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TANGENCY COUNTEREXAMPLES IN l^2

Abstract

In [6], an infinite dimensional curve is constructed which is fairly smooth near an accumulation point of its graph, but has a null tangent set near the accumulation point. We construct extremely smooth curves which still yield such an anomalous tangency behavior.

Let X be a Hilbert space, let $f : (r, +\infty) \to X$ be a function, and assume there exists an $x \in X$ such that $\liminf_{t \downarrow r} \|f(t) - x\| = 0$. Then $(r, x) \in \overline{\text{graph}(f)}$ and so $(0, 0) \in K_{\text{graph}(f)}(r, x)$. Here the overbar ($\overline{\ }$) denotes the closure operator whereas K denotes a tangency concept of Bouligand and Severi.

The K-roots can be tracked down in the 1931 issue of Annales de la Société Polonaise de Mathématique, namely in the papers by Bouligand [2, p. 32] and Severi [5, p. 99]. At the beginning, the K-items were sets of *half-lines*. Later the K-items became sets of *points*. Rigorous translations of the half-line definitions of Bouligand and Severi into point definitions are made in [3, p. 240]. There it is also proved that the translated definitions are equivalent in normed spaces. Their equivalence in linear topological spaces follows from [7, pp. 567,8]. For further details on the history of the subject we refer to [4, p. 133] and [8, p. 342].

Currently, if T is a linear topological space, $S \subseteq T$, and $p \in T$ then $K_S(p)$ denotes the set of all points $q \in T$ such that

$$(0, q) \in \overline{\{(\rho, \tau) \in \mathbb{R} \times T; \rho > 0, p + \rho\tau \in S\}}.$$

If T is a sequential space, i.e. there exists a sequence of points $\tau_n \in S$ converging to p whenever $S \subseteq T$ and $p \in \overline{S}$ (see [1, p. 101, Definition 3.1]),

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then $\mathbb{R} \times T$ is also a sequential space (see again [1, p. 102, Proposition 3.2]). Therefore $q \in K_S(p)$ if and only if there exist a sequence of real numbers $\rho_n > 0$ converging to 0 and a sequence of points $\tau_n \in T$ converging to q such that $p + \rho_n \tau_n \in S$.

If T is not a sequential space then the characterization above may fail since there exist $S \subseteq T$ and $p \in \overline{S}$ such that no sequence $\tau_n \in S$ converges to p. Then $0 \in K_S(p)$ but there exists no sequence $\rho_n > 0$ converging to $0 \in \mathbb{R}$ and no sequence $\tau_n \in T$ converging to $0 \in T$ such that $p + \rho_n \tau_n \in P$.

Now assume there exists $\rho \geq 0$ such that $\liminf_{t\downarrow r} \left| \frac{\|f(t)-x\|}{t-r} - \rho \right| = 0$. If $\rho = 0$ then $(1, 0) \in K_{\text{graph}}(r, x)$. Hence

$$\{(0,0)\} \subset K_{\text{graph}(f)}(r, x). \tag{1}$$

If $\rho > 0$ and the linear space X is finite dimensional then there exists $\xi \in X$ such that $\|\xi\| = \rho$ and $(1, \xi) \in K_{\text{graph}(f)}(r, x)$. Hence the strict inclusion (1) still holds. If $\rho > 0$ but the linear space X is infinite dimensional then the strict inclusion (1) may fail, which means

$$\{(0,0)\} = K_{\text{graph}(f)}(r, x). \tag{2}$$

In this regard a counterexample is given in [6, p. 273-4]. Turowska constructed a continuous function $f : (0, +\infty) \to l^2$ which satisfies the equality $\{(0,0)\} = K_{\text{graph}(f)}(0,0)$ but yields $\lim_{t\downarrow 0} \|f(t)\| = 0$ and $\liminf_{t\downarrow 0} \left| \frac{\|f(t)\|}{t} - \rho \right| = 0$ for all $\rho \in [\frac{1}{\sqrt{2}}, 1]$. That function is also fairly smooth in that it is piecewise affine and $\|\dot{f}(t)\| = \sqrt{5}$ for almost all $t > 0$.

The question arises whether a counterexample could be still given in the case of an extremely smooth function. The answer is affirmative. In the following an infinitely differentiable function $f : (0, +\infty) \to l^2$ is constructed which satisfies the equality $\{(0,0)\} = K_{\text{graph}(f)}(0,0)$ but $\frac{\|f(t)\|}{t} = 1$ and $1 \leq \|\dot{f}(t)\| \leq \sqrt{5}$ for all $t > 0$. In fact we show that for every $L > 1$ there exists an infinitely differentiable function $f : (r, +\infty) \to l^2$ (in short, $f \in C^\infty((r, +\infty); l^2)$) which satisfies the equality (2) but has

$$\frac{\|f(t) - x\|}{t - r} = 1 \tag{3}$$

$$\text{and } 1 \leq \|\dot{f}(t)\| \leq L \tag{4}$$

for all $t > r$ (see Theorem 1 below).

The condition $L > 1$ cannot be replaced with the condition $L = 1$. In fact if the function $f : (r, +\infty) \to l^2$ is locally absolutely continuous (in short, $f \in AC_{loc}((r, +\infty); l^2)$), if

$$\|\dot{f}(t)\| \leq 1 \tag{5}$$

for almost all $t > r$, if $x = \lim_{t\downarrow r} f(t)$, and if the equality (3) holds for all $t > r$ then f satisfies the strict inclusion (1) (see the first remark following Theorem 2 below).

Theorem 1. *For every $L > 1$ there exists $f \in C^\infty((r, +\infty); l^2)$ which satisfies (2) but yields (3) and (4) for all $t > r$.*

PROOF. We can suppose, replacing the function f with the function $t \in (0, +\infty) \to f(r + t) - x \in l^2$ if necessary, that $(r, x) = (0, 0)$.

The proof of the theorem relies on an auxiliary result which concerns several items: a Hilbert space X; two points $x' \in X$ and $x \in X$ such that $\|x'\| = 1$, $\|x\| = 1$, and $\langle x', x\rangle = 0$; two real numbers $\alpha' \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ such that $0 < \alpha' < \alpha$; a real number $\beta \in (0, (\alpha - \alpha')/2)$; and a real number $L > 0$ such that $1 + \frac{\pi^2}{4}\big[\ln\big(\frac{\alpha-\beta}{\alpha'+\beta}\big)\big]^{-2} < L^2$. The auxiliary result states that there exists a function $f \in C^\infty(\mathbb{R}; X)$ which satisfies the equality (3), the inequalities (4) on $\mathbb{R}$, the affine equality $f(t) = tx'$ on $(-\infty, \alpha' + \beta]$, and the affine equality $f(t) = tx$ on $[\alpha - \beta, +\infty)$.

To prove the auxiliary result, choose $\gamma \in (\beta, (\alpha - \alpha')/2)$ such that $1 + \frac{\pi^2}{4}\big[\big(\frac{\alpha-\gamma}{\alpha'+\gamma}\big)\big]^{-2} < L^2$ and consider a function $h \in C^\infty(\mathbb{R}; \mathbb{R})$ with $h(\mathbb{R}) = [0, 1]$, such that $h(t) = 0$ on both $(-\infty, \alpha' + \beta]$ and $[\alpha - \beta, +\infty)$, and such that $h(t) = 1$ on $[\alpha' + \gamma, \alpha - \gamma]$. Furthermore define $g \in C^\infty(\mathbb{R}; \mathbb{R})$ through $g(t) = \big[\int_{\alpha'}^{t} \frac{h(s)}{s}\,ds\big]\big[\int_{\alpha'}^{\alpha} \frac{h(s)}{s}\,ds\big]^{-1}$ and note that $g(t) = 0$ on $(-\infty, \alpha' + \beta]$ whereas $g(t) = 1$ on $[\alpha - \beta, +\infty)$. Moreover $t\dot g(t) \leq \big[\int_{\alpha'}^{\alpha} \frac{h(s)}{s}\,ds\big]^{-1} \leq \big[\int_{\alpha'+\gamma}^{\alpha-\gamma} \frac{h(s)}{s}\,ds\big]^{-1} = \big[\ln\big(\frac{\alpha-\gamma}{\alpha'+\gamma}\big)\big]^{-1}$. Hence $1 + \frac{\pi^2}{4}[t\dot g(t)]^2 < L^2$. Now define $f \in C^\infty(\mathbb{R}; X)$ through $f(t) = t\cos\big[g(t)\frac{\pi}{2}\big]x' + t\sin\big[g(t)\frac{\pi}{2}\big]x$ and note that f satisfies (3) as well as both of the required affine equalities. Additionally $\dot f(t) = \big[\cos\big(g(t)\frac{\pi}{2}\big) - \frac{\pi}{2}t\dot g(t)\sin\big(g(t)\frac{\pi}{2}\big)\big]x' + \big[\sin\big(g(t)\frac{\pi}{2}\big) + \frac{\pi}{2}t\dot g(t)\cos\big(g(t)\frac{\pi}{2}\big)\big]x$. Hence $\|\dot f(t)\|^2 = 1 + \frac{\pi^2}{4}[t\dot g(t)]^2$, f satisfies (4), and the proof of the auxiliary result is accomplished.

We proceed now with the proof of the theorem. Let $L > 1$, choose $\nu > 1$ such that $1 + \frac{\pi^2}{4}[\ln(\nu)]^{-2} < L^2$, define the real sequence α_n through $\alpha_1 = 1$ and $\alpha_{(n+1)} = \frac{\alpha_n}{\nu}$, and observe $(0, +\infty) = \cup_{n\in\mathbb{N}}[\alpha_{(n+1)}, \alpha_n) \cup [1, +\infty)$. Next we construct a function $f : (0, +\infty) \to l^2$ by using the above partition of the interval $(0, +\infty)$ and the standard orthonormal system $\{e_n\}$ in l^2.

On the interval $[1, +\infty)$, we define f through $f(t) = te_1$. Let $n \in \mathbb{N}$. In order to define f on the interval $[\alpha_{(n+1)}, \alpha_n]$, observe $\frac{\alpha_n}{\alpha_{(n+1)}} = \nu$ and choose $\beta_n \in (0, \frac{\alpha_n - \alpha_{(n+1)}}{2})$ sufficiently small so that

$$1 + \frac{\pi^2}{4}\Big[\ln\Big(\frac{\alpha_n - \beta_n}{\alpha_{(n+1)} + \beta_n}\Big)\Big]^{-2} < L^2.$$

According to the auxiliary result above there exists $f_n \in C^\infty(\mathbb{R}; l^2)$ which satisfies the equality $\|f_n(t)\| = t$ on $\mathbb{R}$, the inequality $1 \le \|\dot f_n(t)\| < L$ on $\mathbb{R}$, the affine equality $f_{(n+1)}(t) = te_{(n+1)}$ on $(-\infty, \alpha_{(n+1)} + \beta_n]$, and the affine equality $f_n(t) = te_n$ on $[\alpha_n - \beta_n, +\infty)$. On the interval $[\alpha_{(n+1)}, \alpha_n]$ we define f through $f(t) = f_n(t)$. Due to the affine equalities satisfied by each f_n we get $f \in C^\infty((0, +\infty); l^2)$.

Finally, let $(\rho, \xi) \in K_{\mathrm{graph}(f)}(0, 0)$. We have to show that $(\rho, \xi) = (0, 0)$. Consider a sequence $\sigma_i > 0$ converging to 0, a sequence $\rho_i \in \mathbb{R}$ converging to ρ, and a sequence ξ_i converging to ξ such that $(0, 0) + \sigma_i(\rho_i, \xi_i) \in \mathrm{graph}(f)$, which means $\sigma_i\rho_i > 0$ and $f(\sigma_i\rho_i) = \sigma_i\xi_i$. In view of (3) $\sigma_i\rho_i = \sigma_i\|\xi_i\|$ and so $\rho = \|\xi\|$. Furthermore there exist a sequence n_i such that $\sigma_i\rho_i \in [\alpha_{(n_i+1)}, \alpha_{n_i})$. Hence $\sigma_i\xi_i = f_{n_i}(\sigma_i\xi_i)$. Since the sequence $\sigma_i\rho_i$ converges to 0, we can suppose, taking a subsequence of $(\sigma_i, \rho_i, \xi_i)$ if necessary, that the intervals $[\alpha_{(n_i+1)}, \alpha_{n_i}]$ are mutually disjoint and so the sets $\{e_{(n_i+1)}, e_{n_i}\}$ are also mutually disjoint. Therefore $\langle f_{n_i}(\sigma_i\rho_i), f_{n_j}(\sigma_j\rho_j)\rangle = 0$ whenever $i \neq j$. Finally $\langle \xi_i, \xi_j\rangle = 0$ whenever $i \neq j$ and so $\|\xi\| = 0$, $(\rho, \xi) = (0, 0)$ and the proof of the theorem is accomplished. □

Theorem 2. *Let $f \in AC_{loc}((r, +\infty); l^2)$ satisfy the inequality (5) for almost all $t > r$, let $x = \lim_{t\downarrow r} f(t)$, and let f and x satisfy the equality (3) for all $t > r$. Then there exists $\xi \in l^2$ such that $\|\xi\| = 1$ and $f(t) = x + (t - r)\xi$ for all $t > r$.*

Proof. Let $g(t) = \frac{f(t)-x}{t-r}$ so that $g(t) + (t-r)\dot g(t) = \dot f(t)$ and note $\|g(t)\| = 1$ as well as $\|g(t) + (t - r)\dot g(t)\| \le 1$ for almost all $t > r$. Then $\langle g(t), \dot g(t)\rangle = 0$ and so $\|\dot g(t)\| = 0$ for almost all $t > r$. Finally there exists $\xi \in l^2$ such that $\|\xi\| = 1$ and $g(t) = \xi$ for all $t > r$ and the conclusion follows. □

In the setting of Theorem 2, since $(1, \xi) \in K_{\mathrm{graph}(f)}(r, x)$, the function f satisfies the strict inclusion (1). Theorem 2 remains valid if the particular (infinite dimensional) space l^2 is replaced with a general (finite or infinite dimensional) Hilbert space X, but it may fail if l^2 is replaced with a normed space X. For example, let X denote the vector space $\mathbb{R}^2$ endowed with the l^1-norm, namely $\|x\| = |x_1| + |x_2|$, and let $f : (0, +\infty) \to X$ be given by $f_1(t) = \arctan t$ and $f_2(t) = t - \arctan t$. Then f is not a linear function although $\|f(t)\| = t$ and $\|\dot f(t)\| = 1$.

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