x - \hat{x}\|_g = \infty$ but

$$\begin{aligned}\|\vartheta'(x) - \vartheta'(\hat{x})\|_f &= \sum_{j<\omega} \|f_j(x(j)) - f_j(\hat{x}(j))\|_f \\ &= \sum_{m<\omega} \|f_{j_m}(1) - f_{j_m}(0)\|_f \leq \sum_{m<\omega} 2^{-m} < \infty,\end{aligned}$$

contradicting (18). □

Lemma 4.6. *Let $c > 0$, $N < \omega$ be as in Lemma 4.5. For every $0 < D < \omega$ there exists $N_D > \max\{N, D\}$ such that for every $i \geq N_D$ there is a $0 \leq k < 2^{b_{N_D}}$ with*

$$\|f_i((k+1)/2^{b_{N_D}}) - f_i(k/2^{b_{N_D}})\|_f \geq Dg(1/2^{b_{N_D}}). \tag{21}$$

Proof. Let $\varepsilon > 0$ and $M < \omega$ be as in the assumptions of Theorem 4.1. Fix $0 < D < \omega$; by (A_2) there exists $N_D > \max\{M, N, D\}$ such that with $n = 2^{b_{N_D}}$, $2D/c < \varepsilon\varphi(1/n)/\psi(1/n)$. Fix $i \geq N_D$, set $l = l_{i+1} - l_i$ and

$$\gamma_j = |\mathrm{Pr}_j(f_i(1) - f_i(0))| \ (j < l).$$

For every $x = (x_j)_{j<l} \in [-1,1]^l$ set

$$\|x\|_\Delta = \left(\sum_{j<l} |x_j|^\alpha \varphi(\gamma_j)\right)^{1/\alpha};$$

then $\|\cdot\|_\Delta$ satisfies the triangle inequality on $[0,1]^l$. Since $\|f_i(1) - f_i(0)\|_f = \|f_i(1) - f_i(0)\|_\Delta^\alpha = \sum_{j<l} \gamma_j^\alpha \varphi(\gamma_j)$, by the triangle inequality there is a $0 \leq k < n$ such that

$$\|f_i((k+1)/n) - f_i(k/n)\|_\Delta \geq \frac{1}{n}\|f_i(1) - f_i(0)\|_f^{1/\alpha}.$$

With such a k, set

$$\delta_j = |\mathrm{Pr}_j(f_i((k+1)/n) - f_i(k/n))| \ (j < l);$$

i.e. we have

$$\sum_{j<l} \delta_j^\alpha \varphi(\gamma_j) \geq \frac{1}{n^\alpha}\|f_i(1) - f_i(0)\|_f. \tag{22}$$

Set $J = \{j < l \colon \varphi(\gamma_j) \leq \varphi(\delta_j)c/(2D\psi(1/n))\}$. Then

$$\sum_{j<l} \delta_j^\alpha \varphi(\gamma_j) \leq \sum_{j\in J} \delta_j^\alpha \varphi(\delta_j)\frac{c}{2D\psi(1/n)} + \sum_{j\notin J} \delta_j^\alpha \varphi(\gamma_j). \tag{23}$$

By the choice of N_D, $2D/c < \varepsilon\varphi(1/n)/\psi(1/n)$. So for $j \notin J$, $\varphi(\delta_j) < \varepsilon\varphi(\gamma_j)\varphi(1/n)$. This, by (A_1) and by $b_{N_D} \geq N_D > M$, implies $\delta_j \leq \gamma_j/(2n)$ $(j \notin J)$. Hence

$$\sum_{j\notin J} \delta_j^\alpha \varphi(\gamma_j) \leq \frac{1}{(2n)^\alpha}\sum_{j\notin J} \gamma_j^\alpha \varphi(\gamma_j) = 2^{-\alpha}\frac{\|f_i(1) - f_i(0)\|_f}{n^\alpha}.$$

So by (22) and (23),

$$\sum_{j\in J} \delta_j^\alpha \varphi(\delta_j)\frac{c}{2D\psi(1/n)} \geq (1 - 2^{-\alpha})\frac{\|f_i(1) - f_i(0)\|_f}{n^\alpha}$$

which implies

$$\|f_i((k+1)/n) - f_i(k/n)\|_f = \sum_{j<l} \delta_j^\alpha \varphi(\delta_j) \geq D\frac{\psi(1/n)}{n^\alpha} = Dg(1/n),$$

as required. $\square$

For every $0 < D < \omega$ let N_D be as in Lemma 4.6. Since $g(0) = 0$ and g is continuous, by reassigning N_D we can assume $g(1/2^{b_{N_D}}) \leq 1/D^2$ $(0 < D < \omega)$. Let $I_D \subseteq \omega \setminus N_D$ $(0 < D < \omega)$ be pairwise disjoint sets such that $1/D \leq |I_D|Dg(1/2^{b_{N_D}}) < 2/D$. For every $0 < D < \omega$ and $i \in I_D$ pick a $0 \leq k_{i,D} < 2^{b_{N_D}}$ satisfying (21). Define $x, \hat{x} \in Z'$ by $x(i) = k_{i,D}/2^{b_{N_D}}$, $\hat{x}(i) = (k_{i,D}+1)/2^{b_{N_D}}$ $(i \in I_D,\ 0 < D < \omega)$, else $x(i) = \hat{x}(i) = 0$. Then

$$\begin{aligned}\|\vartheta'(\hat{x}) - \vartheta'(x)\|_f &= \sum_{0<D<\omega} \sum_{i \in I_D} \|f_i((k_{i,D}+1)/2^{b_{N_D}}) - f_i(k_{i,D}/2^{b_{N_D}})\|_f \\ &\geq \sum_{0<D<\omega} D|I_D|g(1/2^{b_{N_D}}) = \infty\end{aligned}$$

while

$$\|\hat{x} - x\|_g = \sum_{0<D<\omega} |I_D|g(1/2^{b_{N_D}}) < \sum_{0<D<\omega} \frac{2}{D^2} < \infty;$$

i.e. $x\mathbf{E}_g\hat{x}$ but $\vartheta'(x) \not\mathbf{E}_f \vartheta'(\hat{x})$. This contradiction completes the proof. $\square$

5 Applications.

In this section we construct several families of functions for which our reducibility and nonreducibility results can be applied. Let $1 \leq \alpha \leq \beta < \infty$, let $\varphi\colon (0,1] \to \mathbb{R}$, $\psi\colon [0,1] \to \mathbb{R}$ be continuous functions and set $f = \mathrm{Id}^\alpha \varphi$, $f(0) = 0$ and $g = \mathrm{Id}^\beta \psi$.

5.1 Definition of φ from ψ and κ.

In order to facilitate the checking of the conditions of Theorem 3.4, we may use the following approach. Instead of defining κ from φ and ψ, we may define φ from ψ and κ. To this end we set $\kappa(1/2^n) = \mu(n)/2^{n\alpha/\beta}$ $(n < \omega)$ where μ will be specified later. We assume $\mu(0) = \varphi(1) = \psi(1) = 1$. Then (8) and (9) read as

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^{n} 2^{(\alpha-\beta)(n-i)} \mu(i)^\beta \psi\left(\frac{2^{(1-\alpha/\beta)i}\mu(i)}{2^n}\right) \ (n < \omega), \tag{24}$$

$$\sum_{i=n}^{\infty} \frac{1}{2^{i\alpha}} \mu(i)^\beta \psi\left(\frac{\mu(i)}{2^{i\alpha/\beta}}\right) \leq L \sum_{i=0}^{n} 2^{i(\beta-\alpha)} \frac{\mu(i)^\beta}{2^{n\beta}} \psi\left(\frac{2^{(1-\alpha/\beta)i}\mu(i)}{2^n}\right). \tag{25}$$

Given μ and ψ, we can define $\varphi(1/2^n)$ $(n < \omega)$ by (24) and then extend φ to $(0,1]$ to be a continuous function which is affine on $[1/2^{n+1}, 1/2^n]$ $(n < \omega)$. When we say below "we define φ from μ, α, β and ψ," we mean this definition.

We show that for a φ defined this way, if there exist $\varepsilon > 0$, $M < \omega$ such that for $n > M$,

$$\varphi(1/2^i) \leq \varepsilon\varphi(1/2^j)\varphi(1/2^n) \Rightarrow i \geq j + n + 3 \ (i, j < \omega), \tag{26}$$

then (A_1) of Theorem 4.1 holds. Let $x, y \in (0, 1]$, say $1/2^{i+1} < x \leq 1/2^i$ and $1/2^{j+1} < y \leq 1/2^j$. We have

$$\varphi(x) \in [\varphi(1/2^{i+1}), \varphi(1/2^i)],\ \varphi(y) \in [\varphi(1/2^{j+1}), \varphi(1/2^j)],$$

thus $\varphi(x) \leq \varepsilon\varphi(y)\varphi(1/2^n)$ implies

$$\min\{\varphi(1/2^{i+1}), \varphi(1/2^i)\} \leq \varepsilon \max\{\varphi(1/2^{j+1}), \varphi(1/2^j)\}\varphi(1/2^n).$$

So by (26), for $n > M$ we have $i \geq n + j + 2$, which implies $x \leq y/2^{n+1}$, as required.

5.2 Explicit Examples.

We introduce a family of functions for which our theorems can be applied and whose growth order is easy to calibrate. For $n < \omega$, let $t_n\colon (0, 1] \to \mathbb{R}$,

$$t_n(x) = \underbrace{1 + \log(1 + \ldots \log}_{n}(1 - \log(x)) \ldots)\ (0 < x \leq 1).$$

For $\eta \in [0, 1)^{<\omega}$ we define $l_\eta\colon (0, 1] \to \mathbb{R}$, $l_\eta(x) = \prod_{i<|\eta|} t_i^{\eta_i}$ $(0 < x \leq 1)$; e.g.,

$$l_\emptyset(x) = 1,\ l_{(\eta_0)}(x) = (1 - \log(x))^{\eta_0},$$
$$l_{(\eta_0\eta_1)}(x) = (1 - \log(x))^{\eta_0}(1 + \log(1 - \log(x)))^{\eta_1},$$

etc. Let $<_{\text{lex}}$ denote the lexicographic order. We summarize some elementary properties of the functions l_η, which will be used in the sequel.

Lemma 5.1. *For every $\eta, \eta' \in [0, 1)^{<\omega}$ with $\eta <_{lex} \eta'$, $1 \leq \alpha < \infty$ and $\delta > 0$,*

(a) $1 \leq l_\eta(xy) \leq l_\eta(x)l_\eta(y)$ $(0 < x, y \leq 1)$;

(b) $l_\eta \circ \mathrm{Id}^\delta \approx l_\eta$ *and* $l_\eta \lesssim \mathrm{Id}^{-\delta}$;

(c) $l_\eta(1/2^{n+1}) - l_\eta(1/2^n) \leq 1$ *for every* $n < \omega$ *sufficiently large;*

(d) l_η *is continuous and strictly decreasing, moreover if* $\eta <_{\text{lex}} \eta'$, *then* $l_\eta/l_{\eta'}$ *is strictly increasing in a neighborhood of 0, so by* $l_\eta(x)/l_{\eta'}(x) > 0$ $(x > 0)$, $l_\eta/l_{\eta'}$ *is essentially increasing and* $\lim_{x\to+0} l_\eta(x)/l_{\eta'}(x) = 0$;

(e) $\mathrm{Id}^\delta l_\eta$ *is bounded and* $\lim_{x\to+0} x^\delta l_\eta(x)=0$*;*

(f) $f(x)=x^\delta l_\eta(x)$ $(0<x<1)$*,* $f(0)=0$ *is continuous, strictly increasing in a neighborhood of 0, so by* $f(x)>0$ $(x>0)$*,* f *is essentially increasing;*

(g) $f(x)=x^\alpha l_\eta(x)$ $(0<x<1)$*,* $f(0)=0$ *is continuous, satisfies* (R_1) *and* (R_2) *hence* $\mathbf{E}_f$ *is an equivalence relation;*

(h) $f(x)=x^\alpha/l_\eta(x)$ $(0<x<1)$*,* $f(0)=0$ *is continuous and strictly increasing, satisfies* (R_1) *and* (R_2) *hence* $\mathbf{E}_f$ *is an equivalence relation;*

(i) $\varphi=1/l_\eta$ *satisfies* (A_1) *of Theorem 4.1.*

Proof. It is enough to prove (a) for t_n $(n<\omega)$. We do this by induction on n. For $n=0$, the statement follows from

$$\begin{aligned}1 &\leq 1-\log(xy)=1-\log(x)-\log(y)\\ &\leq 1-\log(x)-\log(y)+\log(x)\log(y)=(1-\log(x))(1-\log(y)).\end{aligned}$$

Let now $n>1$; then $t_n=1+\log t_{n-1}$, hence $1\leq t_n$. By the inductive hypothesis,

$$\begin{aligned}t_n(xy)&=1+\log t_{n-1}(xy)\leq 1+\log t_{n-1}(x)+\log t_{n-1}(y)\\ &\leq (1+\log t_{n-1}(x))(1+t_{n-1}(y))=t_n(x)t_n(y),\end{aligned}$$

as required.

Similarly, it is enough to show (b) for t_n $(n<\omega)$; we use induction on n. For $n=0$, the first statement follows from $1-\log(x^\delta)=1-\delta\log(x)$ $(0<x\leq 1)$, while $t_0\lesssim \mathrm{Id}^{-\delta}$ is elementary analysis. Let now $n>1$; we have $t_n=1+\log t_{n-1}$. By the inductive hypothesis and $t_{n-1}\geq 1$, $1+\log(t_{n-1}\circ \mathrm{Id}^\delta)\approx 1+\log t_{n-1}$, so the first statement follows. Also by the inductive hypothesis, $1+\log t_{n-1}\lesssim 1-\delta\log \lesssim \mathrm{Id}^{-\delta}$, so the proof is complete.

We show $(l_\eta(1/2^{n+1})-l_\eta(1/2^n))_{n<\omega}$ is a null sequence; then (c) follows. By elementary analysis, for every $\delta\in[0,1)$ and $m<\omega$, $(t_m^\delta(1/2^{n+1})-t_m^\delta(1/2^n))_{n<\omega}$ is a null sequence. Since l_η is a finite product of t_m^δs, the statement follows.

Statements (d), (e) and (f) are elementary analysis. For (g), (R_1) is immediate; (R_2a) follows from $(x+y)^\alpha\lesssim x^\alpha+y^\alpha$ $(0\leq x,y\leq 1)$ and l_η being decreasing; while (R_2b) follows from $\mathrm{Id}^\alpha l_\eta$ being essentially increasing.

Consider now (h). Since l_η is strictly decreasing, $\mathrm{Id}^\alpha/l_\eta$ is strictly increasing. So (R_1) is immediate and (R_2b) holds. To see (R_2a), observe that by (a), for $0<v/2\leq u\leq v\leq 1$ we have

$$l_\eta(u)\leq l_\eta(v/2)\leq l_\eta(1/2)l_\eta(v).$$

So for $0 < x, y \leq 1$,

$$(x+y)^\alpha / l_\eta(x+y) \lesssim l_\eta(1/2)(x^\alpha / l_\eta(x) + y^\alpha / l_\eta(y)),$$

as required.

It remains to prove (i). It is enough to show that for every $n < \omega$,

$$l_\eta(x) \geq l_\eta(1/2) l_\eta(y) l_\eta(1/2^n) \Rightarrow x \leq \frac{y}{2^{n+1}} \ (i, j < \omega).$$

By (a), $l_\eta(y/2^{n+1}) \leq l_\eta(1/2) l_\eta(y) l_\eta(1/2^n)$, so since l_η is decreasing, the statement follows. □

Corollary 5.2. *Let $1 \leq \alpha < \infty$ and let $\eta, \eta' \in [0,1)^{<\omega}$ satisfy $\eta <_{lex} \eta'$.*

1. *The functions $\psi = l_\eta$, $g(x) = x^\alpha l_\eta(x)$ $(0 < x \leq 1)$, $g(0) = 0$ satisfy the conditions of Theorem 3.3.*

2. *The functions $\varphi(x) = 1/l_\eta(x)$, $\psi(x) = 1/l_{\eta'}(x)$ $(0 < x \leq 1)$, $\varphi(0) = \psi(0) = 0$ and $f = \mathrm{Id}^\alpha / l_\eta$, $g = \mathrm{Id}^\alpha / l_{\eta'}$ satisfy the conditions of Theorem 4.1.*

Proof. Statement 1 follows from (d), (e) and (g) of Lemma 5.1. For 2, $\mathbf{E}_f$ and $\mathbf{E}_g$ are equivalence relations by (h) of Lemma 5.1; while (A_1) and (A_2) follow from (i) and (d) of Lemma 5.1. This completes the proof. □

5.3 The Counterintuitive Case.

In this section we present an example illustrating that the comparison of the growth order of functions does not decide Borel reducibility. Let $\alpha = \beta$ and $\psi \equiv 1$. Then (24) turns to $\varphi(1/2^n) = \sum_{i=0}^n \mu(i)^\alpha$; i.e.

$$\mu(n)^\alpha = \varphi(1/2^n) - \varphi(1/2^{n-1}) \ (0 < n < \omega), \tag{27}$$

and (25) reads as

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \leq L\varphi(1/2^n). \tag{28}$$

Since $\mu(n)^\alpha \leq \varphi(1/2^n)$, (28) holds if

$$\sum_{i=0}^{\infty} 1/2^{i\alpha} \varphi(1/2^{n+i}) \leq L\varphi(1/2^n). \tag{29}$$

Corollary 5.3. *Let $\varphi\colon (0,1] \to (0,+\infty)$ be an essentially decreasing continuous function such that $\mathrm{Id}^\alpha\varphi$ is essentially increasing, $\mathbf{E}_{\mathrm{Id}^\alpha\varphi}$ is an equivalence relation, for every $\delta > 0$, $\liminf_{x\to+0} x^\delta\varphi(x) = 0$ and (29) holds. Then $\mathbf{E}_{\mathrm{Id}^\alpha}$ and $\mathbf{E}_{\mathrm{Id}^\alpha\varphi}$ are Borel equivalent.*

PROOF. By Theorem 3.3, $\mathbf{E}_{\mathrm{Id}^\alpha} \leq_B \mathbf{E}_{\mathrm{Id}^\alpha\varphi}$. By Lemma 3.2, we can assume in addition that φ is decreasing. Then the definition of μ in (27) is valid. So by Theorem 3.4, $\mathbf{E}_{\mathrm{Id}^\alpha\varphi} \leq_B \mathbf{E}_{\mathrm{Id}^\alpha}$. $\square$

We show that (29) holds if for some $\varepsilon > 0$, $\mathrm{Id}^{\alpha-\varepsilon}\varphi$ is essentially increasing. Then

$$1/2^{(n+i)(\alpha-\varepsilon)}\varphi(1/2^{n+i}) \lesssim 1/2^{n(\alpha-\varepsilon)}\varphi(1/2^n)\ (i < \omega);$$

i.e. $1/2^{i\alpha}\varphi(1/2^{n+i}) \lesssim 1/2^{i\varepsilon}\varphi(1/2^n)$ $(i < \omega)$, so the statement follows. In particular, by Corollary 5.2.1 and by (d), (f) and (g) of Lemma 5.1, $\varphi = l_\eta$ fulfills these requirements for every $\eta \in [0,1)^{<\omega}$, that is $\mathbf{E}_{\mathrm{Id}^\alpha l_\eta}$ and $\mathbf{E}_{\mathrm{Id}^\alpha}$ are Borel equivalent. We will see below in (33) that for every $\eta \in [0,1)^{<\omega}$, $\mathbf{E}_{\mathrm{Id}^\alpha} <_B \mathbf{E}_{\mathrm{Id}^\alpha/l_\eta}$. So the comparison of the growth order of functions does not decide Borel reducibility.

5.4 The $\alpha < \beta$ Case.

Since the previous and following sections contain the analysis of the reducibility of $\mathbf{E}_{\mathrm{Id}^\beta}$ to $\mathbf{E}_{\mathrm{Id}^\beta\psi}$, in the $\alpha < \beta$ case we assume $\psi \equiv 1$. Then (24) and (25) turn to

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^{n} 2^{(\alpha-\beta)(n-i)}\mu(i)^\beta\ (n < \omega), \tag{30}$$

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}}\mu(n+i)^\beta \leq L\sum_{i=0}^{n} 2^{(\alpha-\beta)(n-i)}\mu(i)^\beta. \tag{31}$$

To satisfy (30), we have to define

$$\mu(n)^\beta = \varphi(1/2^n) - \varphi(1/2^{n-1})/2^{\beta-\alpha}\ (0 < n < \omega), \tag{32}$$

and then (31) follows from (29).

Corollary 5.4. *Let $1 \leq \alpha < \beta < \infty$. Suppose $\varphi\colon [0,1] \to \mathbb{R}^+$ is continuous, essentially increasing, $\varphi/\mathrm{Id}^{\beta-\alpha}$ is essentially decreasing and $\mathbf{E}_{\mathrm{Id}^\alpha\varphi}$ is an equivalence relation. Then $\mathbf{E}_{\mathrm{Id}^\alpha\varphi} \leq_B \mathbf{E}_{\mathrm{Id}^\beta}$.*

PROOF. By Lemma 3.2, we can assume $\varphi/\mathrm{Id}^{\beta-\alpha}$ is decreasing, so that (32) is valid; while φ being essentially increasing implies (29). So $\mathbf{E}_{\mathrm{Id}^\alpha\varphi} \leq_B \mathbf{E}_{\mathrm{Id}^\beta}$ follows from Theorem 3.4. $\square$

The assumptions of Corollary 5.4 are affordable:

- if φ is essentially decreasing, Corollary 5.3 gives the Borel equivalence of $\mathbf{E}_{\mathrm{Id}^\alpha\varphi}$ and $\mathbf{E}_{\mathrm{Id}^\beta}$ under suitable assumptions;

- in order not to be in the counterintuitive case, we may assume that $\varphi/\mathrm{Id}^{\beta-\alpha-\delta}$ is decreasing for some $\delta > 0$, so by Corollary 5.4, $\mathbf{E}_{\mathrm{Id}^\alpha\varphi} \leq_B \mathbf{E}_{\mathrm{Id}^{\beta-\delta}} <_B \mathbf{E}_{\mathrm{Id}^\beta}$;

So Corollary 5.4 indicates that in the $\alpha < \beta$ case growth order decides Borel reducibility. Moreover, in the next section we will see that in order to guarantee $\mathbf{E}_{\mathrm{Id}^\alpha\varphi} \leq_B \mathbf{E}_{\mathrm{Id}^\beta}$ by growth order estimates, we need $\mathrm{Id}^\beta/(\mathrm{Id}^\alpha\varphi)$ to be bounded; the assumptions of Corollary 5.4 reflect this constraint.

By (d), (f) and (h) of Lemma 5.1, the function $\varphi(0) = 0$, $\varphi(x) = 1/l'_\eta(x)$ $(0 < x \leq 1)$ satisfies the assumptions of Corollary 5.4, so

$$\mathbf{E}_{\mathrm{Id}^\alpha/l_\eta} \leq_B \mathbf{E}_{\mathrm{Id}^\gamma} <_B \mathbf{E}_{\mathrm{Id}^\beta} \ (\eta \in [0,1)^{<\omega},\ 1 \leq \alpha < \gamma < \beta < \infty). \tag{33}$$

5.5 The $\alpha = \beta$ Case.

This is the most interesting case for us. Now (24) and (25) turn to

$$\varphi\left(\frac{1}{2^n}\right) = \sum_{i=0}^{n} \mu(i)^\alpha \psi\left(\frac{\mu(i)}{2^n}\right) \ (n < \omega), \tag{34}$$

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \psi\left(\frac{\mu(n+i)}{2^{n+i}}\right) \leq L \sum_{i=0}^{n} \mu(i)^\alpha \psi\left(\frac{\mu(i)}{2^n}\right) \ (n < \omega). \tag{35}$$

We obtain a sufficient condition for (35).

Lemma 5.5. *Assume ψ is essentially increasing, $\psi(x) > 0$ for $x > 0$ and $\mu(n) \leq 1$ for every $n < \omega$ sufficiently large. Then (35) holds.*

PROOF. Since ψ is essentially increasing, for every n sufficiently large we have $\psi(\mu(n+i)/2^{n+i}) \lesssim \psi(1/2^n)$ $(0 \leq i < \omega)$. Hence

$$\sum_{i=0}^{\infty} \frac{1}{2^{i\alpha}} \mu(n+i)^\alpha \psi\left(\frac{\mu(n+i)}{2^{n+i}}\right) \lesssim \frac{1}{(1-1/2^\alpha)} \psi(1/2^n) \ (n < \omega), \tag{36}$$

thus by $\mu(0) = 1$, (35) follows. $\square$

5.5.1 The Question of S. Gao.

In this section, in the spirit of (3), we give the negative answer to the question of S. Gao mentioned in the introduction.

Corollary 5.6. *Let $1 \leq \alpha < \infty$ be arbitrary. Let $\mu\colon \omega \to [0,\infty)$ be such that $\mu(0) = 1$. Let $\psi\colon [0,1] \to [0,\infty)$ be a continuous essentially increasing function such that $\psi(1) = 1$, (35) holds and there is a $K > 0$ for which*

$$\frac{1}{K}\psi(1/2^n) \leq \psi\left(\frac{\mu(i)}{2^n}\right) \leq K\psi(1/2^n) \ (0 \leq i \leq n < \omega). \tag{37}$$

Set $\sigma_{\mu^\alpha}(n) = \sum_{i=1}^{n} \mu^\alpha(i)$ $(n < \omega)$. Suppose $((1+\sigma_{\mu^\alpha}(n))\,\psi\,(1/2^n))_{n<\omega}$ is essentially decreasing.

Define φ from μ, α and ψ. Set $f(x) = x^\alpha\varphi(x)$ $(0 < x \leq 1)$, $f(0) = 0$ and $g = \mathrm{Id}^\alpha\psi$ and suppose $\mathbf{E}_f$ and $\mathbf{E}_g$ are equivalence relations. Then $\mathbf{E}_f \leq_B \mathbf{E}_g$.

If, in addition, φ satisfies A_1 of Theorem 4.1, or equivalently φ satisfies (26), and $\lim_{n\to\infty} \sigma_{\mu^\alpha}(n) = \infty$, then $\mathbf{E}_g \not\leq_B \mathbf{E}_f$.

PROOF. By (37), from (34) we get

$$\frac{1}{K}\left(1+\sigma_{\mu^\alpha}(n)\right)\psi\left(\frac{1}{2^n}\right) \leq \varphi\left(\frac{1}{2^n}\right) \leq K\left(1+\sigma_{\mu^\alpha}(n)\right)\psi\left(\frac{1}{2^n}\right). \tag{38}$$

Since $((1+\sigma_{\mu^\alpha}(n))\psi(1/2^n))_{n<\omega}$ is essentially decreasing, φ is essentially increasing. So by Theorem 3.4, $\mathbf{E}_f \leq_B \mathbf{E}_g$.

Moreover, if φ satisfies A_1 of Theorem 4.1, which follows e.g. if φ satisfies (26), then since $\lim_{n\to\infty}\sigma_{\mu^\alpha}(n) = \infty$ implies (A_2) of Theorem 4.1, $\mathbf{E}_g \not\leq_B \mathbf{E}_f$. This completes the proof. □

Many natural functions satisfy the conditions of Corollary 5.6 for both φ and ψ, in particular the functions $1/l_\eta$. By Lemma 2.3, the following result gives the negative answer to the question of S. Gao.

Corollary 5.7. *For every $1 \leq \alpha < \beta < \infty$ and $\eta, \eta' \in [0,1)^{<\omega}$ with $\eta <_{lex} \eta'$,*

$$\mathbf{E}_{\mathrm{Id}^\alpha} <_B \mathbf{E}_{\mathrm{Id}^\alpha/l_\eta} <_B \mathbf{E}_{\mathrm{Id}^\alpha/l_{\eta'}} <_B \mathbf{E}_{\mathrm{Id}^\beta}.$$

PROOF. By Lemma 5.1 (h), $\mathbf{E}_{\mathrm{Id}^\alpha/l_\eta}$ is an equivalence relation, and in (33) we obtained $\mathbf{E}_{\mathrm{Id}^\alpha/l_\eta} <_B \mathbf{E}_{\mathrm{Id}^\beta}$. By Lemma 5.1 (d), $(l_{\eta'}(1/2^n)/l_\eta(1/2^n))_{n<\omega}$ is strictly increasing for n sufficiently large. Thus there is a function $\mu\colon \omega \to \mathbb{R}^+$ such that $\mu(0) = 1$, and for every $n < \omega$ sufficiently large,

$$\mu^\alpha(n) = l_{\eta'}(1/2^n)/l_\eta(1/2^n) - l_{\eta'}(1/2^{n-1})/l_\eta(1/2^{n-1}).$$

Let $\psi = 1/l_{\eta'}$ $(0 < x \le 1)$, $\psi(0) = 0$ and define φ from μ, α and ψ. We check the conditions of Corollary 5.6.

First we show that for every $\varepsilon > 0$,

$$2^{-n\varepsilon} \le \mu^\alpha(n) \le 1 \tag{39}$$

holds for n sufficiently large. By Lemma 5.1 (a), (c) and (d), for every n sufficiently large,

$$\begin{aligned}\mu^\alpha(n) &= \frac{l_{\eta'}(1/2^n)}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})}\\ &= \frac{l_{\eta'}(1/2^n) - l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} + \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})}\\ &\le \frac{l_{\eta'}(1/2^n) - l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^n)} \le \frac{1}{l_\eta(1/2^n)} \le 1.\end{aligned}$$

For the lower bound, take an $m > |\eta'|$ and consider t_m. By Lemma 5.1 (d), $l_{\eta'}(1/2^n)/(l_\eta(1/2^n)t_m(1/2^n))$ is still strictly increasing for n sufficiently large. So for n sufficiently large,

$$\mu^\alpha(n) = \frac{l_{\eta'}(1/2^n)}{l_\eta(1/2^n)} - \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} \ge \frac{l_{\eta'}(1/2^{n-1})}{l_\eta(1/2^{n-1})} \frac{t_m(1/2^n) - t_m(1/2^{n-1})}{t_m(1/2^{n-1})}.$$

It is elementary analysis that $t_m(1/2^n) - t_m(1/2^{n-1}) \ge 1/n^2$ for n sufficiently large, so the statement follows.

By Lemma 5.1 (d), ψ is continuous, essentially increasing and $\psi(1) = 1$. Lemma 5.5 gives (35). Also, (37) follows from Lemma 5.1 (b) using that $2^{-n/2} \le \mu^\alpha(n) \le 2^{n/2}$ holds for n sufficiently large.

We have

$$(1 + \sigma_{\mu^\alpha}(n))\,\psi\,(1/2^n) \approx 1/l_\eta(1/2^n)\ (n < \omega),$$

so $((1 + \sigma_{\mu^\alpha}(n))\,\psi\,(1/2^n))_{n<\omega}$ is essentially decreasing.

By (38),

$$\varphi\left(\frac{1}{2^n}\right) \approx (1 + \sigma_{\mu^\alpha}(n))\psi\left(\frac{1}{2^n}\right) \approx l_{\eta'}(1/2^n)/l_\eta(1/2^n)\psi\left(\frac{1}{2^n}\right) \approx 1/l_\eta(1/2^n),$$

so by Corollary 5.6, $\mathbf{E}_{\mathrm{Id}^\alpha/l_\eta} \le_B \mathbf{E}_{\mathrm{Id}^\alpha/l_{\eta'}}$.

By Lemma 5.1 (d), $\lim_{x\to+0} l_{\eta'}(x)/l_\eta(x) = \infty$; i.e. $\lim_{n\to\infty} \sigma_{\mu^\alpha}(n) = \infty$. By Lemma 5.1 (i), $1/l_\eta$ satisfies A_1 of Theorem 4.1, so again by Corollary 5.6, $\mathbf{E}_{\mathrm{Id}^\alpha/l_{\eta'}} \not\le_B \mathbf{E}_{\mathrm{Id}^\alpha/l_\eta}$. The $\eta = \emptyset$ special case gives $\mathbf{E}_{\mathrm{Id}^\alpha} <_B \mathbf{E}_{\mathrm{Id}^\alpha/l_\eta}$, so the proof is complete. $\square$

5.5.2 Embedding Long Linear Orders.

In this section we show that every linear order which can be embedded into $(\mathcal{P}(\omega)/\mathrm{fin}, \subset)$ also embeds into the set of Borel equivalence relations $\mathbf{E}_f$ satisfying $\mathbf{E}_{\mathrm{Id}^\alpha} \le_B \mathbf{E}_f \le_B \mathbf{E}_{\mathrm{Id}^\alpha/(1-\log)}$ ordered by $<_B$. We refer to [1] for results on embedding ordered sets into $(\mathcal{P}(\omega)/\mathrm{fin}, \subset)$, here we only remark that it is consistent with ZFC, e.g. under the Continuum Hypothesis, that every ordered set of size continuum embeds into $(\mathcal{P}(\omega)/\mathrm{fin}, \subset)$.

Corollary 5.8. *Let $1 \le \alpha < \infty$ be fixed. There is a mapping $\mathcal{F}\colon \mathcal{P}(\omega)/\mathrm{fin} \to C[0,1]$ such that for every $U, V \in \mathcal{P}(\omega)/\mathrm{fin}$, $\mathbf{E}_{\mathcal{F}(U)}$ is an equivalence relation satisfying $\mathbf{E}_{\mathrm{Id}^\alpha} \le_B \mathbf{E}_{\mathcal{F}(U)} \le_B \mathbf{E}_{\mathrm{Id}^\alpha/(1-\log)}$ and $U \subset V \Rightarrow \mathbf{E}_{\mathcal{F}(V)} <_B \mathbf{E}_{\mathcal{F}(U)}$.*

PROOF. Let $\gamma = 17/16$. For every $U \in \mathcal{P}(\omega)$ set

$$\mu_U(0) = 1,\ \mu_U^\alpha(n) = \gamma^{|U \cap \lfloor 1+\log(n) \rfloor|}\ (0 < n < \omega).$$

Let $\psi_0(x) = 1/(1-\log(x))^2$ $(0 < x \le 1)$, $\psi_0(0) = 0$. For every $U \in \mathcal{P}(\omega)$ we define φ_U from μ_U, α and ψ_0, and we set $\mathcal{F}(U) = \mathrm{Id}^\alpha \varphi_U$.

First we show that for every $U \in \mathcal{P}(\omega)$, μ_U and ψ_0 satisfy (37). By definition, $1 \le \mu_U^\alpha(n) \le \gamma^{1+\log(n)} \le \gamma n$ $(0 < n < \omega)$, so (37) follows.

Next we show that for every $U \in \mathcal{P}(\omega)$, φ_U is essentially increasing. Since (37) holds, by (38) it is enough to show that $((1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n))_{n<\omega}$ is essentially decreasing. We have $\psi_0(1/2^n) \approx 1/n^2$ $(0 < n < \omega)$. Let $0 < n < m < \omega$ be fixed, say $m = \rho n$ for some $\rho > 1$. If $\mu_U^\alpha(n) = \gamma^k$, then $\sigma_{\mu_U^\alpha}(n) \ge n\gamma^{k-1}/2$, and

$$\sigma_{\mu_U^\alpha}(m) \le \sigma_{\mu_U^\alpha}(n) + (m-n)\gamma^{k+1+\log(m)-\log(n)} \le \sigma_{\mu_U^\alpha}(n) + (\rho-1)n\gamma^{k+1+\log(\rho)}.$$

Hence

$$\begin{aligned} \frac{\sigma_{\mu_U^\alpha}(m)}{m^2} &\le \frac{\sigma_{\mu_U^\alpha}(n) + (\rho-1)n\gamma^{k+1+\log(\rho)}}{(\rho n)^2} \le \frac{\sigma_{\mu_U^\alpha}(n)}{n^2} + \frac{\gamma^{k+1+\log(\rho)}}{\rho n} \\ &\le \frac{\sigma_{\mu_U^\alpha}(n)}{n^2}\left(1 + \frac{2\gamma^{2+\log(\rho)}}{\rho}\right) \le 9\frac{\sigma_{\mu_U^\alpha}(n)}{n^2}. \end{aligned}$$

This shows $((1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n))_{n<\omega}$ is essentially decreasing.

Next we check that for every $U \in \mathcal{P}(\omega)$, $\mathbf{E}_{\mathcal{F}(U)}$ is an equivalence relation. By definition, (R_1) holds; (R_2a) holds for $\mathrm{Id}^\alpha\psi_0$ with $C = 8\alpha$, so since φ_U/ψ_0 is decreasing, (R_2a) holds for $\mathrm{Id}^\alpha\varphi_U$, as well. Finally (R_2b) follows from $\mathrm{Id}^\alpha\varphi_U$ is essentially increasing.

Our task is to prove that if $U, V \in \mathcal{P}(\omega)$ satisfy $U \subseteq^\star V$, $|V \setminus U| = \infty$, then $\mathbf{E}_{\mathcal{F}(V)} <_B \mathbf{E}_{\mathcal{F}(U)}$. Observe that if $U, U' \in \mathcal{P}(\omega)$ differ only by a finite set, then $\mathcal{F}(U) \approx \mathcal{F}(U')$ hence $\mathbf{E}_{\mathcal{F}(U)} = \mathbf{E}_{\mathcal{F}(U')}$. So we can assume $U \subseteq V$, $0 \in V \setminus U$.

Our strategy is to show that $\varphi = \varphi_V$ can be obtained from $\psi = \varphi_U$ as in (34) with a μ satisfying the assumptions of Corollary 5.6. Set $\mu(0) = 1$,

$$\mu^\alpha(n+1) = \frac{1 + \sigma_{\mu_V^\alpha}(n+1)}{1 + \sigma_{\mu_U^\alpha}(n+1)} - \frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_U^\alpha}(n)} \ (n < \omega).$$

Later on we will prove

$$\frac{\gamma - 1}{(n+2)^3} \leq \mu^\alpha(n) \leq 1 \ (n < \omega); \tag{40}$$

now we assume (40) and verify the conditions of Corollary 5.6.

We have φ_U is continuous and $\varphi_U(0) = 1$. As we have seen above, φ_U is essentially increasing. By $\mu \leq 1$, Lemma 5.5 gives (35). By (40),

$$\varphi_U\left(\frac{1}{2^{n+3\lfloor\log(n+2)\rfloor+7}}\right) \lesssim \varphi_U\left(\frac{\mu(i)}{2^n}\right) \leq \varphi_U\left(\frac{1}{2^n}\right) \ (0 \leq i \leq n < \omega),$$

so (37) follows from

$$\begin{aligned} \varphi_U\left(1/2^n\right) &\approx (1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n) \\ &\approx (1 + \sigma_{\mu_U^\alpha}(n + 3\lfloor\log(n+2)\rfloor + 7))\psi_0(1/2^{n+3\lfloor\log(n+2)\rfloor+7}) \\ &\approx \varphi_U\left(1/2^{n+3\lfloor\log(n+2)\rfloor+7}\right). \end{aligned}$$

Let φ be defined from μ, α and φ_U. We have

$$1 + \sigma_{\mu^\alpha}(n) = (1 + \sigma_{\mu_V^\alpha}(n))/(1 + \sigma_{\mu_U^\alpha}(n)) \ (n < \omega),$$

so by (38),

$$\begin{aligned} \varphi\left(\frac{1}{2^n}\right) &\approx \frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_U^\alpha}(n)}\varphi_U(n) \\ &\approx \frac{1 + \sigma_{\mu_V^\alpha}(n)}{1 + \sigma_{\mu_U^\alpha}(n)}(1 + \sigma_{\mu_U^\alpha}(n))\psi_0\left(\frac{1}{2^n}\right) = \varphi_V\left(\frac{1}{2^n}\right). \end{aligned}$$

Thus $\mathbf{E}_{\mathcal{F}(V)} = \mathbf{E}_{\mathrm{Id}^\alpha\varphi}$; and $((1 + \sigma_{\mu^\alpha}(n))\,\psi\,(1/2^n))_{n<\omega}$ is essentially decreasing. So by Corollary 5.6, $\mathbf{E}_{\mathcal{F}(V)} \leq_B \mathbf{E}_{\mathcal{F}(U)}$.

Observe that ψ_0 satisfies (26) with $M = 0$ and $\varepsilon = 1/8$. Since $(1 + \sigma_{\mu_U^\alpha}(n))_{n<\omega}$ is increasing, $\varphi_U(1/2^n) \approx (1 + \sigma_{\mu_U^\alpha}(n))\psi_0(1/2^n)$ $(n < \omega)$ also satisfies (26) with the same M and a smaller ε. Thus φ_U satisfies A_1 of Theorem 4.1.

Since $|U \cap \log(n)| + k \leq |V \cap \log(n)|$ implies $\gamma^k \mu_U^\alpha(n) \leq \mu_V^\alpha(n)$, we have

$$\lim_{n\to\infty} (1+\sigma_{\mu_U^\alpha}(n))/(1+\sigma_{\mu_V^\alpha}(n)) = 0,$$

hence $\lim_{n\to\infty} \sigma_{\mu^\alpha}(n) = \infty$. So again by Corollary 5.6, $\mathbf{E}_{\mathcal{F}(U)} \not\leq_B \mathbf{E}_{\mathcal{F}(V)}$. For $U = \emptyset$ and $V = \omega$, $\varphi_U \approx 1/(1-\log)$ and $\varphi_V \approx 1$, so $\mathbf{E}_{\mathrm{Id}^\alpha} \leq_B \mathbf{E}_{\mathcal{F}(U)} \leq_B \mathbf{E}_{\mathrm{Id}^\alpha/(1-\log)}$ $(U \in \mathcal{P}(\omega))$.

It remains to prove (40). For $n = 1$, $\mu^\alpha(1) = (1+\gamma)/2 - 1$; for $n = 2$, $\mu^\alpha(2) = (1+2\gamma)/3 - (1+\gamma)/2$. So (40) holds for $n = 1, 2$. Let $n \geq 2$; then

$$1+\sigma_{\mu_V^\alpha}(n) = 1+\gamma+a_n,\ 1+\sigma_{\mu_U^\alpha}(n) = 2+b_n$$

and

$$1+\sigma_{\mu_V^\alpha}(n+1) = 1+\gamma+a_n+\gamma^c,\ 1+\sigma_{\mu_U^\alpha}(n+1) = 2+b_n+\gamma^d,$$

where $c \geq d+1$ and $\gamma \leq a_n/b_n \leq \gamma^{c-d}$ $(1 < n < \omega)$. Then for every $2 \leq n < \omega$,

$$\begin{aligned}
\frac{1+\sigma_{\mu_V^\alpha}(n+1)}{1+\sigma_{\mu_U^\alpha}(n+1)} - \frac{1+\sigma_{\mu_V^\alpha}(n)}{1+\sigma_{\mu_U^\alpha}(n)} &= \frac{1+\gamma+a_n+\gamma^c}{2+b_n+\gamma^d} - \frac{1+\gamma+a_n}{2+b_n} \\
&= \frac{\gamma^c - \gamma^d \frac{1+\gamma+a_n}{2+b_n}}{2+b_n+\gamma^d} = \frac{\gamma^c - \gamma^d \left(\frac{a_n}{b_n} - \frac{\frac{2a_n}{b_n}-(1+\gamma)}{2+b_n}\right)}{2+b_n+\gamma^d} \\
&\geq \frac{\gamma^c - \gamma^d \left(\gamma^{c-d} - \frac{2\gamma-(1+\gamma)}{2+b_n}\right)}{2+b_n+\gamma^d} = \gamma^d \frac{\gamma-1}{(2+b_n)(2+b_n+\gamma^d)}. \qquad (41)
\end{aligned}$$

We have $d \leq \lfloor \log(n) \rfloor$, so $b_n \leq n\gamma^d \leq n^2$ $(2 \leq n < \omega)$. So (41) can be estimated from below by

$$\frac{\gamma-1}{(2/\gamma^d + b_n/\gamma^d)(2+b_n+\gamma^d)} \geq \frac{\gamma-1}{(2/\gamma^d+n)(2+n^2+n)} \geq \frac{\gamma-1}{(n+2)^3},$$

as stated.

For the upper bound, as we have seen in (41), it is enough to show

$$\gamma^c - \gamma^d \frac{1+\gamma+a_n}{2+b_n} \leq 2+b_n+\gamma^d\ (2 \leq n < \omega).$$

Since $c \leq 1+\log(n+1)$ and $n-2 \leq b_n$,

$$\gamma^c \leq \gamma(n+1)^{\log(\gamma)} \leq n \leq 2+b_n\ (2 \leq n < \omega)$$

for our γ. This completes the proof. $\square$

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