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ON MONOTONE PRESENTATIONS OF BOREL SETS

Abstract

If A is a $\mathbf{\Sigma}^0_\xi$ set and A_n $(n < \omega)$ are Borel sets then we call $\{A_n : n < \omega\}$ a presentation of A if $A = \bigcup_{n<\omega} A_n$ and A_n $(n < \omega)$ have lower Borel class than A has. We show that for $2 \leq \xi < \omega_1$ it is not possible to assign a presentation to $\mathbf{\Sigma}^0_\xi$ sets in a monotone way; i.e., it is not possible to define functions $f_n \colon \mathbf{\Sigma}^0_\xi \to \mathbf{\Pi}^0_\xi$ $(n < \omega)$ such that for every $A \in \mathbf{\Sigma}^0_\xi$ we have $A = \bigcup_{n<\omega} f_n(A)$ and $A, A' \in \mathbf{\Sigma}^0_\xi$, $A \subseteq A'$ implies $f_n(A) \subseteq f_n(A')$ $(n < \omega)$. This answers a question of Márton Elekes in the negative. We also show the nonexistence of monotone presentation for Borel functions.

1 Introduction.

If a set A is of Borel class $\mathbf{\Sigma}^0_\xi$ then by definition there exists $\{A_n : n < \omega\} \subseteq \bigcup_{\vartheta<\xi} \mathbf{\Pi}^0_\vartheta$ such that $A = \bigcup_{n<\omega} A_n$. A natural question arises: can the sets $\{A_n : n < \omega\}$ be assigned to A in a "canonical" way?

Ironically, up to our knowledge, the answer to this question was always negative when "canonical" meant something "useful". In [6] G. Hjorth obtained that for every $\eta < \xi < \omega_1$ there is no injection $f \colon \mathbf{\Pi}^0_\xi \to \mathbf{\Pi}^0_\eta$ which

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is Borel in a fixed, natural coding of Borel sets. Roughly speaking, this is because there are “many more” $\mathbf{\Pi}^0_\xi$ sets than $\mathbf{\Pi}^0_\eta$ sets (see [6, Corollary 4.4, p. 677]). The cardinality concept of [6] was further developed in [7] and [1]; in particular in [1], under the assumption of $AD + DC(\mathbb{R})$, the nonexistence of bijection between various classes of the Wadge hierarchy is obtained.

In [5] the authors examine whether there exist representing sets with given sections for certain families of sets; i.e., whether for given Polish spaces X and Y, pointclass $\mathbf{\Gamma}$ and $\mathcal{C}_1, \mathcal{C}_2 \subseteq 2^Y$ there exists $R \in \mathbf{\Gamma}(X \times Y)$ satisfying

- $\forall x \in X\colon\ \{y \in Y\colon (x,y) \in R\} \in \mathcal{C}_1$ and
- $\forall C \in \mathcal{C}_2\ \exists x \in X\colon\ C = \{y \in Y\colon (x,y) \in R\}$.

Apart from several independence results, in [5, Theorem 5.4, p. 957] it is shown that for every $\alpha < \omega_1$ there is no Borel set with $\mathbf{\Delta}^0_{\alpha+1}$ sections which represents the $\mathbf{\Delta}^0_{\alpha+1}$ sets.

In the present note we provide another situation where the answer to the “canonical presentation” question is negative, namely we show that for $\xi \geq 2$ no monotone presentations exist, as follows.

Theorem 1. *Let (X,τ) be an uncountable Polish space and let ξ be an ordinal satisfying $2 \leq \xi < \omega_1$. Then there are no functions $f_n\colon \mathbf{\Sigma}^0_\xi(\tau) \to \mathbf{\Pi}^0_\xi(\tau)$ $(n < \omega)$ such that*

(M$_1$) *for every $Q, Q' \in \mathbf{\Sigma}^0_\xi(\tau)$, $Q \subseteq Q'$ implies $f_n(Q) \subseteq f_n(Q')$ $(n < \omega)$;*

(M$_2$) *for every $Q \in \mathbf{\Sigma}^0_\xi(\tau)$ we have $Q = \bigcup_{n<\omega} f_n(Q)$.*

Equivalently, for multiplicative Borel classes we have the following.

Corollary 2. *Let (X,τ) be an uncountable Polish space and let ξ be an ordinal satisfying $2 \leq \xi < \omega_1$. Then there are no functions $g_n\colon \mathbf{\Pi}^0_\xi(\tau) \to \mathbf{\Sigma}^0_\xi(\tau)$ $(n < \omega)$ such that*

$(M_1^\star)$ *for every $P, P' \in \mathbf{\Pi}^0_\xi(\tau)$, $P \subseteq P'$ implies $g_n(P) \subseteq g_n(P')$ $(n < \omega)$;*

$(M_2^\star)$ *for every $P \in \mathbf{\Pi}^0_\xi(\tau)$ we have $P = \bigcap_{n<\omega} g_n(P)$.*

Note that we do not assume any definability on the functions f_n. As we will see below, asking monotonicity is a very strong restriction and in fact, even much weaker monotonicity-like properties cannot be satisfied. The problem of the existence of monotone presentations was raised by Márton Elekes who in [3] studied whether measurable hulls can be assigned in a monotone way. It is important to point out that all the results obtained in [3] are consistencies:

they show the independence of ZFC of every variant of the problem of assigning measurable hulls in a monotone way. At the end of this note we will see that the problem of finding monotone presentations for Borel sets is strongly related to the problem whether a given partial order can be embedded into the partially ordered set of Borel sets with a given Borel class or into the partially ordered set of Borel functions with a given Baire class (see e.g. [2] or [9]). However, this problem is also independent of ZFC (see [4], [9] and [10]). In view of these facts it is surprising that we get the nonexistence of monotone presentations as a ZFC result. We remark that Theorem 1 in the special $\xi = 2$ case was first proved independently by Márton Elekes and Viktor Harangi.

As we mentioned above, presentations with much weaker monotonicity properties cannot be found. Instead of Theorem 1, we will prove the following theorem, which implies Theorem 1 and is the key element of the proof of the nonexistence of monotone presentations of Borel functions.

Theorem 3. *Let (X, τ) be an uncountable Polish space and let ξ be an ordinal satisfying $2 \leq \xi < \omega_1$. If the functions $f_n \colon \mathbf{\Sigma}^0_\xi(\tau) \to \mathbf{\Pi}^0_\xi(\tau)$ $(n < \omega)$ satisfy*

(M_3) *for every $n < \omega$, $Q \in \mathbf{\Sigma}^0_\xi(\tau)$ and $x \in Q$, $x \in f_n(\{x\})$ implies $x \in f_n(Q)$;*

(M_4) *for every $Q \in \mathbf{\Sigma}^0_\xi(\tau)$ we have $\bigcup_{n<\omega} \bigcap_{n \leq i < \omega} f_i(Q) \subseteq Q$;*

then there exists an $x \in X$ such that $\bigcup_{n<\omega} \bigcap_{n \leq i < \omega} f_i(\{x\}) = \emptyset$.

In Theorem 3, (M_1) of Theorem 1 is weakened to (M_3) requiring monotonicity only in relation of $\mathbf{\Sigma}^0_\xi$ sets and the points therein, and (M_2) is weakened to (M_4) requiring only $\liminf_{n\to\infty} f_n(Q) \subseteq Q$. As we will see at the end of this note, Theorem 3 has the following corollary, which is the natural formulation of Theorem 1 for Borel functions.

Corollary 4. *Let (X, τ) be an uncountable Polish space and let ξ be an ordinal satisfying $1 \leq \xi < \omega_1$. For $0 \leq \eta < \omega_1$ let $\mathcal{B}_\eta(X)$ denote the family of Baire-η functions $f \colon X \to \mathbb{R}$, starting with $\mathcal{B}_0(X)$=continuous functions. Then there are no mappings $\mathcal{F}_n \colon \mathcal{B}_\xi(X) \to \bigcup_{\vartheta<\xi} \mathcal{B}_\vartheta(X)$ $(n < \omega)$ satisfying*

(M_5) *for every $f, f' \in \mathcal{B}_\xi(X)$, $f \leq f'$ implies $\mathcal{F}_n(f) \leq \mathcal{F}_n(f')$ $(n < \omega)$;*

(M_6) *for every $f \in \mathcal{B}_\xi(X)$ we have $\lim_{n\to\infty} \mathcal{F}_n(f) = f$ pointwise.*

In the proof of Corollary 4 we will see how the existence of monotone presentations is related to the problem of embedding partial orders into the partially ordered set of Borel sets or functions. In particular, the $\xi = 1$ case of Corollary 4 will follow from some results in [2].

Our terminology and notation follow [8]. From now on (X, τ) denotes a Polish space. As usual, $\mathbf{\Pi}^0_\xi(\tau)$ and $\mathbf{\Sigma}^0_\xi(\tau)$ $(0 < \xi < \omega_1)$ stand for the ξ^{th} multiplicative and additive Borel classes in the Polish space (X, τ), starting with $\mathbf{\Pi}^0_1(\tau) =$ closed sets, $\mathbf{\Sigma}^0_1(\tau) =$ open sets.

2 Nonexistence of Monotone Presentations.

We start with the proof of Theorem 3.

Proof of Theorem 3. Let the functions $f_n\colon \mathbf{\Sigma}^0_\xi(\tau) \to \mathbf{\Pi}^0_\xi(\tau)$ $(n < \omega)$ satisfy (M_3) and (M_4). Since (X, τ) is an uncountable Polish space, there exists a set $D \in \mathbf{\Pi}^0_{\xi+1}(\tau) \setminus \mathbf{\Sigma}^0_{\xi+1}(\tau)$. We show that there exists an $x \in D$ such that $\bigcup_{n<\omega} \bigcap_{n \le i < \omega} f_i(\{x\}) = \emptyset$.

Suppose this is not the case; i.e.,

$$x \in \bigcup_{n<\omega} \bigcap_{n \le i < \omega} f_i(\{x\}) \text{ for every } x \in D. \tag{1}$$

Let $D_j \in \mathbf{\Sigma}^0_\xi(\tau)$ $(j < \omega)$ be such that $D = \bigcap_{j<\omega} D_j$ and set

$$D^- = \bigcap_{j<\omega} \bigcup_{n<\omega} \bigcap_{n \le i < \omega} f_i(D_j),$$

$$D^+ = \bigcup_{n<\omega} \bigcap_{j<\omega} \bigcap_{n \le i < \omega} f_i(D_j).$$

By (M_4) we have $D^- \subseteq D$. To see $D \subseteq D^+$, pick an $x \in D$. By (1), there is an $n_x < \omega$ such that $x \in \bigcap_{n_x \le i < \omega} f_i(\{x\})$. Since $x \in D_j$ $(j < \omega)$, by (M_3) we have $x \in \bigcap_{n_x \le i < \omega} f_i(D_j)$ for every $j < \omega$. So $x \in \bigcap_{j<\omega} \bigcap_{n_x \le i < \omega} f_i(D_j)$ which shows $x \in D^+$.

But $D^+ \subseteq D^-$ is obvious: if $x \in D^+$ then there is an $n_x < \omega$ for which $x \in \bigcap_{n_x \le i < \omega} f_i(D_j)$ for every $j < \omega$; in particular $x \in D^-$, as well. So we obtained $D = D^- = D^+$. Since D^+ is a $\mathbf{\Sigma}^0_{\xi+1}(\tau)$ set, this contradicts the choice of D and completes the proof. □

Next we show that Theorem 3 implies Theorem 1; then by taking complements, Corollary 2 follows.

Proof of Theorem 1. Suppose there are functions f_n $(n < \omega)$ satisfying both (M_1) and (M_2). For every $n < \omega$ and $Q \in \mathbf{\Sigma}^0_\xi(\tau)$ we define $\tilde{f}_n(Q) = \bigcup_{i \le n} f_i(Q)$. Then the functions $\tilde{f}_n$ $(n < \omega)$ satisfy (M_3) and (M_4) so by

Theorem 3, there is an $x \in X$ such that $\bigcup_{n<\omega} \bigcap_{n \le i < \omega} \tilde{f}_i(\{x\}) = \emptyset$. However, this implies $f_n(\{x\}) = \emptyset$ $(n < \omega)$, which contradicts (M$_2$). $\square$

It remains to prove Corollary 4 from Theorem 3. As we mentioned in the introduction, we prove the $\xi = 1$ case of Corollary 4 from Theorem 4.2 of [2]. We recall the results we need; the first one is folklore (see the introduction of [2]). In the second one, $\le_{\text{lex}}$ denotes the lexicographic order. For every $0 \le \xi < \omega_1$ we consider $\mathcal{B}_\xi(X)$ as a partially ordered set where for $f, g \in \mathcal{B}_\xi(X)$, $f \le g$ if and only if $f(x) \le g(x)$ $(x \in X)$, and $f < g$ if and only if $f \le g$ but $f \ne g$.

Proposition 5. *Let (X, τ) be an uncountable Polish space.*

1. *An ordered set $(O, \le)$ is similar to a subset of $(\mathcal{B}_0(X), \le)$ if and only if $(O, \le)$ is similar to a subset of $(\mathbb{R}, \le)$.*
2. *For all $\alpha < \omega_1$, $([0,1]^\alpha, \le_{lex})$ is similar to a subset of $(\mathcal{B}_1(X), \le)$.*

We remark that we will use Proposition 5.2 only in the special $\alpha = 2$ case. Then the mapping $(x, y) \mapsto f_{x,y}$ where

$$f_{x,y}(t) = \begin{cases} 1 \text{ if } t < x, \\ y \text{ if } t = x, \\ 0, \text{ if } t > x, \end{cases}$$

gives a particularly simple embedding of $([0,1]^2, \le_{\text{lex}})$ into $(\mathcal{B}_1(X), \le)$.

Proof of Corollary 4. Let first $2 \le \xi < \omega_1$ and suppose that $\mathcal{F}_n \colon \mathcal{B}_\xi(X) \to \bigcup_{\vartheta < \xi} \mathcal{B}_\vartheta(X)$ $(n < \omega)$ satisfy (M$_5$) and (M$_6$). For every $n < \omega$ and $\mathbf{\Sigma}^0_\xi(\tau)$ set $Q \subseteq X$ let

$$f_n(Q) = \{x \in X \colon\ [\mathcal{F}_n(\chi_Q)](x) \ge 1/2\}.$$

By [8, Chapter II, Theorem 24.3, p. 190], the definition makes sense and f_n maps $\mathbf{\Sigma}^0_\xi(\tau)$ into $\mathbf{\Pi}^0_\xi(\tau)$ $(n < \omega)$. Moreover by (M$_5$) and (M$_6$), $(f_n)_{n<\omega}$ satisfies (M$_3$) and (M$_4$). Hence there is a point $x \in X$ for which $\bigcap_{n \le i < \omega} f_i(\{x\}) = \emptyset$ $(n < \omega)$. This means $\mathcal{F}_n(\chi_{\{x\}}) \le 1/2$ holds for infinitely many $n < \omega$, which contradicts (M$_6$).

It remains to treat the $\xi = 1$ case. Let (X_n, τ_n) $(n < \omega)$ be disjoint copies of the Polish space (X, τ) and set $(Y, \sigma) = \bigcup_{n<\omega} (X_n, \tau_n)$. Consider the mapping $\mathcal{F} \colon \mathcal{B}_1(X) \to \mathcal{B}_0(Y)$,

$$\forall n < \omega\ \forall x \in X_n\ ([\mathcal{F}(f)](x) = [\mathcal{F}_n(f)](x)).$$

Then by (M_5) and (M_6), $\mathcal{F}$ is an order preserving injection. By Proposition 5.2 there is an ordered set $\mathcal{A} \subseteq (\mathcal{B}_1(X), \leq)$ which is similar to $([0,1]^2, \leq_{\mathrm{lex}})$. Then $\mathcal{F}(\mathcal{A}) \subseteq (\mathcal{B}_0(Y), \leq)$ is also similar to $([0,1]^2, \leq_{\mathrm{lex}})$, but by Proposition 5.1, it is similar to a subset of $\mathbb{R}$. This contradiction completes the proof. $\square$

Note that the previous argument, valid in the $\xi = 1$ case, is *consistently not applicable* for $2 \leq \xi < \omega_1$. It is an easy exercise that every partially ordered set of cardinality ω_1 can be embedded into $(\omega^\omega, \leq^*)$ (see [9]). Moreover, as observed in [9] (see also [4, 26Kf and 21Nb] and [10]), it is consistent with ZFC+$\neg$CH that every ordered set of cardinality continuum and every partially ordered set of cardinality less than continuum can be embedded into $(\omega^\omega, \leq^*)$. Since the mapping $f \mapsto \{g \in \omega^\omega : g \leq^* f\}$ is an embedding of $(\omega^\omega, \leq^*)$ into the partially ordered set $(\mathbf{\Sigma}^0_2(\tau_{\omega^\omega}), \subseteq)$ we get that consistently every (partial) order of cardinality continuum can be embedded into $(\mathcal{B}_2, \leq)$ using characteristic functions of $\mathbf{\Sigma}^0_2$ sets. In view of these facts it is somewhat surprising that we get the nonexistence of monotone presentations as a ZFC result.

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