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CONSTRUCTING $\mathbf{\Delta}_3^0$ USING TOPOLOGICALLY RESTRICTIVE COUNTABLE DISJOINT UNIONS

Abstract

In a zero-dimensional Polish space, the Borel sets are generated from the clopen sets by repeatedly applying the operations of countable disjoint union and complementation. Here we look at topologically restrictive versions of the general countable disjoint union of sets, and obtain "construction principles" for $\mathbf{\Delta}_3^0$, i.e., sets which are both $\mathcal{F}_{\sigma\delta}$ and $\mathcal{G}_{\delta\sigma}$.

Throughout this paper, we work in an arbitrary but fixed zero-dimensional Polish space X; e.g., the Cantor space or the space of irrationals. Suppose that a set A is the *countable disjoint union* of the sets $A_1, A_2, \ldots, A_n, \ldots$:

$$A = \bigcup_{n=1}^{\infty} A_n \ \ (A_i \cap A_j = \emptyset \text{ for } i \neq j,\ i, j = 1, 2, \ldots). \tag{1}$$

In the following definition, we put successively stronger topological restrictions on how the above countable disjoint union (1) is formed.

Definition. We say that the countable disjoint union in (1) above is:

(a) a *separated union* if the sets $(A_n)_{n=1}^{\infty}$ are pairwise separated; i.e.,

$$\overline{A_i} \cap A_j = \emptyset = A_i \cap \overline{A_j} \quad \text{for } i \neq j \quad (i, j = 1, 2, \ldots);$$

(b) a *strongly separated union* if the sets $(A_n)_{n=1}^{\infty}$ have pairwise disjoint closures; i.e.,

$$\overline{A_i} \cap \overline{A_j} = \emptyset \quad \text{for } i \neq j \quad (i, j = 1, 2, \ldots); \text{ and}$$

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(c) a *uniformly clopen separated union* if there is a pairwise disjoint sequence of clopen sets $(B_n)_{n=1}^{\infty}$ with each B_n containing the corresponding A_n; i.e.,

$$A_n \subseteq B_n,\ B_n \text{ clopen },(n = 1, 2, \ldots),$$
$$B_i \cap B_j = \emptyset \text{ for } i \neq j\ (i, j = 1, 2, \ldots).$$

It is a standard fact that the class of sets generated from the clopen sets by repeatedly forming countable disjoint unions and taking complements is the class of Borel sets. If we now restrict the operation of forming countable disjoint unions to one of its restricted versions listed above (instead of allowing arbitrary countable disjoint unions), then of course some subclass of the Borel sets is generated.

In particular, a known result due to Hausdorff, Steel and Van Wesep (see [3]) is that the smallest class of sets containing the clopen sets and closed under uniformly clopen separated unions and complements is the class of $\boldsymbol{\Delta}^0_2$ sets, that is, sets which are both $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$.

In this paper we show that for both separated unions and strongly separated unions, the generated class is $\boldsymbol{\Delta}^0_3$, sets which are both $\mathcal{F}_{\sigma\delta}$ and $\mathcal{G}_{\delta\sigma}$. This may be viewed as a "construction principle" for $\boldsymbol{\Delta}^0_3$.

Theorem 1. *In a zero-dimensional Polish space X, let $\mathcal{S}$ be the smallest class of sets containing the clopen sets and closed under separated unions and complements, and let $\mathcal{C}$ denote the smallest class of sets containing the clopen sets and closed under strongly separated unions and complements. Then $\mathcal{S} = \mathcal{C} = \boldsymbol{\Delta}^0_3$.*

PROOF. Clearly $\mathcal{C} \subseteq \mathcal{S}$. Also every uniformly clopen separated union is also a strongly separated union, so by the Hausdorff-Van Wesep-Steel result, $\boldsymbol{\Delta}^0_2 \subseteq \mathcal{C}$. We also have the following assertion.

Lemma 2. $\boldsymbol{\Delta}^0_3$ *is closed under separated unions.*

PROOF OF LEMMA 2. Let $(A_n)_{n=1}^{\infty}$ be a sequence of $\boldsymbol{\Delta}^0_3$ sets such that $\overline{A_i} \cap A_j = \emptyset = A_i \cap \overline{A_j}$ for all $i < j$. Let $D_n = \overline{A_n} \setminus \cup_{i<n} \overline{A_i}$. Thus the D_n's form a disjoint sequence of $\boldsymbol{\Delta}^0_2$ sets with each D_n containing A_n, so $\cup_n A_n$ is $\boldsymbol{\Delta}^0_3$. □

From this lemma it follows that $\mathcal{S} \subseteq \boldsymbol{\Delta}^0_3$. Thus we have

$$\boldsymbol{\Delta}^0_2 \subseteq \mathcal{C} \subseteq \mathcal{S} \subseteq \boldsymbol{\Delta}^0_3.$$

Thus to prove the theorem, it is enough to show that $\mathbf{\Delta_3^0} \subseteq \mathcal{C}$.

For this, we use a more delicate version of the usual proof of the Lusin separation theorem, and actually show that any two disjoint $\mathbf{\Pi_3^0}$ sets can be separated by a set in $\mathcal{C}$.

Lemma 3. *In a zero-dimensional Polish space, given any $\epsilon > 0$, any $\mathcal{F}_\sigma$ set is a countable disjoint union of closed sets each of diameter $< \epsilon$.*

PROOF OF LEMMA 3. The proof is routine. □

Now let P and Q be disjoint $\mathbf{\Pi_3^0}$ ($\mathcal{F}_{\sigma\delta}$) sets. We may write

$$P = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} P_{m,n} \text{ and } Q = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} Q_{m,n},$$

where $P_{m,n}$ and $Q_{m,n}$ are closed sets for $m, n = 1, 2, \ldots$. By Lemma 3, we may assume that for each n, the sets $P_{1,n}, P_{2,n}, \ldots, P_{m,n}, \ldots$ form a pairwise disjoint sequence of closed sets; and similarly, that for each n, $Q_{1,n}, Q_{2,n}, \ldots, Q_{m,n}$, $\ldots$ are pairwise disjoint closed sets. Moreover, we can assume that each of the sets $P_{m,n}$ and $Q_{m,n}$ has diameter less than $1/n$.

Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of all finite sequences of positive integers. For any finite sequence $[m_1, m_2, \ldots, m_k] \in \mathbb{N}^{<\mathbb{N}}$, put

$$S([m_1, m_2, \ldots, m_k]) = \bigcap_{i=1}^{k} P_{m_i,i} \text{ and } T([m_1, m_2, \ldots, m_k]) = \bigcap_{i=1}^{k} Q_{m_i,i}.$$

Now each of S and T is a *Suslin scheme* in the sense that it assigns a closed set to each $[m_1, m_2, \ldots, m_k] \in \mathbb{N}^{<\mathbb{N}}$; also, we have $\operatorname{diam} S([m_1, m_2, \ldots, m_k]) \to 0$ as $k \to \infty$. Similarly for $\operatorname{diam} T([m_1, m_2, \ldots, m_k])$.

Note also that for $i \neq j$, $S([m_1, m_2, \ldots, m_k, i])$ and $S([m_1, m_2, \ldots, m_k, j])$ are disjoint closed sets, and similarly for T. We apply the Suslin operation ($\mathcal{A}$-operation) to the Suslin schemes S and T.

$$\mathcal{A}[S] = \left\{ x \mid x \in \bigcap_{k=1}^{\infty} S([m_1, m_2, \ldots, m_k]) \text{ for some sequence } (m_k)_{k=1}^{\infty} \right\},$$

and similarly set $\mathcal{A}[T]$.

It is then easy to verify that

$$P = \mathcal{A}[S] \text{ and } Q = \mathcal{A}[T].$$

We also define, for each $[m_1, m_2, \dots, m_k] \in \mathbb{N}^{<\mathbb{N}}$, the set $P^*_{[m_1,m_2,\dots,m_k]}$ to be the set of all x such that

$$x \in \bigcap_{j=1}^{\infty} S([m_1, m_2, \dots, m_k, n_1, n_2, \dots, n_j]) \text{ for some sequence } (n_j)_{j=1}^{\infty},$$

and similarly define the set $Q^*_{[m_1,m_2,\dots,m_k]}$.

Notice that $P^*_{[]} = P$, $Q^*_{[]} = Q$, and for any $[m_1, m_2, \dots, m_k] \in \mathbb{N}^{<\mathbb{N}}$,

$$P^*_{[m_1,m_2,\dots,m_k]} = \bigcup_{j=1}^{\infty} P^*_{[m_1,m_2,\dots,m_k,j]}, \text{ and } Q^*_{[m_1,m_2,\dots,m_k]} = \bigcup_{j=1}^{\infty} Q^*_{[m_1,m_2,\dots,m_k,j]},$$

where each of the last two countable unions is a strongly separated union.

Lemma 4. *The class $\mathcal{C}$ is closed under intersection with closed sets, that is if $E \in \mathcal{C}$ and F is closed, then $E \cap F \in \mathcal{C}$.*

Proof of Lemma 4. The proof is a simple inductive argument. □

Lemma 5. *Suppose that $(A_m)_{m=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ are two sequences of sets each of which is pairwise strongly separated. Suppose that each A_m can be separated from each B_n using a $\mathcal{C}$ set; that is, there is a set $C_{m,n} \in \mathcal{C}$ separating A_m and B_n. Then there is a set $C \in \mathcal{C}$ which separates $\cup_m A_m$ and $\cup_n B_n$.*

Proof of Lemma 5. The set

$$C = X \setminus \bigcup_{n=1}^{\infty} \left(\overline{B_n} \setminus \bigcup_{m=1}^{\infty} \left(C_{m,n} \cap \overline{A_m} \right) \right)$$

separates $\cup_m A_m$ and $\cup_n B_n$. Now using Lemma 4 and the fact that each of the sequence of sets $(A_m)_{m=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ is pairwise strongly separated, we see that $C \in \mathcal{C}$. □

The result now follows by contradiction when we assume that the sets $P = P^*_{[]}$ and $Q = Q^*_{[]}$ cannot be separated by a set in $\mathcal{C}$. (The rest of the proof is exactly like the proof by contradiction of the Lusin separation theorem.) □

We note that we could also give a more constructive proof using bar-recursion as done in [1] or [2].

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