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Christian Richter, Friedrich-Schiller-Universität Jena, Mathematisches Institut, D-07740 Jena, Germany. e-mail: `richterc@minet.uni-jena.de`

REPRESENTING CLIQUISH FUNCTIONS AS QUASIUNIFORM LIMITS OF QUASICONTINUOUS FUNCTIONS

Abstract

It is shown that every cliquish function f mapping a pseudometrizable space X into a separable metric space Y can be expressed as the quasiuniform limit of a sequence of quasicontinuous functions f_k.

1 Representing Cliquish Functions on Pseudometrizable Spaces

Let X be a topological space and let (Y, d_Y) be a metric space. A function $f : X \to Y$ is called *quasicontinuous at the point* $x_0 \in X$ if, for every neighborhood U of x_0 and every $\varepsilon > 0$, there exists a non-empty open set $G \subseteq U$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ for all $x \in G$ (cf. [10]). The function f is said to be *cliquish at* x_0 if, under the same conditions as above, $d_Y(f(x), f(y)) < \varepsilon$ for all $x, y \in G$ (cf. [15]). Accordingly, f is called *quasicontinuous* or *cliquish* if f is quasicontinuous or cliquish, respectively, at every point $x_0 \in X$.

Quasicontinuous functions in general form a proper subclass of the class of all cliquish functions. However, under reasonable restrictions on X and Y it turns out that cliquish functions can be represented as pointwise or even quasiuniform limits of quasicontinuous functions. We recall that a sequence $(f_k)_{k=1}^{\infty}$ of functions $f_k : X \to Y$ *quasiuniformly converges to* $f : X \to Y$ if f is the pointwise limit of $(f_k)_{k=1}^{\infty}$ and

$$\begin{aligned}&\forall \varepsilon > 0\, \forall m \geq 1\, \exists p \geq 1\, \forall x \in X : \\ &\qquad \min\{d_Y(f_{m+1}(x), f(x)), \dots, d_Y(f_{m+p}(x), f(x))\} < \varepsilon\end{aligned}$$

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(cf. [13], p. 143)[1]. The origin of this concept goes back to Arzelá's theorem concerning the continuity of pointwise limits of continuous functions on compact spaces.

In [6] Grande proved that a function $f : \mathbb{R} \to \mathbb{R}$ is cliquish if and only if it is the pointwise limit of a sequence of quasicontinuous functions, provided that the domain $\mathbb{R}$ is equipped with the usual topology or with the density topology. In [8] the analogous result was obtained for real-valued functions f on the topological space $\mathbb{R}^m$ with the density topology. In his paper [3] Borsík showed that every cliquish function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the quasiuniform limit of a sequence of quasicontinuous functions f_k. He considered the usual topology on $\mathbb{R}$. In [7] the last result was sharpened in so far as the functions f_k can be assumed to have the Darboux property.

The main result of the present paper is the following fairly general representation theorem. The proof will be given in a separate section.

Theorem 1.1. *Let $f : X \to Y$ be a cliquish function mapping a pseudometrizable space X into a separable metric space (Y, d_Y). Then f is the quasiuniform limit of a sequence of quasicontinuous functions $f_k : X \to Y$. In particular,*

$$\forall m \geq 1 \, \forall x \in X : \min\{d_Y(f_{2m}(x), f(x)), d_Y(f_{2m+1}(x), f(x))\} < \tfrac{1}{m}. \tag{1}$$

If f is bounded, then one can require the functions f_k to be bounded, too.[2]

The separability of (Y, d_Y) is a necessary assumption in Theorem 1.1. In fact, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(\xi_1, \xi_2) = \begin{cases} \xi_1 & \text{if} \quad \xi_2 = 0, \\ 0 & \text{if} \quad \xi_2 \neq 0, \end{cases}$$

the domain $\mathbb{R}^2$ being the Euclidean plane, the range $\mathbb{R}$, however, being equipped with the metric $d_{\mathbb{R}}(\xi, \eta) = 1$ if $\xi \neq \eta$. Then $(\mathbb{R}, d_{\mathbb{R}})$ is non-separable.

[1] In the literature different definitions of the concept of a quasiuniform limit appear. In [1], p. 265, a real-valued function f on a topological space X is called the quasiuniform limit of functions f_k, $k \geq 1$, if f is the pointwise limit of $(f_k)_{k=1}^{\infty}$ and if, for all $\varepsilon > 0$ and all $m \geq 1$, there exist an at most countable open cover $\{G_i : i \in I\}$ of X and a corresponding set $\{p_i : i \in I\}$ of natural numbers such that, for all $i \in I$ and all $x \in G_i$, $|f(x) - f_{m+p_i}(x)| < \varepsilon$. This is equivalent to Sikorski's definition if X is compact and all functions f_k are continuous.

[2] Theorem 1.1 becomes false if one uses Aleksandroff's concept of a quasiuniform limit instead of Sikorski's definition. Indeed, one easily checks that the cliquish function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$ can not be expressed as the quasiuniform limit of a sequence of quasicontinuous functions in the sense of Aleksandroff.

Of course, f is cliquish. We assume f to be the pointwise limit of a sequence $(f_k)_{k=1}^{\infty}$ of quasicontinuous functions. Then $\mathbb{R} = \bigcup_{k=1}^{\infty} R_k$ where $R_k = \{\xi \in \mathbb{R} : d_{\mathbb{R}}(f_k(\xi,0), f(\xi,0)) < 1\} = \{\xi \in \mathbb{R} : f_k(\xi,0) = \xi\}$. Hence there exists an uncountable set R_{k_0}. The quasicontinuity of f_{k_0} at a point $(\xi,0)$, $\xi \in R_{k_0}$, yields that there exists an non-empty open set $G_\xi \subseteq \mathbb{R}^2$ such that $d_{\mathbb{R}}(f_{k_0}(\eta_1,\eta_2),\xi) = d_{\mathbb{R}}(f_{k_0}(\eta_1,\eta_2), f_{k_0}(\xi,0)) < 1$ for all $(\eta_1,\eta_2) \in G_\xi$. That is, $f_{k_0}|_{G_\xi} = \xi$, $f_{k_0}|_{G_\xi}$ denoting the restriction of f_{k_0} to G_ξ. This way we have found uncountably many non-empty open sets $G_\xi \subseteq \mathbb{R}^2$, $\xi \in R_{k_0}$, which are pairwise disjoint. This clearly is impossible, since every set G_ξ must contain points from the countable set $\mathbb{Q}^2$, where $\mathbb{Q}$ denotes the set of rational numbers.

Theorem 1.1 gives rise to a characterization of cliquish functions on pseudometrizable Baire spaces.

Corollary 1.2. *Let $f : X \to Y$ be a function mapping a pseudometrizable Baire space X into a separable metric space Y. Then the following are equivalent.*

(i) f is cliquish.

(ii) f is the quasiuniform limit of a sequence of quasicontinuous functions $f_k : X \to Y$.

(iii) f is the pointwise limit of a sequence of quasicontinuous functions $f_k : X \to Y$.

PROOF. The implication (i)⇒(ii) rests on Theorem 1.1. (ii)⇒(iii) is trivial.

Let us assume that f is represented as claimed under (iii). In [2] it is shown that then the discontinuity points of f constitute a set of the first category, provided that X is metrizable. However, the same proof applies to a pseudometrizable space X. Hence the continuity points of f are dense in X, since X is a Baire space. This obviously implies that f is cliquish. □

In Corollary 1.2 the supposition "Baire" can not be omitted. For instance, *every* function $f : \mathbb{Q} \to \{0,1\}$ on the rational numbers $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$ equipped with the usual distance $d_{\mathbb{Q}}(q_i,q_j) = |q_i - q_j|$ can be expressed as the quasiuniform limit of a sequence of quasicontinuous functions $f_k : \mathbb{Q} \to \{0,1\}$. Indeed, if $\left\{r_1^{(m)} < r_2^{(m)} < \ldots < r_{2m}^{(m)}\right\} = \{q_1, q_2, \ldots, q_{2m}\}$ for $m \geq 1$, then the functions

$$f_{2m}(q) = \begin{cases} f(q) & \text{if } q \in \left\{r_1^{(m)}, r_2^{(m)}, \ldots, r_{2m}^{(m)}\right\} = \{q_1, q_2, \ldots, q_{2m}\}, \\ 0 & \text{if } q \in \left(r_1^{(m)}, r_2^{(m)}\right) \cup \left(r_3^{(m)}, r_4^{(m)}\right) \cup \ldots \cup \left(r_{2m-1}^{(m)}, r_{2m}^{(m)}\right), \\ 1 & \text{if } q \in \left(-\infty, r_1^{(m)}\right) \cup \left(r_2^{(m)}, r_3^{(m)}\right) \cup \ldots \cup \left(r_{2m}^{(m)}, +\infty\right) \end{cases}$$

and

$$f_{2m+1}(q) = \begin{cases} f(q) & \text{if } q \in \left\{r_1^{(m)}, r_2^{(m)}, \ldots, r_{2m}^{(m)}\right\} = \{q_1, q_2, \ldots, q_{2m}\}, \\ 1 & \text{if } q \in \left(r_1^{(m)}, r_2^{(m)}\right) \cup \left(r_3^{(m)}, r_4^{(m)}\right) \cup \ldots \cup \left(r_{2m-1}^{(m)}, r_{2m}^{(m)}\right), \\ 0 & \text{if } q \in \left(-\infty, r_1^{(m)}\right) \cup \left(r_2^{(m)}, r_3^{(m)}\right) \cup \ldots \cup \left(r_{2m}^{(m)}, +\infty\right) \end{cases}$$

are quasicontinuous and we obtain $\lim\limits_{k\to\infty} f_k(q) = f(q)$ for all $q \in \mathbb{Q}$, since $f_{2m}|_{\{q_1,q_2,\ldots,q_{2m}\}} = f_{2m+1}|_{\{q_1,q_2,\ldots,q_{2m}\}} = f|_{\{q_1,q_2,\ldots,q_{2m}\}}$. Moreover,

$$\min\{|f_{2m}(q) - f(q)|, |f_{2m+1}(q) - f(q)|\} = 0$$

for all $q \in \mathbb{Q}$ and $m \geq 1$. Hence f is the quasiuniform limit of the sequence $(f_k)_{k=2}^{\infty}$.

2 Quasicontinuous and Cliquish Functions on More General Topological Spaces

Obviously, a non-constant cliquish function from a space X into a space Y can not be represented as the limit of a sequence of quasicontinuous functions if all quasicontinuous functions are constant. This section is devoted to spaces X on which all quasicontinuous or cliquish functions, respectively, are constant. We restrict our considerations to the case $Y = \mathbb{R}$, though most of the claims obviously can be proved for more general spaces Y.

Quasicontinuous and cliquish functions on an arbitrary topological space X can be expressed by the aid of functions from particular basic subclasses, which have been introduced and studied in [12]. We recall some important concepts and claims from this paper. A partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ of X into pairwise disjoint subsets P_ι is called *semi-open* if all partition sets P_ι are semi-open; that is, $P_\iota \subseteq \mathrm{cl}(\mathrm{int}(P_\iota))$ (cf. [11]), $\mathrm{cl}(\cdot)$ and $\mathrm{int}(\cdot)$ denoting the closure operator and the interior operator, respectively. The partition $\mathcal{P}$ is said to be *almost semi-open* if $\bigcup\limits_{\iota\in I} \mathrm{int}(P_\iota)$ is dense in X. A function $\varphi : X \to \mathbb{R}$ is called a *semi-open step function* or an *almost semi-open step function* if it is piecewise constant on the sets of a semi-open or an almost semi-open partition $\mathcal{P}$ of X, respectively. Every semi-open step function is quasicontinuous and every real-valued quasicontinuous function on X is the uniform limit of a sequence of semi-open step functions. Similarly, every almost semi-open step function is cliquish and every real-valued cliquish function can be uniformly approached by almost semi-open step functions.

Proposition 2.1. *A topological space X admits a non-constant quasicontinuous function $f : X \to \mathbb{R}$ if and only if there exist two non-empty disjoint open subsets $G_1, G_2 \subseteq X$.*

PROOF. If $G_1, G_2 \subseteq X$ are non-empty, disjoint, and open then $\mathcal{P} = \{\mathrm{cl}(G_1), X \setminus \mathrm{cl}(G_1)\}$ is a semi-open partition, since $\mathrm{cl}(G_1)$ is semi-open being the closure of an open set and since $X \setminus \mathrm{cl}(G_1)$ is open. Hence the characteristic function $\mathbf{I}_{\mathrm{cl}(G_1)}$ is quasicontinuous, where $\mathbf{I}_{\mathrm{cl}(G_1)}|_{G_1} = 1$ and $\mathbf{I}_{\mathrm{cl}(G_1)}|_{G_2} = 0$.

Conversely, if f is a non-constant quasicontinuous function, say $f(x_1) \neq f(x_2)$, then the existence of two non-empty disjoint open sets $G_1, G_2 \subseteq X$ is a direct consequence of the definition of quasicontinuity. $\square$

Proposition 2.2. *A topological space X admits a non-constant cliquish function $f : X \to \mathbb{R}$ if and only if X is not connected or there exists a non-empty nowhere dense subset $N \subseteq X$.*

PROOF. If X is not connected, then X can be decomposed into two non-empty open sets G_1 and G_2. In this case the characteristic function $\mathbf{I}_{G_1}$ even is an example of a non-constant continuous function on X. If X contains a non-empty nowhere dense subset N, then $\mathbf{I}_N$ is a non-constant cliquish function.

Now we suppose that there is a non-constant cliquish function $f : X \to \mathbb{R}$. Then there must exist a non-constant almost semi-open step function φ, since f is the uniform limit of functions of this type. The function φ is defined on an almost semi-open partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ consisting of at least two sets. We fix an index $\iota_0 \in I$. Then $\mathcal{Q} = \{Q_1, Q_2\}$ with $Q_1 = P_{\iota_0}$ and $Q_2 = \bigcup_{\iota \neq \iota_0} P_\iota$ is an almost semi-open partition, too. Hence $\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2)$ is a dense open subset of X. The complement $N = X \setminus (\mathrm{int}(Q_1) \cup \mathrm{int}(Q_2))$ is nowhere dense. If $N \neq \emptyset$, we have found the required non-empty nowhere dense subset of X. In the case $N = \emptyset$ we note that $\emptyset = N = \mathrm{cl}(Q_1) \cap \mathrm{cl}(Q_2)$ is the boundary of Q_1 as well as of Q_2. Then Q_1 and Q_2 are open. Hence X is not connected being the union of Q_1 and Q_2. $\square$

In [4] Borsík considered the space $X = \mathbb{R}$ with the system of open sets $\{\mathbb{R}, \emptyset\} \cup \{(a, +\infty) : a \in \mathbb{R}\}$. This is a second countable T_4-space, but does not fulfil T_1. By Proposition 2.1, all real-valued quasicontinuous functions on X are constant. However, Proposition 2.2 says that there exist non-constant cliquish functions $f : X \to \mathbb{R}$, since a set $A \subseteq \mathbb{R}$ is nowhere dense in X provided that $\sup A < +\infty$. In fact, one easily checks that $f : X \to \mathbb{R}$ is cliquish if and only if $\lim\limits_{x \to +\infty} f(x)$ exists in $\mathbb{R}$.

An example of a T_1-space is given by the cofinite topology on an arbitrary infinite set X. Here a subset $G \subseteq X$ is open if and only if $G = \emptyset$ or $X \setminus G$ is finite (cf. [14], p. 49). Again Proposition 2.1 yields that there exist constant quasicontinuous functions only. By Proposition 2.2 we obtain non-constant cliquish functions, because every finite subset of X is nowhere dense. However, in the case of an infinite T_1-space one can show a stronger result.

Proposition 2.3. *Every infinite T_1-space X admits a cliquish function $f : X \to \mathbb{R}$ with infinite range.*

Proof. Let $(x_i)_{i=1}^{\infty}$ be a sequence of mutually distinct points in X and let $(\lambda_i)_{i=0}^{\infty}$ be a sequence of reals such that $\lim\limits_{i\to\infty} \lambda_i = 0$. We shall see that the function $f = \lambda_0 + \sum\limits_{i=1}^{\infty} \lambda_i \mathbf{I}_{\{x_i\}} : X \to \mathbb{R}$ is cliquish. This yields the above claim if the values λ_i, $i \geq 1$, are mutually different.

Every partition $\mathcal{P} = \{X \setminus \{x_0\}, \{x_0\}\}$ of X with arbitrary $x_0 \in X$ is almost semi-open. Indeed, if x_0 is an isolated point; i.e., $\{x_0\}$ is open, then $\mathrm{int}(X\setminus\{x_0\}) \cup \mathrm{int}(\{x_0\}) = (X\setminus\{x_0\}) \cup \{x_0\}$ is dense in X. If x_0 is not isolated, then $\mathrm{int}(X \setminus \{x_0\}) \cup \mathrm{int}(\{x_0\}) = (X \setminus \{x_0\}) \cup \emptyset$ is dense as well. Consequently, every function $\mathbf{I}_{\{x_0\}}$ is an almost semi-open step function. Hence the functions $f_k = \lambda_0 + \sum\limits_{i=1}^{k} \lambda_i \mathbf{I}_{\{x_i\}}$, $k \geq 1$, are cliquish, since they are sums of cliquish functions. Then the uniform limit $f = \lim\limits_{k\to\infty} f_k$ is cliquish, too. $\square$

Passing from T_1-spaces to Hausdorff spaces we obtain a similar result concerning quasicontinuous functions.

Proposition 2.4. *Every infinite Hausdorff space X admits a quasicontinuous function $f : X \to \mathbb{R}$ with infinite range.*

Proof. First we show that every semi-open set $A \subseteq X$ containing at least two points can be decomposed into two non-empty semi-open subsets A_1 and A_2; that is, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. Since $A \subseteq \mathrm{cl}(\mathrm{int}(A))$, we can find two distinct points $x_1, x_2 \in \mathrm{int}(A)$. These two points can be separated by two open sets $G_1, G_2 \subseteq \mathrm{int}(A)$. Now let $A_1 = \mathrm{cl}(G_1) \cap A$ and $A_2 = A \setminus \mathrm{cl}(G_1)$. The first set is semi-open, because $A_1 \subseteq \mathrm{cl}(G_1) \subseteq \mathrm{cl}(\mathrm{int}(A_1))$. The set A_2 is semi-open, since $A_2 = A \cap (X \setminus \mathrm{cl}(G_1))$ is the intersection of a semi-open set and an open set. Moreover, A_1 and A_2 are non-empty, for $G_1 \subseteq A_1$ and $G_2 \subseteq A_2$.

Let $(\lambda_i)_{i=1}^{\infty}$ be a sequence of reals such that $\lim\limits_{i\to\infty} \lambda_i = \lambda$ exists in $\mathbb{R}$. We want to construct a function $f : X \to \mathbb{R}$ with $(\lambda_i)_{i=1}^{\infty} \subseteq f(X)$. Therefore we inductively define a sequence of semi-open partitions of the form $\mathcal{P}_k = \{P_1, P_2, \ldots, P_{k-1}, Q_k\}$, $k \geq 1$, where the sets Q_k are infinite. We start with

$\mathcal{P}_1 = \{Q_1\} = \{X\}$. Given $\mathcal{P}_k$, the above argument shows that Q_k can be decomposed into two non-empty semi-open subsets P_k and Q_{k+1}. We can assume that Q_{k+1} is infinite, since Q_k is an infinite set. This way we obtain the desired partition $\mathcal{P}_{k+1}$. Every partition $\mathcal{P}_k$ allows the definition of a semi-open step function $\varphi_k = \lambda \mathbf{I}_{Q_k} + \sum_{i=1}^{k-1} \lambda_i \mathbf{I}_{P_i}$. By $\lim_{i\to\infty} \lambda_i = \lambda$, the sequence $(\varphi_k)_{k=1}^{\infty}$ converges uniformly to a function $f : X \to \mathbb{R}$. Thus f is quasicontinuous and $(\lambda_i)_{i=1}^{\infty} \subseteq f(X)$, for $f|_{P_i} = \lambda_i$. □

Proposition 2.4 illustrates that every Hausdorff space gives rise to a large variety of quasicontinuous functions. This is remarkable in so far as there exist infinite regular Hausdorff spaces on which all real-valued continuous functions are constant (cf. [14], pp. 111-113). We do not know if Theorem 1.1 can be generalized in so far as the assumption on X to be pseudometrizable can be weakened.

3 Proof of Theorem 1.1

We use the following notation. Given a function f mapping a topological space X into a metric space (Y, d_Y), the *oscillation of f on a set $A \subseteq X$* is given by $\omega_f(A) = \sup\{d_Y(f(x_1), f(x_2)) : x_1, x_2 \in A\}$. The *oscillation of f at a point $x_0 \in X$* is defined by $\omega_f(x_0) = \inf\{\omega_f(U) : U$ is a neighborhood of $x_0\}$. If the space X is equipped with a pseudometric d_X, we denote the open ball of radius $r > 0$ centered at the point $x_0 \in X$ by $B_X(x_0, r) = \{x \in X : d_X(x, x_0) < r\}$. For a subset $A \subseteq X$ and a radius $r > 0$, we define a corresponding neighborhood of A by $B_X(A, r) = \bigcup_{x\in A} B_X(x, r)$. The proof of Theorem 1.1 is based on two technical lemmas.

Lemma 3.1. *Let $f : X \to Y$ be a cliquish function mapping a pseudometrizable space X into a metric space (Y, d_Y). Then there exist functions $g_m : X \to Y$, $m \geq 1$, and an increasing sequence of nowhere dense closed subsets $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots \subseteq X$ such that*

(i) $d_Y(g_m(x), f(x)) < \frac{1}{m}$ *for all* $m \geq 1$, $x \in X$,

(ii) $g_m|_{F_m} = f|_{F_m}$ *for all* $m \geq 1$,

(iii) $g_m|_{X\setminus F_m}$ *is quasicontinuous for all* $m \geq 1$,

(iv) $\omega_f(x) < \frac{1}{m}$ *for all* $m \geq 1$, $x \in X \setminus F_m$.

PROOF. Let $m \geq 1$ be fixed. We define $F_m = \left\{x \in X : \omega_f(x) \geq \frac{1}{m}\right\}$. Then F_m is closed and nowhere dense, because f is cliquish. Now we consider a point $x \in X \setminus F_m$. Since $\omega_f(x) < \frac{1}{m}$, there exists an open neighborhood $U_x \subseteq X \setminus F_m$ of x such that $\omega_f(U_x) < \frac{1}{m}$. We can choose another open neighborhood V_x of x such that $\mathrm{cl}(V_x) \subseteq U_x$, for X is a normal space (see [9], p. 120). Then $\mathcal{V} = \{V_x : x \in X \setminus F_m\}$ is an open cover of $X \setminus F_m$ such that $\mathrm{cl}(V_x) \subseteq X \setminus F_m$ for all $x \in X \setminus F_m$. The open subset $X \setminus F_m$ of X is a pseudometrizable space itself and hence paracompact (see [9], p. 160). Thus there exists a locally finite open cover $\mathcal{W} = \{W_\iota : \iota \in I\}$ of the space $X \setminus F_m$ which is a refinement of $\mathcal{V}$. Let the index set I be well-ordered. Then we define a locally finite partition $\mathcal{P} = \{P_\iota : \iota \in I\} \setminus \{\emptyset\}$ of $X \setminus F_m$ by $P_\iota = \mathrm{cl}(W_\iota) \setminus \bigcup\limits_{\kappa < \iota} \mathrm{cl}(W_\kappa)$. We fix a point $x_\iota \in P_\iota$ for every $P_\iota \in \mathcal{P}$. Now we define $g_m : X \to Y$ by

$$g_m(x) = \begin{cases} f(x) & \text{if } x \in F_m, \\ f(x_\iota) & \text{if } x \in P_\iota. \end{cases}$$

We have $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots$, (ii), and (iv) as immediate consequences of the definitions of F_m and g_m. In order to prove (i), let $x_0 \in X \setminus F_m$ be fixed. Then there exist an index $\iota \in I$ and a set $V_x \in \mathcal{V}$ such that $x_0, x_\iota \in P_\iota \subseteq \mathrm{cl}(W_\iota) \subseteq \mathrm{cl}(V_x) \subseteq U_x$. Hence

$$d_Y(g_m(x_0), f(x_0)) = d_Y(f(x_\iota), f(x_0)) \leq \omega_f(U_x) < \frac{1}{m},$$

which shows (i). Finally, every set $P_\iota = \mathrm{cl}(W_\iota) \cap \left(X \setminus \bigcup\limits_{\kappa < \iota} \mathrm{cl}(W_\kappa)\right)$ is semi-open, since $\mathrm{cl}(W_\iota)$ is semi-open being the closure of the open set W_ι and since $X \setminus \bigcup\limits_{\kappa < \iota} \mathrm{cl}(W_\kappa)$ is open, because $\mathcal{W}$ is locally finite. Thus $g_m|_{X \setminus F_m}$ is piecewise constant on the semi-open partition $\mathcal{P}$ of $X \setminus F_m$. Obviously, a function of this type is quasicontinuous. This proves (iii). □

The following claim is taken from Borsík's paper [5].

Lemma 3.2. *Let X be a pseudometrizable space, $F \subseteq X$ a nowhere dense closed subset of X, and $G \subseteq X$ an open set such that $F \subseteq \mathrm{cl}(G)$. Then there exist pairwise disjoint classes $\mathcal{K}_n$ of non-empty open sets, $n \geq 1$, such that the sets K belonging to the family $\mathcal{K} = \bigcup\limits_{n=1}^{\infty} \mathcal{K}_n$ are subject to the following conditions.*

(i) $\mathrm{cl}(K) \subseteq G \setminus F$ *for all* $K \in \mathcal{K}$,

(ii) for every $x \in X \setminus F$, there exists a neighborhood V of x such that the set $\{K \in \mathcal{K} : V \cap \mathrm{cl}(K) \neq \emptyset\}$ has at most one element,

(iii) for every $x \in F$ and for every neighborhood U of x, there is a number $n_0 \geq 1$ such that, for all $n \geq n_0$, there exists $K \in \mathcal{K}_n$ with $\mathrm{cl}(K) \subseteq U$.

PROOF OF THEOREM 1.1 *1. Preliminaries.* We assume that f is represented as the limit of functions g_m according to Lemma 3.1. For technical reasons we add the set $F_0 = \emptyset$ to the sequence $(F_m)_{m=1}^{\infty}$.

Given $m \geq 1$, we apply Lemma 3.2 to the nowhere dense closed set $F = F_m$ and the open set $G = X$ in order to obtain a corresponding family $\mathcal{K}^{(m)} = \bigcup_{n=1}^{\infty} \mathcal{K}_n^{(m)}$. In every set $K \in \mathcal{K}^{(m)}$ we fix a point $x_K \in K$. In addition to the claims of Lemma 3.2 we can assume that

$$\omega_f(\mathrm{cl}(K)) < \tfrac{1}{m} \text{ for all } K \in \mathcal{K}^{(m)}. \tag{2}$$

Indeed, if $\omega_f(\mathrm{cl}(K)) \geq \frac{1}{m}$, then we choose an open neighborhood $K' \subseteq K$ of x_K with $\omega_f(K') < \frac{1}{m}$. This is possible by Lemma 3.1 (iv), because $x_K \in K \subseteq X \setminus F_m$. Now we can fix an open neighborhood K'' of x_K such that $\mathrm{cl}(K'') \subseteq K'$, since X is normal. Then $x_K \in K'' \subseteq K$ and $\omega_f(\mathrm{cl}(K'')) < \frac{1}{m}$. Hence we can replace K by K'' without affecting the claims of Lemma 3.2, which justifies (2).

Let $(y_l)_{l=0}^{\infty}$ be dense in Y. We put $(z_l)_{l=0}^{\infty} = (y_0,\ y_0, y_1,\ y_0, y_1, y_2, y_0, y_1, y_2, y_3, \ldots)$. Then the sequence $(z_l)_{l=l_0}^{\infty}$ is dense in Y for every $l_0 \geq 0$.

We assume that X is equipped with a pseudometric d_X.

2. Definition of the functions f_{2m+p}, $m \geq 1$, $p \in \{0,1\}$.

$$f_{2m+p}(x_0) = \begin{cases} z_l & \text{if there exist } l \geq 0,\ i \in \{1,2,\ldots,m\}, \\ & \text{and } K \in \mathcal{K}^{(m)}_{2(lm+i)+p} \text{ such that} \\ & x_0 \in \mathrm{cl}(K),\ \mathrm{cl}(K) \subseteq B_X\left(F_i, \frac{1}{m}\right), \text{ and} \\ & d_Y(z_l, f(x_K)) < \frac{1}{m} + \sup\limits_{x \in X \setminus F_{i-1}} \omega_f(x), \\ g_m(x_0) & \text{otherwise.} \end{cases} \tag{3}$$

This definition is correct, since the sets $\mathrm{cl}(K)$, $K \in \mathcal{K}^{(m)}$, are pairwise disjoint according to Lemma 3.2 (ii). We have $f_{2m+p}|_{X\setminus L_{m,p}} = g_m|_{X\setminus L_{m,p}}$ where

$$L_{m,p} = \bigcup \Big\{ \mathrm{cl}(K) : \text{there exist } l \geq 0,\ i \in \{1,2,\ldots,m\}, \text{ and } K \in \mathcal{K}^{(m)}_{2(lm+i)+p} \text{ such that } \mathrm{cl}(K) \subseteq B_X\left(F_i, \tfrac{1}{m}\right) \text{ and } d_Y(z_l, f(x_K)) < \tfrac{1}{m} + \sup_{x\in X\setminus F_{i-1}} \omega_f(x) \Big\}.$$

3. Quasicontinuity of f_{2m+p}. Let $x_0 \in X$ be fixed. In case $x_0 \in L_{m,p}$, we find a set $K \in \mathcal{K}^{(m)}$ and a number $l \geq 0$ such that $x_0 \in \mathrm{cl}(K)$ and $f_{2m+p}|_{\mathrm{cl}(K)} = z_l$. Hence every open neighborhood U of x_0 has a non-empty open intersection $G = U \cap K$ with K such that $f_{2m+p}|_G = z_l = f_{2m+p}(x_0)$. Thus f_{2m+p} is quasicontinuous at x_0.

Now let $x_0 \in X \setminus L_{m,p}$. If $x_0 \in X \setminus F_m$, then, by Lemma 3.2 (ii), there exists an open neighborhood $U \subseteq X \setminus F_m$ of x_0 such that $U \cap L_{m,p} = \emptyset$. Hence $f_{2m+p}|_U = g_m|_U$ and $g_m|_U$ is quasicontinuous by Lemma 3.1 (iii). Accordingly, f_{2m+p} is quasicontinuous at x_0.

It remains to show that f_{2m+p} is quasicontinuous at an arbitrary point $x_0 \in F_m$. Let a neighborhood U of x_0 and a bound $\varepsilon > 0$ be fixed. We have $F_m \subseteq X \setminus L_{m,p}$, since $L_{m,p} \subseteq \bigcup\left\{\mathrm{cl}(K) : K \in \mathcal{K}^{(m)}\right\}$ and since $F_m \subseteq X \setminus \bigcup\left\{\mathrm{cl}(K) : K \in \mathcal{K}^{(m)}\right\}$ by Lemma 3.2 (i). Hence $f_{2m+p}|_{F_m} = g_m|_{F_m}$ and, by Lemma 3.1 (ii),

$$f_{2m+p}|_{F_m} = f|_{F_m}. \tag{4}$$

There exists a uniquely determined $i \in \{1,2,\ldots,m\}$ such that $x_0 \in F_i \setminus F_{i-1}$, for $\emptyset = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_m$. We choose a neighborhood U' of x_0 such that $\omega_f(U') < \omega_f(x_0) + \frac{1}{2m}$. Now we define an additional neighborhood U'' of x_0 by $U'' = U \cap U' \cap B_X\left(F_i, \frac{1}{m}\right)$. By Lemma 3.2 (iii), there exists $l_0 \geq 0$ such that, for all $l \geq l_0$, there is a set $K \in \mathcal{K}^{(m)}_{2(lm+i)+p}$ with $\mathrm{cl}(K) \subseteq U''$. We can pick $l_1 \geq l_0$ such that $d_Y(z_{l_1}, f(x_0)) < \min\left\{\frac{1}{2m}, \varepsilon\right\}$, because $(z_l)_{l=l_0}^{\infty}$ is dense in Y. Then we find a corresponding set $K_1 \in \mathcal{K}^{(m)}_{2(l_1m+i)+p}$ with $\mathrm{cl}(K_1) \subseteq U''$. Hence in particular

$$\mathrm{cl}(K_1) \subseteq B_X\left(F_i, \tfrac{1}{m}\right). \tag{5}$$

The inclusions $x_0 \in U'$ and $x_{K_1} \in K_1 \subseteq U'' \subseteq U'$ yield $d_Y(f(x_0), f(x_{K_1})) \leq \omega_f(U') < \omega_f(x_0) + \frac{1}{2m}$. Hence $d_Y(f(x_0), f(x_{K_1})) < \sup_{x\in X\setminus F_{i-1}} \omega_f(x) + \frac{1}{2m}$, since

$x_0 \in X \setminus F_{i-1}$. This gives rise to

$$\begin{aligned} d_Y(z_{l_1}, f(x_{K_1})) &\le d_Y(z_{l_1}, f(x_0)) + d_Y(f(x_0), f(x_{K_1})) \\ &< \frac{1}{2m} + \sup_{x\in X\setminus F_{i-1}} \omega_f(x) + \frac{1}{2m} = \frac{1}{m} + \sup_{x\in X\setminus F_{i-1}} \omega_f(x). \end{aligned} \tag{6}$$

Properties (5) and (6) together with definition (3) show that $f_{2m+p}|_{\mathrm{cl}(K_1)} = z_{l_1}$. Thus we have found a non-empty open set K_1 such that $K_1 \subseteq U$, since $\mathrm{cl}(K_1) \subseteq U''$, and, by applying (4) to $x_0 \in F_m$,

$$d_Y(f_{2m+p}(x), f_{2m+p}(x_0)) = d_Y(z_{l_1}, f(x_0)) < \varepsilon$$

for all $x \in K_1$. Hence f_{2m+p} is quasicontinuous at x_0.

4. Pointwise convergence of $(f_k)_{k=2}^{\infty}$ *to* f. Let $x_0 \in X$. If $x_0 \in \bigcup_{m=1}^{\infty} F_m$, say $x_0 \in F_{m_0}$, then $x_0 \in F_m$ for all $m \ge m_0$. Hence, by (4), $f_{2m+p}(x_0) = f(x_0)$ whenever $m \ge m_0$, $p \in \{0,1\}$, so that trivially $\lim_{k\to\infty} f_k(x_0) = f(x_0)$.

Now let $x_0 \notin \bigcup_{m=1}^{\infty} F_m$ and let $\varepsilon > 0$ be fixed. We choose $m_0 \ge 1$ with $\frac{3}{m_0} \le \varepsilon$. Since $x_0 \notin F_{m_0}$, we find $m_1 \ge m_0$ such that $x_0 \notin B_X\left(F_{m_0}, \frac{1}{m_1}\right)$. We shall show that

$$d_Y(f_{2m+p}(x_0), f(x_0)) < \varepsilon \text{ for all } m \ge m_1,\ p \in \{0,1\}. \tag{7}$$

In the case $x_0 \in X \setminus L_{m,p}$ we have $f_{2m+p}(x_0) = g_m(x_0)$. Then claim (7) is a consequence of Lemma 3.1 (i), namely $d_Y(f_{2m+p}(x_0), f(x_0)) = d_Y(g_m(x_0), f(x_0)) < \frac{1}{m} < \frac{3}{m_0} \le \varepsilon$.

Next we assume that $x_0 \in L_{m,p}$. Then there exist $l \ge 0$, $i \in \{1, 2, \dots, m\}$, and $K \in \mathcal{K}^{(m)}_{2(lm+i)+p}$ such that $x_0 \in \mathrm{cl}(K)$, $\mathrm{cl}(K) \subseteq B_X\left(F_i, \frac{1}{m}\right)$, and also $d_Y(z_l,\ f(x_K)) < \frac{1}{m} + \sup_{x\in X\setminus F_{i-1}} \omega_f(x)$. Definition (3) yields $f_{2m+p}(x_0) = z_l$. Hence

$$d_Y(f_{2m+p}(x_0), f(x_0)) \le d_Y(z_l, f(x_K)) + d_Y(f(x_K), f(x_0)). \tag{8}$$

We obtain $i > m_0$, because $i \le m_0$ would yield $x_0 \in \mathrm{cl}(K) \subseteq B_X\left(F_i, \frac{1}{m}\right) \subseteq B_X\left(F_i, \frac{1}{m_1}\right) \subseteq B_X\left(F_{m_0}, \frac{1}{m_1}\right)$ contrary to $x_0 \notin B_X\left(F_{m_0}, \frac{1}{m_1}\right)$. Thus by Lemma 3.1 (iv) $\sup_{x\in X\setminus F_{i-1}} \omega_f(x) \le \sup_{x\in X\setminus F_{m_0}} \omega_f(x) \le \frac{1}{m_0}$ and

$$d_Y(z_l, f(x_K)) < \frac{1}{m} + \sup_{x\in X\setminus F_{i-1}} \omega_f(x) \le \frac{1}{m} + \frac{1}{m_0} \le \frac{2}{m_0}. \tag{9}$$

On the other hand,

$$d_Y(f(x_K), f(x_0)) < \tfrac{1}{m} \leq \tfrac{1}{m_0}, \tag{10}$$

since $x_K, x_0 \in \mathrm{cl}(K)$ and $\omega_f(\mathrm{cl}(K)) < \frac{1}{m}$ by (2). Inequalities (8), (9), and (10) amount to $d_Y(f_{2m+p}(x_0), f(x_0)) < \frac{3}{m_0} \leq \varepsilon$. This proves (7).

5. Quasiuniform convergence of $(f_k)_{k=2}^{\infty}$ *to* f *in the sense of (1).* Let $m \geq 1$ be fixed. Lemma 3.2 (ii) shows that $\mathrm{cl}(K_1) \cap \mathrm{cl}(K_2) = \emptyset$ for all $K_1, K_2 \in \mathcal{K}^{(m)}$, $K_1 \neq K_2$. Hence

$$L_{m,0} \cap L_{m,1} \subseteq \bigcup \left\{ \mathrm{cl}(K) : K \in \bigcup_{j=1}^{\infty} \mathcal{K}_{2j}^{(m)} \right\} \cap \bigcup \left\{ \mathrm{cl}(K) : K \in \bigcup_{j=1}^{\infty} \mathcal{K}_{2j+1}^{(m)} \right\} = \emptyset$$

and thus $(X \setminus L_{m,0}) \cup (X \setminus L_{m,1}) = X$. Now $f_{2m+p}|_{X \setminus L_{m,p}} = g_m|_{X \setminus L_{m,p}}$ implies that, for every $x \in X$, $f_{2m}(x) = g_m(x)$ or $f_{2m+1}(x) = g_m(x)$. In this situation Lemma 3.1 (i) yields (1): namely, $\min\{d_Y(f_{2m}(x), f(x)), d_Y(f_{2m+1}(x), f(x))\} \leq d_Y(g_m(x), f(x)) < \frac{1}{m}$.

6. Boundedness of f_{2m+p}. We assume that $f(X)$ is bounded. It will turn out that $f_{2m+p}(X) \subseteq B_Y\left(f(X), 1 + \omega_f(X)\right)$, which obviously implies that f_{2m+p} is bounded, too.

Let $x_0 \in X$. If $x_0 \in X \setminus L_{m,p}$, then, by Lemma 3.1 (i),

$$f_{2m+p}(x_0) = g_m(x_0) \in B_Y\left(f(x_0), \frac{1}{m}\right) \subseteq B_Y\left(f(X), 1 + \omega_f(X)\right).$$

In the case $x_0 \in L_{m,p}$ definition (3) says that $f_{2m+p}(x_0) = z_l$, where in particular $d_Y(z_l, f(x_K)) < \frac{1}{m} + \sup\limits_{x \in X \setminus F_{i-1}} \omega_f(x)$ for certain $l \geq 0$, $i \in \{1, 2 \ldots, m\}$, and $K \in \mathcal{K}^{(m)}$. Then

$$f_{2m+p}(x_0) = z_l \in B_Y\left(f(x_K), \frac{1}{m} + \sup_{x \in X \setminus F_{i-1}} \omega_f(x)\right) \subseteq B_Y(f(X), 1 + \omega_f(X)).$$

□

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