Real Analysis Exchange
ISSN:0147-1937
Vol. 31(1), 2005/2006, pp. 253–270

Tomasz Natkaniec, Institute of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland. email: `mattn@math.univ.gda.pl`

Harvey Rosen, Department of Mathematics, Box 870350, University of Alabama, Tuscaloosa, AL 35487-0350, USA.email: `hrosen@gp.as.ua.edu`

ADDITIVE SIERPIŃSKI-ZYGMUND FUNCTIONS

Abstract

In the paper we present an exhaustive discussion of the relations between Darboux-like functions within the class of additive Sierpiński-Zygmund (SZ) functions. In particular, we give an example of an additive Sierpiński-Zygmund (SZ) injection $f\colon \mathbb{R} \to \mathbb{R}$ such that f^{-1} is not an SZ function. Under the assumption that $\mathbb{R}$ cannot be covered by less than $\mathfrak{c}$-many meager sets we give examples of an additive SZ bijection $f\colon \mathbb{R} \to \mathbb{R}$ such that f^{-1} is not SZ and of an additive injection $f\colon \mathbb{R} \to \mathbb{R}$ such that both f and f^{-1} are SZ.

A function $f\colon \mathbb{R} \to \mathbb{R}$ belongs to the class of *Sierpiński-Zygmund functions* (abbr. $f \in \mathrm{SZ}$) if the restriction $f{\restriction}A$ is discontinuous for each $A \subset \mathbb{R}$ of size $\mathfrak{c}$. This concept was introduced in [SZ]. A function $f\colon \mathbb{R} \to \mathbb{R}$ is *additive* if $f(x+y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$.

In this paper we will construct several examples of additive SZ functions. The paper has two main goals. The first of them is to show that almost all inclusions from Gibson's diagram remain strict in the class of additive SZ functions. The second one is to examine when the inverses of additive SZ injections are also of SZ type.

Our terminology is standard. In particular, symbols $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of all rationals and reals, respectively. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. For a cardinal number κ the symbol $[X]^{\kappa}$ will denote the family of all subsets Y of X with $\mathrm{card}\,(Y) = \kappa$. If

---

Key Words: additive function, Sierpiński-Zygmund function, Darboux like function, almost continuous functions, connectivity functions, functions with perfect road, peripherally continuous functions, CIVP-functions, SCIVP-functions

Mathematical Reviews subject classification: Primary: 26A15; Secondary: 03E50

Received by the editors April 16, 2005

Communicated by: Krzysztof Chris Ciesielski

A is a planar set, we denote its x-projection by $\mathrm{dom}\,(A)$. For $x \in \mathbb{R}$ and $A \subset \mathbb{R}^2$ we denote the x-section of A by A_x. The closure of a set $A \subset \mathbb{R}$ is denoted by $\mathrm{cl}\,(A)$, its interior by $\mathrm{int}\,(A)$, and its boundary by $\mathrm{bd}\,(A)$. $\mathcal{M}$ denotes the ideal of meager subsets of the real line and $\mathrm{cov}\,(\mathcal{M})$ is the minimal cardinality of a family of meager sets which cover $\mathbb{R}$. If $A \subset \mathbb{R}$ (or $A \subset \mathbb{R}^2$), then $\mathrm{LIN}\,(A)$ denotes the linear subspace of $\mathbb{R}$ ($\mathbb{R}^2$, respectively) over $\mathbb{Q}$ generated by A. (Note that if $A \subset \mathbb{R}^2$, then $\mathrm{dom}\,(\mathrm{LIN}\,(A))$ is a linear subspace of $\mathbb{R}$.) In particular, if $q \in \mathbb{Q}$ and $\langle x, y\rangle \in \mathbb{R}^2$, then $q\langle x, y\rangle = \langle qx, qy\rangle$ and if $q \in \mathbb{Q}$ and $A \subset \mathbb{R}^2$, then $qA = \{qa \colon a \in A\}$.

Let $\mathcal{C}_{G_\delta}$ be the collection of all real-valued continuous functions defined on G_δ subsets of $\mathbb{R}$, and $\mathcal{C}^*_{G_\delta}$ be the family of all nowhere constant functions $g \in \mathcal{C}_{G_\delta}$. It is well known that f is an SZ function iff $\mathrm{card}\,(f \cap g) < \mathfrak{c}$ for every $g \in \mathcal{C}_{G_\delta}$ [SZ]. We will need also the following lemma. (We will use it for one-to-one and for countable-to-one functions.)

Lemma 1. ([CN, Lemma 4.24]) *Let $X \in [\mathbb{R}]^{\mathfrak{c}}$ and $f \colon X \to \mathbb{R}$ have all level sets of size less than $\mathfrak{c}$. Then $f \in \mathrm{SZ}$ iff* $\mathrm{card}\,(f \cap g) < \mathfrak{c}$ *for every $g \in \mathcal{C}^*_{G_\delta}$.*

1 Additive Darboux Like Sierpiński-Zygmund Functions.

In several papers Darboux like properties in the class of SZ functions were considered. (See Section 4 in [GN].) In the first of them, [UD], Darji constructs in ZFC an SZ function having a perfect road at each point, and in [BCN], Balcerzak, Ciesielski, and Natkaniec give an additive example of such a function. In [NR], Natkaniec and Rosen under the assumption that $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$ constructed an example of an additive almost continuous SZ function which is PR but not CIVP. Note that some additional set-theoretic assumptions are here necessary, because the existence of an SZ Darboux function is independent of ZFC axioms [BCN, Section 5]. (For example, this is one of the consequences of CPA Axiom introduced by K. Ciesielski and J. Pawlikowski [CP, Paragraph 6.2].)

Let us flash back to several definitions. All but the second definition are given for functions $f : \mathbb{R} \to \mathbb{R}$.

D – f is a *Darboux function* if $f(C)$ is connected whenever C is connected in $\mathbb{R}$;

Conn – $f \colon X \to \mathbb{R}$ is a *connectivity function* if the graph of f restricted to C is connected in $X \times \mathbb{R}$ whenever C is connected subset of X;

AC – f is an *almost continuous function* in the sense of Stallings, if each open subset of $\mathbb{R}^2$ containing the graph of f contains also the graph of a continuous function from $\mathbb{R}$ to $\mathbb{R}$;

Ext – f is an *extendable function* if there exists a connectivity function $g\colon \mathbb{R}\times[0,1]\to\mathbb{R}$ such that $f(x)=g(x,0)$ for all $x\in\mathbb{R}$;

PR – f has a *perfect road* if for every $x\in\mathbb{R}$, there exists a perfect set P having x as a bilateral limit point such that $f{\restriction}P$ is continuous at x;

CIVP – *Cantor Intermediate Value Property*: $f\in \mathrm{CIVP}$ if for all $p,q\in\mathbb{R}$ with $p\neq q$ and $f(p)\neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C)\subset K$;

SCIVP – *Strong Cantor Intermediate Value Property*: $f\in \mathrm{SCIVP}$ if for all $p,q\in\mathbb{R}$ with $p\neq q$ and $f(p)\neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C)\subset K$ and $f{\restriction}C$ is continuous;

PC – f is *peripherally continuous* if for every $x\in\mathbb{R}$ there exist two sequences $s_n\nearrow x$ and $t_n\searrow x$ such that $\lim_{n\to\infty}f(s_n)=f(x)=\lim_{n\to\infty}f(t_n)$.

The basic relations between these classes for the functions from $\mathbb{R}$ to $\mathbb{R}$ are given in Gibson's diagram, in which arrows $\longrightarrow$ denote strict inclusions, and the symbol C denotes the class of all continuous functions. (See [GN].)

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}$$

We will show that almost all inclusions from Gibson's diagram remain strict in the class of all additive Sierpiński-Zygmund functions. Moreover, examples from the lower line of this diagram (SCIVP → CIVP → PR → PC) can be found in ZFC, in the class of one-to-one functions. In the examples from the upper line (Ext → AC → Conn → $\mathcal{D}$) we need some additional set-theoretic assumptions, like CH or $\mathrm{cov}(\mathcal{M})=\mathfrak{c}$. Additionally, such examples cannot be $1-1$, because of the well known fact that a one-to-one function $f\colon\mathbb{R}\to\mathbb{R}$ satisfying the intermediate value property must be continuous.

We start with two easy observations.

Remark 2. *No* SZ *function has the* SCIVP*, and therefore there is no* SZ *extendable function.*

Remark 3. *Every additive* SZ *function* $f\colon \mathbb{R}\to\mathbb{R}$ *is* PC.

Proof. This is a consequence of the following facts. Every SZ function is discontinuous, each additive discontinuous function is dense in $\mathbb{R}^2$, and all dense functions are PC. □

Example 4. *There is an additive* SZ *injection having a perfect road at no* $x\in\mathbb{R}$.

Proof. Let $\mathcal{C}_{G_\delta} = \{g_\xi\colon \xi < \mathfrak{c}\}$, and let $\{I_\alpha\colon \alpha<\mathfrak{c}\}$ be the family of all proper open intervals. Let $\{H_\alpha\colon \alpha<\mathfrak{c}\}$ be a family of pairwise disjoint sets such that $H=\bigcup_{\alpha<\mathfrak{c}} H_\alpha$ is a Hamel basis and each H_α is a Bernstein set. (See e.g., [KC, Theorem 7.3.4, p. 113].) Let $H=\{h_\alpha\colon \alpha<\mathfrak{c}\}$. First we define inductively an injection $\tilde{f}\colon H\to H$. Suppose $\tilde{f}$ is defined on $\{h_\beta\colon \beta<\alpha\}$. Let $V_\alpha = \mathrm{LIN}\,(\{h_\beta\colon \beta<\alpha\})$ and $W_\alpha=\mathrm{LIN}\,(\{\tilde{f}(h_\beta)\colon \beta<\alpha\})$. Choose

$$\tilde{f}(h_\alpha)\in H\setminus\Big(\mathrm{LIN}\Big(W_\alpha+\bigcup_{\xi\le\alpha} g_\xi(V_{\alpha+1})\Big)\cup I_\beta\Big)$$

where $h_\alpha\in H_\beta$, $\beta<\mathfrak{c}$.

Let f be an additive extension of $\tilde{f}$. Then f is $1-1$, and $\mathrm{dom}\,(f\cap g_\xi)\subset V_\xi$ for any $\xi<\mathfrak{c}$, so $f\in\mathrm{SZ}$. To see that f has a perfect road at no x, fix $x\in\mathbb{R}$ and $\varepsilon>0$. Let β be the number of the interval $(f(x)-\varepsilon, f(x)+\varepsilon)$. Then for each perfect set P there is $h_\alpha\in P\cap H_\beta$ with $f(h_\alpha)\notin I_\beta$, so P is not a perfect road of f at x. □

In the next examples we need the following lemma.

Lemma 5. ([BCN, Lemma 2]) *There exists a collection* $\{\langle H_\alpha, p_\alpha\rangle\colon \alpha<\mathfrak{c}\}$ *such that:*

1. $H_\alpha\cup\{p_\alpha\}$ *is a compact perfect subset of* $\mathbb{R}$ *and* p_α *is a bilateral limit point of* H_α,
2. $H=\bigcup_{\alpha<\mathfrak{c}} H_\alpha$ *is a linearly independent set,*
3. $H_\alpha\cap H_\beta=\emptyset$ *for all* $\alpha\ne\beta$,
4. *for every* $x\in\mathbb{R}$ *there exists* $\mathfrak{c}$*-many* $\gamma<\mathfrak{c}$ *such that* $x=p_\gamma$.

Example 6. *There exists an additive SZ injection* $f\colon\mathbb{R}\to\mathbb{R}$ *with the* CIVP. *(Note that* f *is not Darboux.)*

Proof. Let $\{I_n\colon n<\omega\}$ be the family of all open intervals with rational end-points, and let $\{C_\beta\colon \beta<\mathfrak{c}\}$ be the family of all Cantor sets. Fix a Hamel basis B which is a Bernstein set (See e.g., [KC, Theorem 7.3.4], p. 113.), and a family $\{H_{\alpha,n}\colon \alpha<\mathfrak{c}, n<\omega\}$ of pairwise disjoint perfect sets such that

(i) $\bigcup_{\alpha<\mathfrak{c}}\bigcup_{n<\omega} H_{\alpha,n}$ is linearly independent;

(ii) $\bigcup_{\alpha<\mathfrak{c}} H_{\alpha,n} \subset I_n$ for all $n<\omega$.

(The existence of such sets is an easy consequence of Lemma 5. Cf. [KC1, Lemma 4.2].) Let $H=\{h_\alpha\colon \alpha<\mathfrak{c}\}$ be a Hamel basis containing all sets $H_{\alpha,n}$. Fix $b\in B$ and put $B_0=B\setminus\{b\}$. We will define inductively a $1-1$ function $\tilde{f}\colon H\to B$. Assume $\tilde{f}$ is defined on $\{h_\beta\colon \beta<\alpha\}$. Put $V_\alpha=\mathrm{LIN}\,(\{h_\beta\colon \beta<\alpha\})$ and $W_\alpha=\mathrm{LIN}\,(\{\tilde{f}(h_\beta)\colon \beta<\alpha\})$. At the step α choose:

(1) $\tilde{f}(h_\alpha)\in B_0\cap C_\beta\setminus \mathrm{LIN}\,(W_\alpha+\bigcup_{\xi\le\alpha} g_\xi(V_{\alpha+1}))$ if $h_\alpha\in\bigcup_{n<\omega}H_{\beta,n}$;

(2) $\tilde{f}(h_\alpha)\in B_0\setminus \mathrm{LIN}\,(W_\alpha+\bigcup_{\xi\le\alpha} g_\xi(V_{\alpha+1}))$ if $h_\alpha\notin\bigcup_{n<\omega}\bigcup_{\beta<\mathfrak{c}}H_{\beta,n}$.

Let $f\colon\mathbb{R}\to\mathbb{R}$ be the additive extension of $\tilde{f}$.

First observe that the set $\{\tilde{f}(h_\alpha)\colon \alpha<\mathfrak{c}\}$ is linearly independent, so f is an injection.

To verify that $f\in\mathrm{SZ}$ we will show that for a given $\xi<\mathfrak{c}$, $\mathrm{dom}\,(f\cap g_\xi)\subset V_\xi$, so $\mathrm{card}\,(f\cap g_\xi)<\mathfrak{c}$. In fact, fix $x\in\mathbb{R}$ with $f(x)=g_\xi(x)$. Let α be the first ordinal for which $x\in V_{\alpha+1}$. Then $x=v+qh_\alpha$ for some $v\in V_\alpha$ and $q\in\mathbb{Q}\setminus\{0\}$, and consequently, $g_\xi(x)=f(x)=f(v)+q\tilde{f}(h_\alpha)$, so $\tilde{f}(h_\alpha)=-q^{-1}f(v)+q^{-1}g_\xi(x)\in W_\alpha+g_\xi(V_{\alpha+1})$. Thus the statements (1) and (2) give easily $\alpha<\xi$.

Since $\mathrm{dom}\,(f\cap g_\xi)\subset V_\xi$ for any $\xi<\mathfrak{c}$, so $f\in\mathrm{SZ}$. Next, fix $x,y\in\mathbb{R}$ and a Cantor set C between $f(x)$ and $f(y)$. There are $n<\omega$ and $\beta<\mathfrak{c}$ such that $I_n\subset(x,y)$ and $C=C_\beta$. Then $H_{\beta,n}\subset(x,y)$ and $f(H_{\beta,n})\subset C$, so f has the CIVP. □

Example 7. *There exists an additive* SZ *injection* $f\colon\mathbb{R}\to\mathbb{R}$ *such that* $f\in\mathrm{PR}\setminus\mathrm{CIVP}$*.*

PROOF. In [BCN, Theorem 2], an additive SZ function $f\colon\mathbb{R}\to\mathbb{R}$ with a perfect road is constructed as the additive extension of a function $\hat{f}\colon\widehat{H}\to\mathbb{R}$ where $\widehat{H}=\{h_\alpha\colon\alpha<\mathfrak{c}\}$ is a Hamel basis containing the set $H=\bigcup_{\alpha<\mathfrak{c}}H_\alpha$ of Lemma 5. For each $\alpha<\mathfrak{c}$, they chose $\hat{f}(h_\alpha)$ such that

(i) $\hat{f}(h_\alpha)\neq\hat{f}(h_\beta)$ for all $\beta<\alpha$ along with other properties, and they chose a set $\widehat{H}_\alpha=H_\gamma$ for some γ such that $h_\alpha=p_\gamma$ and $\widehat{H}_\alpha\cap\{h_\beta\colon\beta\le\alpha\}=\emptyset$.

But here we also require the following.

(ii) $\hat{f}(h_\alpha)\in\widehat{H}\setminus K$, where $K=\widehat{H}_0\cup\{h_0\}$, and

(iii) $\hat{f}(h_0)<\min(K)<\max(K)<\hat{f}(h_1)$.

By (i) and (ii), $\hat{f}\colon \widehat{H} \to \widehat{H} \setminus K$ is one-to-one and so f is one-to-one. To see $f \notin \mathrm{CIVP}$, let C be a perfect nowhere dense subset of $\mathbb{R}$ between h_0 and h_1. Then $f(C) \subset f(\mathbb{R}) = \mathrm{LIN}\,(\hat{f}(\widehat{H})) \subset \mathrm{LIN}\,(\widehat{H} \setminus K) \subset \mathbb{R} \setminus (K \setminus \{h_0\})$ because $\widehat{H}_0 = K \setminus \{h_0\} \subset \widehat{H}$. $\square$

Example 8. *Assume* $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$. *There exists an additive* SZ *function* $f\colon \mathbb{R} \to \mathbb{R}$ *which is Darboux but not connectivity.*

PROOF. Let $\mathbb{R} = \{r_\alpha\colon \alpha < \mathfrak{c}\}$, $r_0 = 0$, and let $H = \{h_\alpha\colon \alpha < \mathfrak{c}\}$ be a Hamel basis. Let $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathcal{C}^*_{G_\delta}$ with $g_0 = \mathrm{id}_{\mathbb{R}}$. We choose inductively two families of two-element sets $\{\{a_\alpha, b_\alpha\}\colon \alpha < \mathfrak{c}\}$, $\{\{c_\alpha, d_\alpha\}\colon \alpha < \mathfrak{c}\}$ such that

(1) The set $\{a_\alpha, b_\alpha\colon \alpha < \mathfrak{c}\}$ is a Hamel basis.

(2) $c_0 = 0$, and $\{d_0\} \cup \{c_\alpha, d_\alpha\colon 0 < \alpha < \mathfrak{c}\} = H$.

(3) If $f\colon \mathbb{R} \to \mathbb{R}$ is the additive function such that $f(a_\alpha) = c_\alpha$ and $f(b_\alpha) = d_\alpha$ for $\alpha < \mathfrak{c}$, then

  (3a) $\mathrm{dom}\,(f \cap g_\xi) \subset \mathrm{LIN}\,(\{a_\alpha, b_\alpha\colon \alpha < \xi\})$ for every $\xi < \mathfrak{c}$;

  (3b) f is Darboux and all level sets of f are countably dense (Thus f is dense in $\mathbb{R}^2$.);

  (3c) $f(x) \neq x$ for any $x \in \mathbb{R} \setminus \{0\}$.

Let $a_0 = h_0$, $c_0 = 0$, $b_0 = h_1$ and $d_0 = h_0$. Assume that α is fixed and $a_\beta, b_\beta, c_\beta, d_\beta$ are defined for $\beta < \alpha$. Let $V_\alpha = \mathrm{LIN}\,(\{a_\beta, b_\beta\colon \beta < \alpha\})$, $W_\alpha = \mathrm{LIN}\,(\{c_\beta, d_\beta\colon \beta < \alpha\})$, $\widehat{W}_\alpha = \mathrm{LIN}\,(W_\alpha \cup \{h_\alpha\})$, and $f_\alpha\colon V_\alpha \to W_\alpha$ be the linear function defined by $f_\alpha(a_\beta) = c_\beta$, $f_\alpha(b_\beta) = d_\beta$ for $\beta < \alpha$.

STEP I. Let

$$a_\alpha \in \mathbb{R} \setminus \Big(V_\alpha + \bigcup_{\beta \le \alpha} \mathbb{Q} g_\beta^{-1}(\widehat{W}_\alpha)\Big).$$

Such a choice is possible because the assumption $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$ implies the inequality $V_\alpha + \bigcup_{\beta \le \alpha} \mathbb{Q} g^{-1}(\widehat{W}_\alpha) \neq \mathbb{R}$. Put $V'_\alpha = \mathrm{LIN}\,(V_\alpha \cup \{a_\alpha\})$.

STEP II. If $h_\alpha \notin W_\alpha$, then $c_\alpha = h_\alpha$. Otherwise choose

$$c_\alpha \in H \setminus \Big(W_\alpha + \bigcup_{\beta \le \alpha} \mathbb{Q} g_\beta(V'_\alpha)\Big).$$

Put $W'_\alpha = \mathrm{LIN}\,(W_\alpha \cup \{c_\alpha\})$. Then

(i) $f_\alpha(v) + q c_\alpha \neq g_\beta(v + q a_\alpha)$ for $\beta \le \alpha$, $v \in V_\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$.

Step III. If $r_\alpha \notin V'_\alpha$, then $b_\alpha = r_\alpha$. Otherwise pick arbitrary $b_\alpha \in H \setminus V'_\alpha$.
Step IV. Choose

$$d_\alpha \in H \setminus \Big(W'_\alpha + \bigcup_{\beta \le \alpha} \mathbb{Q} g_\beta(V_{\alpha+1}) \Big).$$

Then

(ii) $f_\alpha(v)+q_0c_\alpha+q_1d_\alpha \neq g_\beta(v+q_0a_\alpha+q_1b_\alpha)$ for $\beta \le \alpha$, $v \in V_\alpha$, and $q_0, q_1 \in \mathbb{Q}$, $q_1 \neq 0$.

By construction, the set $H_1 = \{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ is linearly independent and, for each $\alpha < \mathfrak{c}$, $r_\alpha \in \mathrm{LIN}(\{a_\beta, b_\beta \colon \beta \le \alpha\})$, so H_1 is a Hamel basis and (1) is fulfilled. Since $d_0 \in H$, $c_\alpha, d_\alpha \in H$ for $0 < \alpha < \mathfrak{c}$, and $h_\alpha \in W'_\alpha$ for each $\alpha < \mathfrak{c}$, the condition (2) holds.

To see that (3a) holds fix $\xi < \mathfrak{c}$ and assume that $f(x) = g_\xi(x)$. Let α be the first ordinal for which $x \in V_{\alpha+1}$. Then there are $v \in V_\alpha$ and $q_0, q_1 \in \mathbb{Q}$ with $|q_0| + |q_1| \neq 0$ such that $x = v + q_0a_\alpha + q_1b_\alpha$. Now we have two cases to consider.

- $q_1 \neq 0$. Then by (ii), $g_\xi(x) = f(x) \neq g_\beta(x)$ for $\beta \le \alpha$, so $\alpha < \xi$.
- $q_1 = 0$. Then $q_0 \neq 0$, and (i) implies $g_\xi(x) = f(x) \neq g_\beta(x)$ for $\beta \le \alpha$, so $\alpha < \xi$.

In both cases $x \in V_\xi$. Now we will verify that (3b) holds. Since the range of f is a linear subspace of $\mathbb{R}$ and, by (2), $H \subset f(\mathbb{R})$, so $f(\mathbb{R}) = \mathbb{R}$. Hence to prove that f is Darboux it is enough to observe the kernel of f, $f^{-1}(0)$, is dense in $\mathbb{R}$. This is because $\mathbb{Q}h_0 \subset f^{-1}(0)$. To see that level sets of f are countable it is enough to prove that the kernel of f is countable. (Recall that any level set of an additive function f is a translation of the kernel of f. See e.g., [MK, Theorem 1], p. 295.) So, fix $x \in f^{-1}(0)$. There are $t_0, \dots, t_n \in H_1 \setminus \{h_0\}$ such that $t_i \neq t_j$ whenever $i \neq j$, and $q, q_0, \dots, q_n \in \mathbb{Q}$ such that $x = qh_0 + q_0t_0 + \dots + q_nt_n$. Then $0 = f(x) = q_0f(t_0) + \dots + q_nf(t_n)$ is a linear combination of the vectors $f(t_0), \dots, f(t_n)$ from the Hamel basis H. Since f is $1-1$ on H_1, so $f(t_i)$'s are pairwise different. Thus $q_0 = \dots = q_n = 0$. Therefore $x = qh_0$ and consequently, $f^{-1}(0) = \mathbb{Q}h_0$.

The conditions (3b) and (3a) together with Lemma 1 imply $f \in \mathrm{SZ}$. Finally observe that the condition (3a) implies $\mathrm{dom}(f \cap \mathrm{id}_{\mathbb{R}}) = \mathrm{dom}(f \cap g_0) \subset \mathrm{LIN}(\emptyset) = \{0\}$. This and (3b) give $f \notin \mathrm{Conn}$. □

Example 9. *Assume the Continuum Hypothesis CH. Then there exists an additive* SZ *function which is* Conn *but not* AC *.*

PROOF. Let $\mathcal{C}_{G_\delta} = \{g_\alpha \colon \alpha < \mathfrak{c}\}$, $g_0 = \emptyset$, $\mathcal{K} = \{K_\beta \colon \beta < \mathfrak{c}\}$ be the family of all continua $K \subset \mathbb{R}^2$ with $\operatorname{card}(\operatorname{dom}(K)) = \mathfrak{c}$, $K_0 = [0,1]^2$, and let $\mathbb{R} = \{r_\gamma \colon \gamma < \mathfrak{c}\}$ be an enumeration of all reals such that the set $X = \{r_n \colon n < \omega\}$ is linearly independent and dense in the interval $(-1,1)$. We will adapt the proof of [CR, Theorem 2.3], where an example of an additive function $f \in \mathrm{Conn} \setminus \mathrm{AC}$ was constructed. Let F, M and Z be as in [CR, Lemma 2.1]; i.e., $F \colon \mathbb{R} \to (-1,1) \times \mathbb{R}$ is a continuous embedding; $M = F(\mathbb{R})$ is closed in $\mathbb{R}^2$; Z is a closed subset of M and $g \cap Z \neq \emptyset$ for every continuous function $g \colon [-1,1] \to \mathbb{R}$; $Z_x = M_x$ is a singleton for all $x \in (-1,1) \setminus X$; for each $x \in X$ the section M_x is a non-trivial closed interval and Z_x consists of the two endpoints of that interval.

We will define inductively a $\subset$-increasing sequence f_ξ, $\xi < \mathfrak{c}$, of additive functions defined on subspaces of $\mathbb{R}$ such that

(i) $\operatorname{dom} f_0 = \mathrm{LIN}(X)$;

(ii) $r_\xi \in \operatorname{dom} f_\xi$;

(iii) $g_\alpha \cap f_\xi \subset f_\alpha$ for $\alpha < \xi$;

(iv) $K_\xi \cap f_\xi \neq \emptyset$;

(v) $Z \cap f_\xi = \emptyset$.

Simultaneously we will choose a sequence $\{g'_\xi \colon \xi < \mathfrak{c}\} \subset \mathcal{C}_{G_\delta}$ with $g'_\xi \subset K_\xi \setminus \mathbb{Q}Z$.

Let $f = \bigcup_{\beta<\mathfrak{c}} f_\beta$. Notice that f is an additive function. By (ii), f is defined on all of $\mathbb{R}$. By (iii), f is SZ. By (iv), f is Conn, and by (v), it is not AC.

The function f_0 is defined inductively, similarly to f_n's, $n < \omega$, in [CR, Theorem 2.3], such that $\langle r_n, f_0(r_n)\rangle \in M$ and the condition (v) holds; i.e., $\langle r_n, f_0(r_n)\rangle \notin W_{r_n} = \mathbb{Q}Z + \mathrm{LIN}(\{\langle r_i, f_0(r_i)\rangle \colon i < n\})$, so $f_0(r_n) \in M_{r_n} \setminus W_{r_n}$. It is possible because for each $n < \omega$ we have an entire interval of possible choices for $f_0(r_n)$ (the set M_{r_n}), while there is only a countable number of exceptional points we have to avoid, (the set W_{r_n}). Let $g'_0 = \emptyset$. Assume that $\xi < \mathfrak{c}$ and the sequences $\{f_\beta \colon \beta < \xi\}$, $\{g'_\beta \colon \beta < \xi\}$ are constructed. We will construct f_ξ and g'_ξ in 3 steps.

STEP I. If $r_\xi \in \operatorname{dom}(\bigcup_{\beta<\xi} f_\beta)$, then $f'_\xi = f_\xi$. Otherwise choose $y \in \mathbb{R}$ such that

1. $\langle r_\xi, y\rangle \notin \mathbb{Q}Z + \bigcup_{\beta<\xi} f_\beta$;
2. $qy + v_2 \neq g_\beta(qr_\xi + v_1)$ for $\langle v_1, v_2\rangle \in \bigcup_{\beta<\xi} f_\beta$, $\beta \leq \xi$, and $q \in \mathbb{Q} \setminus \{0\}$;

3. $qy + v_2 \neq g'_\beta(qr_\xi + v_1)$ for $\langle v_1, v_2\rangle \in \bigcup_{\beta<\xi} f_\beta$, $\beta < \xi$, and $q \in \mathbb{Q} \setminus \{0\}$,

and set $f'_\xi = \mathrm{LIN}\,(\bigcup_{\beta<\xi} f_\beta \cup \{\langle r_\xi, y\rangle\})$.

STEP II. Let $\{I_n : n < \omega\}$ be a sequence of all intervals with rational end-points. For each $n < \omega$ choose $d_{\xi,n}$ such that

1. either $d_{\xi,n} \in I_n$ or $d_{\xi,n} = 0$;
2. the set $D_\xi = \{d_{\xi,n} : n < \omega\} \setminus \{0\}$ is linearly independent;
3. $D_\xi \cap \mathrm{dom}\, f'_\xi = \emptyset$;
4. $\mathrm{LIN}\,(g_\xi \restriction D_\xi \cup f'_\xi) \cap \left(\bigcup_{\beta<\xi} g_\beta \cup \bigcup_{\beta<\xi} g'_\beta\right) \subset f'_\xi$;
5. $\mathrm{LIN}\,(g_\xi \restriction D_\xi \cup f'_\xi) \cap Z = \emptyset$.

Points $d_{\xi,n}$'s are defined inductively. Assume $d_{\xi,i}$ are defined for $i < n$. Let $D_{\xi,n} = \{d_{\xi,i} : i < n\} \setminus \{0\}$ and $f_{\xi,n} = \mathrm{LIN}\,(f'_\xi \cup g_\xi \restriction D_{\xi,n})$. If $\mathrm{dom}\,(g_\xi \setminus (\mathbb{Q}Z + f_{\xi,n}))$ is residual in I_n and all sets $I_n \cap \mathrm{dom}\,([g_\xi \setminus (\mathbb{Q}Z + f_{\xi,n})] \cap [qg_\beta + w])$, $I_n \cap \mathrm{dom}\,([g_\xi \setminus (\mathbb{Q}Z + f_{\xi,n})] \cap [qg'_\beta + w])$ are nowhere dense for all $\beta < \xi$, $q \in \mathbb{Q}$ and $w \in f_{\xi,n}$, then choose $d_{\xi,n} \in I_n \cap \mathrm{dom}\,(g_\xi \setminus (\mathbb{Q}Z + f_{\xi,n})) \setminus \mathrm{dom}\, f_{\xi,n}$ such that

- $\mathrm{LIN}\left(\{\langle d_{\xi,n}, g_\xi(d_{\xi,n})\rangle\} \cup f_{\xi,n}\right) \cap \left(\bigcup_{\beta\leq\xi} g_\beta \cup \bigcup_{\beta<\xi} g'_\beta\right) \subset f_{\xi,n}$;
- $\mathrm{LIN}\left(\{\langle d_{\xi,n}, g_\xi(d_{\xi,n})\rangle\} \cup f_{\xi,n}\right) \cap Z = \emptyset$.

Otherwise, $d_{\xi,n} = 0$. Put $f''_\xi = \mathrm{LIN}\,(g_\xi \restriction D_\xi \cup f'_\xi)$.

STEP III. If $K_\xi \cap f''_\xi \neq \emptyset$, then $f_\xi = f''_\xi$ and $g'_\xi = \emptyset$. Otherwise, choose

$$\langle x, y\rangle \in K_\xi \setminus \Big((\mathrm{dom}\, f''_\xi \times \mathbb{R}) \cup (\mathbb{Q}Z + f''_\xi) \cup (\bigcup_{\beta\leq\xi} \mathbb{Q}g_\beta + f''_\xi) \cup (\bigcup_{\beta<\xi} \mathbb{Q}g'_\beta + f''_\xi)\Big)$$

To argue for this, we will consider 3 cases.

CASE 1. If $\emptyset \neq (I \times \mathbb{R}) \cap (qM + v) \subset K_\xi$ for some $v \in f''_\xi$, $q \in \mathbb{Q} \setminus \{0\}$ and an open interval I, then $K_\xi \cap f''_\xi \neq \emptyset$. (Cf. the proof of [CR, Theorem 2.3].) Moreover, let $g'_\xi = \emptyset$.

CASE 2. Let $A = \{z \in \mathbb{R} : \mathrm{card}\,((K_\xi)_z) = \mathfrak{c}\}$ be uncountable. Then note that A is analytic (cf Mazurkiewicz-Sierpiński Theorem, [AK, Theorem 29.19, p.231]), so it has cardinality $\mathfrak{c}$, and we can choose $x \in A \setminus \mathrm{dom}\, f''_\xi$ and y such that

$$\langle x, y\rangle \in K_\xi \setminus \Big((\mathbb{Q}Z + f''_\xi) \cup \bigcup_{\beta\leq\xi} (\mathbb{Q}g_\beta + f''_\xi) \cup \bigcup_{\beta<\xi} (\mathbb{Q}g'_\beta + f''_\xi)\Big).$$

In this case also $g'_\xi = \emptyset$.

CASE 3. Neither Case 1 nor Case 2 hold. Then $A = \{z \in \mathbb{R}\colon \mathrm{card}\,((K_\xi)_z) = \mathfrak{c}\}$ is countable. Put $Y = K_\xi \setminus (A \times \mathbb{R})$. Then Y is a Baire space, f''_ξ is countable, and for each $q \in \mathbb{Q}$ and $v \in f''_\xi$ the set $qZ + v$ is nowhere dense in Y. (Cf. the proof of Case 3 in [CR, Theorem 2.3].) We claim that $\mathrm{dom}\,(Y \setminus (\mathbb{Q}Z + f''_\xi))$ is residual in $\mathrm{dom}\,K_\xi$. Let V be a non-empty open subset of $\mathrm{dom}\,K_\xi$. Fix $q \in \mathbb{Q}$ and $v \in f''_\xi$. Pick $w \in [(V \times \mathbb{R}) \cap Y] \setminus (qZ + v)$. Let $U \subset V \times \mathbb{R}$ be a K_ξ-open neighborhood of w such that $\mathrm{cl}\,(U) \subset (V \times \mathbb{R}) \setminus (qZ + v)$. By the boundary bumping theorem (cf. [CR, Proposition 1.1]), there is a continuum L such that $w \in L \subset \mathrm{cl}\,(U) \subset (V \times \mathbb{R}) \setminus (qZ + v)$ and $L \cap \mathrm{bd}\,(U) \neq \emptyset$. Since $\mathrm{dom}\,(w) \notin A$, $L \not\subset \mathrm{dom}\,(w) \times \mathbb{R}$. Thus, $\mathrm{int}\,(\mathrm{dom}\,(L)) \neq \emptyset$ and $\mathrm{dom}\,(L) \subset V \cap \mathrm{dom}\,(K_\xi \setminus (qZ + v))$. Thus, $\mathrm{dom}\,(K_\xi \setminus (qZ + v))$ contains a dense open subset of $\mathrm{dom}\,(K_\xi)$. Since A and f''_ξ are countable, we have $\mathrm{dom}\,(Y \setminus (\mathbb{Q}Z + f''_\xi)) = [\bigcap_{v \in f''_\xi, q \in \mathbb{Q}} \mathrm{dom}\,(K_\xi \setminus (\mathbb{Q}Z + f''_\xi))] \setminus A$ is residual in $\mathrm{dom}\,(K_\xi)$. Since the set $Y \setminus (\mathbb{Q}Z + f''_\xi)$ is Borel (Here we use the CH) with all sections countable, the Lusin-Novikow Theorem (See e.g., [AK, Theorem 18.10, p. 123].) implies that there is a Borel function g defined on a set $\mathrm{dom}\,(Y \setminus (\mathbb{Q}Z + f''_\xi))$. Consequently there is a continuous function g'_ξ defined on a G_δ subset of $\mathbb{R}$ which is residual in some interval I, and such that $g'_\xi \subset Y \setminus (\mathbb{Q}Z + f''_\xi)) \subset K_\xi$. Again, let $\{I_n\colon n < \omega\}$ be a sequence of all open intervals with rational end-points. Define inductively a sequence $d'_{\xi,n}$ (similarly to $d_{\xi,n}$'s from the second step) such that

1. either $d'_{\xi,n} \in I_n$ or $d'_{\xi,n} = 0$;
2. the set $D'_\xi = \{d'_{\xi,n}\colon n < \omega\} \setminus \{0\}$ is linearly independent;
3. $D'_\xi \cap \mathrm{dom}\, f''_\xi = \emptyset$;
4. $\mathrm{LIN}\,(g'_\xi \restriction D'_\xi \cup f''_\xi) \cap \left(\bigcup_{\beta \le \xi} g_\beta \cup \bigcup_{\beta < \xi} g'_\beta\right) \subset f''_\xi$;
5. $\mathrm{LIN}\,(g'_\xi \restriction D'_\xi \cup f''_\xi) \cap Z = \emptyset$.

Let $f_\xi = \mathrm{LIN}\,(f''_\xi \cup g'_\xi \restriction D'_\xi)$. Then f_ξ satisfies the conditions (iv) and (v). First we will verify that $f_\xi \cap Z = \emptyset$. Since $f_\xi = \bigcup_{n<\omega} f'_{\xi,n}$ (where $f'_{\xi,n} = \mathrm{LIN}\,(f''_\xi \cup g'_\xi \restriction \{d'_{\xi,i} \neq 0\colon i < n\})$, it is enough to show that $f'_{\xi,n} \cap Z = \emptyset$ for each $n < \omega$. We work inductively. Assume $f'_{\xi,n} \cap Z = \emptyset$. Since $\langle d'_{\xi,n}, g'_\xi(d'_{\xi,n})\rangle \notin \mathbb{Q}Z + f'_{\xi,n}$, we have $f'_{\xi,n+1} \cap \mathbb{Q}Z = \emptyset$.

Now observe that $f_\xi \cap K_\xi \neq \emptyset$. In fact, if all the sets $I \cap \mathrm{dom}\,[(qg_\beta + w) \cap g'_\xi]$ for $\beta \le \xi$ and $w \in f''_\xi$, and $I \cap \mathrm{dom}\,[(qg'_\beta + w) \cap g'_\xi]$ for $\beta < \xi$ and $w \in f''_\xi$ are nowhere dense, then $\langle d'_{\xi,0}, g'_\xi(d'_{\xi,0})\rangle \in f_\xi \cap K_\xi$. Otherwise, there is $\beta \le \xi$ such that either

- there are $q \in \mathbb{Q}\setminus\{0\}$ and $w \in f''_\xi$ such that $\operatorname{dom}[(qg_\beta+w)\cap g'_\xi]$ is residual in some interval J, or
- $\beta < \xi$, and there are $q \in \mathbb{Q}\setminus\{0\}$ and $w \in f''_\xi$ such that $\operatorname{dom}[(qg'_\beta+w)\cap g'_\xi]$ is residual in some interval J.

Let α be the first ordinal with this property, and pick a $q \in \mathbb{Q} \setminus \{0\}$ and a $w = \langle w_1, w_2\rangle \in f''_\xi$ for which $\operatorname{dom}[(qg_\alpha + w) \cap g'_\xi]$ is residual in some interval J. First of all, observe that $(qg_\alpha + w) \cap (J \times \mathbb{R}) \subset K_\xi$, because $qg_\alpha + w$, g'_ξ are continuous, $g'_\xi \subset K_\xi$, and K_ξ is closed. Moreover, for each $\beta < \alpha$, $q' \in \mathbb{Q} \setminus \{0\}$ and $v \in f''_\xi$ the sets $J \cap \operatorname{dom}[(q'g_\beta + v) \cap (qg_\alpha + w)]$, $J \cap \operatorname{dom}[(q'g'_\beta+v)\cap(qg_\alpha+w)]$ are nowhere dense. Let m be the first integer such that $I_m \subset q^{-1}(J - w_1)$ and there are $q' \in \mathbb{Q}$ and $v \in f_{\alpha,m}$ such that at least one of the sets $I_m \cap \operatorname{dom}(q'g_\beta + v)$, $I_m \cap \operatorname{dom}(q'g'_\beta + v)$ is not nowhere dense. (Hence it is residual in some non-degenerate interval.) Then $\operatorname{dom} g_\alpha$ is residual in I_m and for each $\beta \le \alpha$, $q' \in \mathbb{Q}$ and $v \in f_{\alpha,m}$ the sets $I_m \cap \operatorname{dom}(q'g_\beta + v)$, $I_m \cap \operatorname{dom}(q'g'_\beta+v)$ are nowhere dense. Thus $\langle d_{\alpha,m}, g_\alpha(d_{\alpha,m})\rangle \in g_\alpha \cap (I_m \times \mathbb{R})$, and $q\langle d_{\alpha,m}, g_\alpha(d_{\alpha,m})\rangle + w \in f_\xi \cap [qg_\alpha + w] \cap (J \times \mathbb{R}) \subset f_\xi \cap K_\xi$. □

Problem 1. *Can the example above be constructed under a weaker assumption* $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$*?*

Example 10. *Assume* $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$*. There exists an additive* SZ *function which is* AC *and* CIVP.

PROOF. Let $\{I_n : n < \omega\}$ be a sequence of all open intervals with rational end-points, $\mathcal{C}_{G_\delta} = \{g_\alpha : \alpha < \mathfrak{c}\}$, $\{K_\alpha : \alpha < \mathfrak{c}\}$ be the family of all closed sets $K \subset \mathbb{R}^2$ with $\operatorname{dom}(K)$ having non-empty interior, $\{C_\alpha : \alpha < \mathfrak{c}\}$ be a sequence of all perfect subsets of $\mathbb{R}$, and let $\mathcal{H} = \{H_\alpha : \alpha < \mathfrak{c}\}$ be a sequence of pairwise disjoint sets such that:

- the set $\bigcup_{\alpha<\mathfrak{c}} H_\alpha$ is linearly independent;
- for each non-empty open interval I and $\alpha < \mathfrak{c}$ the set $H_\alpha \cap I$ contains a perfect set.

Such a sequence can be obtained as an easy consequence of Lemma 5. (Cf. [KC1, Lemma 3.3].) Let $H = \{h_\alpha : \alpha < \mathfrak{c}\}$ be a Hamel basis that contains all H_α.

We will define a sequence f_α, $\alpha < \mathfrak{c}$, of additive functions and a sequence $P_\alpha \in \mathcal{H}$, $\alpha < \mathfrak{c}$, with the following properties:

(i) $h_\alpha \in \operatorname{dom}(f_\alpha)$ and $\operatorname{card}(\operatorname{dom}(f_\alpha)) < \mathfrak{c}$;

(ii) if $\alpha \neq \beta$, then $P_\alpha \neq P_\beta$;

(iii) $f_\beta \subset f_\alpha$ if $\beta < \alpha$;

(iv) $f_\alpha \cap g_\beta \subset f_\beta$ for $\beta < \alpha$;

(v) $f_\alpha(x) \in C_\alpha$ for $x \in P_\alpha$ and $\alpha < \mathfrak{c}$;

(vi) $f_\alpha \cap K_\alpha \neq \emptyset$.

Functions f_α are constructed by induction. Suppose α is fixed and all f_β, P_β are defined for $\beta < \alpha$.

STEP I. Let $\bar{f}_\alpha = \mathrm{LIN}\,(\bigcup_{\beta<\alpha} f_\beta)$. We define inductively a sequence $d_{\alpha,n}$, $n < \omega$, in the following way. Let $D_{\alpha,n} = \{d_{\alpha,i} \colon i < n\} \setminus \{0\}$ and $f_{\alpha,n} = \mathrm{LIN}\,(\bar{f}_\alpha \cup (g_\alpha | D_{\alpha,n}))$. If

$(*)$ $\mathrm{dom}\,(g_\alpha)$ is residual in I_n, and for all $\beta < \alpha$, $q \in \mathbb{Q}$ and $w \in f_{\alpha,n}$ the set $I_n \cap \mathrm{dom}\,[(qg_\beta + w) \cap g_\alpha]$ is nowhere dense,

then pick $d_{\alpha,n} \in I_n \cap \mathrm{dom}\,(g_\alpha) \setminus \mathrm{dom}\,(f_{\alpha,n})$ such that

$$\mathrm{LIN}\left(\{\langle d_{\alpha,n}, g_\alpha(d_{\alpha,n})\rangle\} \cup f_{\alpha,n}\right) \cap \bigcup_{\beta<\alpha} g_\beta \subset f_{\alpha,n}. \tag{1}$$

Otherwise $d_{\alpha,n} = 0$.

STEP II. Let $\tilde{f}_\alpha = \bigcup_{n<\omega} f_{\alpha,n}$. Let $\beta(\alpha)$ be the first ordinal $\beta < \mathfrak{c}$ for which $H_\beta \cap \mathrm{dom}\,(\tilde{f}_\alpha) = \emptyset$. (Such β exist because $\mathrm{card}\,(\mathrm{dom}\,(\tilde{f}_\alpha)) < \mathfrak{c}$.) Put $P_\alpha = H_{\beta(\alpha)}$. Now choose a number y_α such that $y_\alpha = \tilde{f}(h_\alpha)$ whenever $h_\alpha \in \mathrm{dom}\,\tilde{f}_\alpha$. Otherwise, choose

$$y_\alpha \in \mathbb{R} \setminus \left\{ g_\beta(v + qh_\alpha) - q^{-1}\tilde{f}_\alpha(v) \colon \beta \leq \alpha,\ q \in \mathbb{Q} \setminus \{0\},\ v \in \mathrm{dom}\,(\tilde{f}_\alpha) \right\}$$

Moreover, if $h_\alpha \in P_\xi$ for some $\xi \leq \alpha$, then we may pick $y_\alpha \in C_\xi$. This will give (v).

Put $f_\alpha = \mathrm{LIN}\,(\tilde{f}_\alpha \cup \{\langle h_\alpha, y_\alpha\rangle\})$ and define f as the union of all f_α. As in [NR, Theorem 1] we can verify that f has the property (vi), so $f \in \mathrm{AC}$, and (iv), so $f \in \mathrm{SZ}$. Finally, the property (v) guarantees that $f \in \mathrm{CIVP}$. □

2 Additive Sierpiński-Zygmund Bijections and Their Inverses.

In this section we examine when the inverses of additive one-to-one SZ functions defined on subspaces of $\mathbb{R}$ are also of SZ type. (Note that the inverse of an additive function is additive again.) Recall that in ZFC there exists a one-to-one SZ function $f\colon \mathbb{R} \to \mathbb{R}$ with $f^{-1} \notin \mathrm{SZ}$ [CN1], which we make additive in Example 11, however the existence of an SZ bijection $f\colon \mathbb{R} \to \mathbb{R}$ is not provable in ZFC [BCN] unless one makes an extra assumption like $\mathbb{R}$ is not the union of less than $\mathfrak{c}$-many meager subsets [CN1]. Recall also that it is consistent with ZFC that there is no bijection f from a set $X \in [\mathbb{R}]^{\mathfrak{c}}$ onto a set $Y \in [\mathbb{R}]^{\mathfrak{c}}$ with $f, f^{-1} \in \mathrm{SZ}$ [CN1, Corollary 9].

Example 11. *There exists an additive injection* $f\colon \mathbb{R} \to \mathbb{R}$ *such that* $f \in \mathrm{SZ}$ *and* $f^{-1} \notin \mathrm{SZ}$*.*

PROOF. To see this, let $H = \{h_\alpha \colon \alpha < \mathfrak{c}\}$ be a Hamel basis which meets every perfect set in $\mathbb{R}$. (See e.g., [KC, Theorem 7.3.4].) For $\alpha < \mathfrak{c}$ set $V_\alpha = \mathrm{LIN}\,(\{h_\beta \colon \beta < \alpha\})$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous nowhere constant function such that $\mathrm{card}\,(g^{-1}(y)) = \mathfrak{c}$ for every $y \in \mathbb{R}$. (See e.g., [AB], p.222, for an example of such a function.) Let $\mathcal{C}^*_{G_\delta} = \{g_\xi \colon \xi < \mathfrak{c}\}$. Since the perfect set $g^{-1}(h_\alpha)$ meets H in $\mathfrak{c}$-many points, then by transfinite induction, for each $\alpha < \mathfrak{c}$ we can choose $\hat{f}(h_\alpha) = y_\alpha$ such that

1. $y_\alpha \in g^{-1}(h_\alpha) \cap H$;
2. $y_\alpha \neq y_\beta$ for $\beta < \alpha$;
3. $y_\alpha \neq p g_\beta(x) - f_\alpha(t)$ for $\beta \leq \alpha$, $p \in \mathbb{Q}$, $t \in V_\alpha$, $x \in V_{\alpha+1}$, and f_α being the additive extension of $\hat{f}\restriction\{h_\beta \colon \beta < \alpha\}$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the additive extension of $\hat{f}$. By the condition (2), f is one-to-one. To verify that $f \in \mathrm{SZ}$ use Lemma 1 and observe that for a given $\xi < \mathfrak{c}$, $\mathrm{dom}\,(f \cap g_\xi) \subset V_\xi$, so $\mathrm{card}\,(f \cap g_\xi) < \mathfrak{c}$. Fix $x \in \mathbb{R}$ with $f(x) = g_\xi(x)$. Let α be the first ordinal for which $x \in V_{\alpha+1}$. Then $x = v + q h_\alpha$ for some $v \in V_\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$. Thus $g_\xi(x) = f(x) = f_\alpha(v) + q y_\alpha$, and the condition (3) gives $\alpha < \xi$. Let $A = f(H)$. Since $f\restriction H$ is one-to-one, $\mathrm{card}\,(A) = \mathfrak{c}$, and $f^{-1}\restriction A \subset g$ because $f^{-1}(y_\alpha) = h_\alpha = g(y_\alpha)$ for every $\alpha < \mathfrak{c}$. Therefore $f^{-1}\restriction A$ is continuous, so $f^{-1} \notin \mathrm{SZ}$. □

Example 12. *Assume* $\mathrm{cov}\,(\mathcal{M}) = \mathfrak{c}$*. There exists an additive bijection* $f\colon \mathbb{R} \to \mathbb{R}$ *such that* $f \in \mathrm{SZ}$ *and* $f^{-1} \notin \mathrm{SZ}$*.*

PROOF. Let $\mathcal{C}^*_{G_\delta} = \{g_\alpha \colon \alpha < \mathfrak{c}\}$, $\mathbb{R} = \{r_\alpha \colon \alpha < \mathfrak{c}\}$, and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous nowhere constant function with $\mathrm{card}\,(g^{-1}(y)) = \mathfrak{c}$ for every $y \in \mathbb{R}$. We will construct inductively two families $\{\{a_\alpha, b_\alpha\} \in [\mathbb{R}]^2 \colon \alpha < \mathfrak{c}\}$, $\{\{c_\alpha, d_\alpha\} \in [\mathbb{R}]^2 \colon \alpha < \mathfrak{c}\}$ such that $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$, $\{c_\alpha, d_\alpha \colon \alpha < \mathfrak{c}\}$ are Hamel bases. Then define $f\colon \mathbb{R} \to \mathbb{R}$ as the additive extension of the set of all pairs $\langle a_\alpha, c_\alpha\rangle$, $\langle b_\alpha, d_\alpha\rangle$ for $\alpha < \mathfrak{c}$.

Thus assume that a_β, b_β, c_β and d_β are chosen for $\beta < \alpha$. Let $V_\alpha = \mathrm{LIN}(\{a_\beta, b_\beta \colon \beta < \alpha\})$, $W_\alpha = \mathrm{LIN}(\{c_\beta, d_\beta \colon \beta < \alpha\})$. We choose a_α, b_α, c_α, d_α.

(i) If $r_\alpha \notin V_\alpha$, then $a_\alpha = r_\alpha$. Otherwise pick arbitrary $a_\alpha \in \mathbb{R} \setminus V_\alpha$. Set $V'_\alpha = \mathrm{LIN}(V_\alpha \cup \{a_\alpha\})$.

(ii) If $r_\alpha \notin W_\alpha$, then $d_\alpha = r_\alpha$. Otherwise pick arbitrary $d_\alpha \in \mathbb{R} \setminus W_\alpha$. Set $W'_\alpha = \mathrm{LIN}(W_\alpha \cup \{d_\alpha\})$.

(iii) $c_\alpha \in g^{-1}(a_\alpha) \setminus (W'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi(V'_\alpha))$.

(iv) $b_\alpha \in \mathbb{R} \setminus (V'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi^{-1}(W_{\alpha+1}))$.

Such a choice is possible because the set $V'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi^{-1}(W_{\alpha+1})$ is the union of less than $\omega \cdot \alpha < \mathfrak{c}$ many of meager sets.

First observe that the sets $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ and $\{c_\alpha, d_\alpha \colon \alpha < \mathfrak{c}\}$ are Hamel bases. In fact, they are linearly independent, and for each $\alpha < \mathfrak{c}$, $r_\alpha \in \mathrm{LIN}(\{a_\beta, b_\beta \colon \beta \le \alpha\}) \cap \mathrm{LIN}(\{c_\beta, d_\beta \colon \beta \le \alpha\})$. Let $\tilde{f}$ be defined on $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ by the equations $\tilde{f}(a_\alpha) = c_\alpha$, $\tilde{f}(b_\alpha) = d_\alpha$ for $\alpha < \mathfrak{c}$, and let f be the additive extension of $\tilde{f}$. Then f is an additive bijection on $\mathbb{R}$.

To verify that $f \in \mathrm{SZ}$ fix $\xi < \mathfrak{c}$. We will show that $\mathrm{dom}(f \cap g_\xi) \subset V_\xi$, so $\mathrm{card}(f \cap g_\xi) < \mathfrak{c}$. Fix $x \in \mathbb{R}$ with $f(x) = g_\xi(x)$. Let α be the first ordinal for which $x \in V_{\alpha+1}$. Then $x = v + q_0 a_\alpha + q_1 b_\alpha$ for some $v \in V_\alpha$ and $q_0, q_1 \in \mathbb{Q}$ with $|q_0| + |q_1| \ne 0$. Two cases are possible.

(a) $q_1 = 0$. Then $x = v + q_0 a_\alpha$, $q_0 \ne 0$, and $g_\xi(x) = f(x) = f(v) + q_0 c_\alpha$. Thus $c_\alpha = -q_0^{-1} f(v) + q_0^{-1} g_\xi(x) \in W_\alpha + \mathbb{Q} g_\xi(V'_\alpha)$, and by (iii), $\alpha < \xi$.

(b) $q_1 \ne 0$. Then $f(x) \in W_{\alpha+1}$ and $g_\xi(x) = f(x) = f(v) + q_0 c_\alpha + q_1 d_\alpha$. Thus $v + q_0 a_\alpha + q_1 b_\alpha \in g_\xi^{-1}(f(x)) \subset g_\xi^{-1}(W_{\alpha+1})$, so $b_\alpha \in V'_\alpha + \mathbb{Q} g_\xi^{-1}(W_{\alpha+1})$, and (iv) implies $\alpha < \xi$.

To see $f^{-1} \notin \mathrm{SZ}$, notice that by (iii), $f^{-1}(c_\alpha) = a_\alpha = g(c_\alpha)$ for every $\alpha < \mathfrak{c}$, so $f^{-1} \restriction \{c_\alpha \colon \alpha < \mathfrak{c}\}$ is continuous. □

Example 13. *Assume* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. *There exists an additive bijection* $f \colon \mathbb{R} \to \mathbb{R}$ *such that* $f, f^{-1} \in \mathrm{SZ}$.

PROOF. Let $\mathcal{C}^*_{G_\delta} = \{g_\alpha \colon \alpha < \mathfrak{c}\}$ and $\mathbb{R} = \{r_\alpha \colon \alpha < \mathfrak{c}\}$.We will construct two families of two-element sets $\{\{a_\alpha, b_\alpha\} \in [\mathbb{R}]^2 \colon \alpha < \mathfrak{c}\}$, $\{\{c_\alpha, d_\alpha\} \in [\mathbb{R}]^2 \colon \alpha < \mathfrak{c}\}$, aiming for defining f on $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ by $f(a_\alpha) = c_\alpha$ and $f(b_\alpha) = d_\alpha$. We work inductively. Assume that for a given $\alpha < \mathfrak{c}$ the sequences $\{\{a_\beta, b_\beta\} \in [\mathbb{R}]^2 \colon \beta < \alpha\}$ and $\{\{c_\beta, d_\beta\} \in [\mathbb{R}]^2 \colon \beta < \alpha\}$ are defined, and the sets $\{a_\beta, b_\beta \colon \beta < \alpha\}$, $\{c_\beta, d_\beta \colon \beta < \alpha\}$ are linearly independent. Put $f_\alpha = \mathrm{LIN}\,(\{\langle a_\beta, c_\beta\rangle, \langle b_\beta, d_\beta\rangle \colon \beta < \alpha\})$, $V_\alpha = \mathrm{LIN}\,(\{a_\beta, b_\beta \colon \beta < \alpha\})$, $W_\alpha = \mathrm{LIN}\,(\{c_\beta, d_\beta \colon \beta < \alpha\})$, and notice that f_α is a linear bijection between V_α and W_α. We will choose $a_\alpha, b_\alpha, c_\alpha, d_\alpha$ in 4 steps.

Step I. If $r_\alpha \notin V_\alpha$, then $a_\alpha = r_\alpha$. Otherwise pick arbitrary $a_\alpha \in \mathbb{R} \setminus V_\alpha$. Set $V'_\alpha = \mathrm{LIN}\,(V_\alpha \cup \{a_\alpha\})$.

Step II. If $r_\alpha \notin W_\alpha$, then $d_\alpha = r_\alpha$. Otherwise pick arbitrary $d_\alpha \in \mathbb{R} \setminus W_\alpha$. Set $W'_\alpha = \mathrm{LIN}\,(W_\alpha \cup \{d_\alpha\})$.

Step III. Choose

$$c_\alpha \in \mathbb{R} \setminus \Big(W'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi(V'_\alpha) + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi^{-1}(V'_\alpha)\Big).$$

Observe that this guarantees that the set $\{c_\beta, d_\beta \colon \beta \le \alpha\}$ is linearly independent and moreover,

(1) $f_\alpha(v) + q c_\alpha \neq g_\xi(v + q a_\alpha)$ for $\xi \le \alpha$, $v \in V_\alpha$ and $q \in \mathbb{Q} \setminus \{0\}$.

Step IV. Finally choose

$$b_\alpha \in \mathbb{R} \setminus \Big(V'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi(W_{\alpha+1}) + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi^{-1}(W_{\alpha+1})\Big).$$

Such a choice is possible because each set $g_\xi^{-1}(W_{\alpha+1})$ is the union of less than $\mathfrak{c}$-many meager sets, and $V'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi(W_{\alpha+1})$ has cardinality less than $\mathfrak{c}$, so $V'_\alpha + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi(W_{\alpha+1}) + \bigcup_{\xi \le \alpha} \mathbb{Q} g_\xi^{-1}(W_{\alpha+1})$ is the union of less than $\mathfrak{c}$ many meager sets and does not cover $\mathbb{R}$. Observe that $\{a_\beta, b_\beta \colon \beta \le \alpha\}$ is linearly independent and the following conditions hold:

(2) $q_0 c_\alpha + q_1 d_\alpha + f_\alpha(v) \neq g_\xi(q_0 a_\alpha + q_1 b_\alpha + v)$ for $v \in V_\alpha$, $q_0, q_1 \in \mathbb{Q}$ with $q_1 \neq 0$, $\xi \le \alpha$;

(3) $q_0 a_\alpha + q_1 b_\alpha + f_\alpha^{-1}(w) \neq g_\xi(q_0 c_\alpha + q_1 d_\alpha + w)$ for $w \in W_\alpha$, $q_0, q_1 \in \mathbb{Q}$ with $q_0 \neq 0$, $\xi \le \alpha$;

(4) $f_\alpha^{-1}(w) + q b_\alpha \neq g_\xi(w + q d_\alpha)$ for $w \in W_\alpha$, $q \in \mathbb{Q} \setminus \{0\}$, $\xi \le \alpha$.

First observe that sets $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ and $\{c_\alpha, d_\alpha \colon \alpha < \mathfrak{c}\}$ are Hamel bases. In fact, they are linearly independent, and for each $\alpha < \mathfrak{c}$,

$$r_\alpha \in \mathrm{LIN}\,(\{a_\beta, b_\beta \colon \beta \le \alpha\}) \cap \mathrm{LIN}\,(\{c_\beta, d_\beta \colon \beta \le \alpha\}).$$

Let $\tilde{f}$ be the function defined on $\{a_\alpha, b_\alpha \colon \alpha < \mathfrak{c}\}$ by the equations $\tilde{f}(a_\alpha) = c_\alpha$, $\tilde{f}(b_\alpha) = d_\alpha$ for $\alpha < \mathfrak{c}$, and let f be the additive extension of $\tilde{f}$. Then f is an additive bijection on $\mathbb{R}$.

To verify that $f \in \mathrm{SZ}$ we will show that for a given $\xi < \mathfrak{c}$, $\mathrm{dom}\,(f \cap g_\xi) \subset V_\xi$, so $\mathrm{card}\,(f \cap g_\xi) < \mathfrak{c}$. Fix $x \in \mathbb{R}$ with $f(x) = g_\xi(x)$. Let α be the first ordinal for which $x \in V_{\alpha+1}$. Then $x = v + q_0 a_\alpha + q_1 b_\alpha$ for some $v \in V_\alpha$ and $q_0, q_1 \in \mathbb{Q}$ with $|q_0| + |q_1| \neq 0$. Two cases are possible.

(a) $q_1 = 0$. Then $x = v + q_0 a_\alpha$, $q_0 \neq 0$ and, by (1), $f(x) = f_\alpha(v) + q_0 c_\alpha \neq g_\xi(x)$ for $\xi \leq \alpha$. Thus $\alpha < \xi$.

(b) $q_1 \neq 0$. Then (2) yields $f(x) \neq g_\xi(x)$ for $\xi \leq \alpha$, so $\alpha < \xi$.

In an analogous way we verify that $f^{-1} \in \mathrm{SZ}$. Fix $\xi < \mathfrak{c}$ and $x \in \mathbb{R}$ with $f^{-1}(x) = g_\xi(x)$. Let α be the first ordinal for which $x \in W_{\alpha+1}$. Then $x = w + q_0 c_\alpha + q_1 d_\alpha$ for $w \in W_\alpha$ and $q_0, q_1 \in \mathbb{Q}$ with $|q_0| + |q_1| \neq 0$. Consider two cases.

(a') $q_0 = 0$. Then (4) implies $f^{-1}(x) \neq g_\xi(x)$ for $\xi \leq \alpha$.

(b') $q_0 \neq 0$. Then (3) gives $f^{-1}(x) \neq g_\xi(x)$ for $\xi \leq \alpha$.

Therefore $\alpha < \xi$, so $x \in V_\xi$. $\square$

References

[BCN] M. Balcerzak, K. Ciesielski and T. Natkaniec, *Sierpiński-Zygmund Functions that are Darboux, Almost Continuous, or Have a Perfect Road*, Arch. Math. Logic, **37** (1997), 29–35.

[AB] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Math. Vol. **659**, Springer-Verlag, Berlin, 1978.

[KC] K. Ciesielski, *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts, **39**, Cambridge Univ. Press, 1997.

[KC1] K. Ciesielski, *Some Additive Darboux Like Functions*, J. App. Anal., **4** (1997), 29–35.

[CJ] K. Ciesielski, J. Jastrzębski, *Darboux Like Functions within the Classes of Baire One, Baire Two, and Additive Functions*, Topology Appl., **103** (2000), 203–219.

[CN] K. Ciesielski, T. Natkaniec, *Algebraic Properties of the Class of Sierpiński-Zygmund Functions*, Topology Appl., **79** (1997), 75–99.

[CN1] K. Ciesielski, T. Natkaniec, *On Sierpiński-Zygmund Bijections and Their Inverses*, Topology Proc., **22** (1997), 155–164.

[CP] K. Ciesielski, J. Pawlikowski, *The Covering Property Axiom. A Combinatorial Core of the Iterated Perfect Set Model*, Cambridge Tracts in Mathematics, **164**, Cambridge Univ. Press, Cambridge, 2004.

[CR] K. Ciesielski, A. Rosłanowski, *Two Examples Concerning Almost Continuous Functions*, Topology Appl., **103** (2000), 187–202.

[UD] U. B. Darji, *A Sierpiński-Zygmund Function Which Has a Perfect Road at Each Point*, Colloq. Math., **64** (1993), 159–162.

[GN] R. Gibson, T. Natkaniec, *Darboux Like Functions*, Real Anal. Exchange, **22(2)** (1996–97), 492–533.

[GN1] R. Gibson, T. Natkaniec, *Darboux Like Functions. Old Problems and New Results*, Real Anal. Exchange, **24(2)** (1998–99), 487–496.

[AK] A. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[MK] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, PWN, Warszawa – Kraków – Katowice, 1985.

[NR] T. Natkaniec, H. Rosen, *An Example of an Additive Almost Continuous Sierpiński-Zygmund Function*, Real Anal. Exchange, **30** (2004–05), 261–266.

[KP] K. Płotka, *Sum of Sierpiński-Zygmund and Darboux Like Functions*, Topology Appl., **122(3)** (2002), 547–564.

[SZ] W. Sierpiński, A. Zygmund, *Sur une Fonction qui est Discontinue sur Tout Ensemble de Puissance du Continu*, Fund. Math., **4** (1923), 316–318.