Real Analysis Exchange
Summer Symposium 2002 pp. 495–496

Pieter C. Allaart, Department of Mathematics, University of North Texas, P.O.Box 311430, Denton, Texas 76203-1430. email: allaart@unt.edu
R. Daniel Mauldin, Department of Mathematics, University of North Texas, P.O.Box 311430, Denton, Texas 76203-1430. email: mauldin@unt.edu

A POLYNOMIAL FIXED-POINT PROBLEM

This problem arose in an earlier, unsuccessful, attempt to answer a question about the Dubins-Freedman construction of random distributions that has in the meantime been answered affirmatively in the paper [1].

For $n \in \mathbf{N}$, let $\mathcal{P}_n$ denote the set of polynomials of the form

$$\sum_{i=0}^{2k} x^{n-s(i)}(1-x)^{s(i)}$$

where $0 \leq k \leq 2^{n-1}-1$ and $s(i)$ is the number of 1's in the binary expansion of i. Thus,

$$\begin{aligned}
\mathcal{P}_1 &= \{x\},\\
\mathcal{P}_2 &= \{x^2, x^2+2x(1-x)\},\\
\mathcal{P}_3 &= \{x^3, x^3+2x^2(1-x), x^3+3x^2(1-x)+x(1-x)^2,\\
&\qquad x^3+3x^2(1-x)+3x(1-x)^2\},
\end{aligned}$$

etc.

Let $\mathcal{P} = \cup_{n=1}^{\infty}\mathcal{P}_n$. Note that all members of $\mathcal{P}$ are partition polynomials which map 0 to 0 and 1 to 1, and are increasing in between. (A *partition polynomial* is a polynomial of the form $\sum_{i=0}^{n} a_i x^i (1-x)^{n-i}$, where each a_i is integer with $0 \leq a_i \leq \binom{n}{i}$.) However, there are many increasing partition polynomials with this property which are not members of $\mathcal{P}$. (For example, $x^3 + x^2(1-x) + x(1-x)^2$.)

Let $\mathcal{L}$ denote the set of those members of $\mathcal{P}$ which are $< x$ on $(0,1)$, and $\mathcal{R}$ the set of those members of $\mathcal{P}$ which are $> x$ on $(0,1)$. Then $\mathcal{P} = \mathcal{L} \cup \{x\} \cup \mathcal{R}$. Furthermore, if $p \in \mathcal{R}$ then $p(x) = x + (1-x)r(x)$ for some $r \in \mathcal{P}$; and if $q \in \mathcal{L}$ then $q(x) = xs(x)$ for some $s \in \mathcal{P}$.

Received by the editors June 12, 2008

Conjecture 1. Let $p \in \mathcal{R}$ and $q \in \mathcal{L}$. Then the equation $p(q(x)) = x$ has a unique root in $(0, 1)$.

Some easy observations:

- The rotation of any member of $\mathcal{R}$ by 180° about $(1/2, 1/2)$ is a member of $\mathcal{L}$, and vice versa. That is, $p \in \mathcal{R} \Rightarrow 1 - p(x) = q(1 - x)$ for some $q \in \mathcal{L}$.
- It is clear that $x^2 | q(x)$ for all $q \in \mathcal{L}$, so $q'(0) = 0$. Similarly, $p'(1) = 0$ for $p \in \mathcal{R}$. As a result, the function $f(x) = p(q(x)) - x$ satisfies $f'(0) = f'(1) = -1$ and $f(0) = f(1) = 0$. Thus, f has *at least* one root in $(0, 1)$, and has an odd number of total roots, counting multiplicity.
- It is clear that the statement of the conjecture is equivalent to the statement that $q(p(x)) - x$ has a unique root in $(0, 1)$.

Conjecture 2. If $p \in \mathcal{R}$ and $q \in \mathcal{L}$, then $q(p(x)) > p(q(x))$ in $(0, 1)$.

(It is not clear how this would help to prove the first conjecture.)

References

[1] P. C. Allaart and R. D. Mauldin (2008), *Injectivity of the Dubins-Freedman construction of random distributions*, preprint, University of North Texas.