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DYNAMICS OF SYMMETRIC HOLOMORPHIC MAPS ON PROJECTIVE SPACES

Kohei Ueno

Abstract

We consider complex dynamics of a *critically finite* holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, which has symmetries associated with the symmetric group S_{k+2} acting on $\mathbf{P}^k$, for each $k \geq 1$. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1. Introduction

For a finite group G acting on $\mathbf{P}^k$ as projective transformations, we say that a rational map f on $\mathbf{P}^k$ is G-*equivariant* if f commutes with each element of G. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. Doyle and McMullen [**4**] introduced the notion of *equivariant* functions on $\mathbf{P}^1$ to solve quintic equations. See also [**11**] for *equivariant* functions on $\mathbf{P}^1$. Crass [**2**] extended Doyle and McMullen's algorithm to higher dimensions to solve sextic equations. Crass [**3**] found a good family of finite groups and *equivariant* maps for which one may say something about global dynamics. Crass [**3**] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Although I do not know whether this family has relation to solving equations or not, our results will give affirmative answers for the conjectures in [**3**].

In Section 2 we shall explain an action of the symmetric group S_{k+2} on $\mathbf{P}^k$ and properties of our S_{k+2}-equivariant map. In Sections 3 and 4 we shall show our results about the Fatou sets and hyperbolicity of our maps by using properties of our maps and Kobayashi metrics.

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2. S_{k+2}-equivariant maps

Crass [**3**] selected the symmetric group S_{k+2} as a finite group acting on $\mathbf{P}^k$ and found an S_{k+2}-*equivariant* map which is holomorphic and *critically finite* for each $k \geq 1$. We denote by $C = C(f)$ the critical set of f and say that f is *critically finite* if each irreducible component of $C(f)$ is periodic or preperiodic. More precisely, S_{k+2}-*equivariant* map g_{k+3} defined in Section 2.2 preserves each irreducible component of $C(g_{k+3})$, which is a projective hyperplane. The complement of $C(g_{k+3})$ is Kobayashi hyperbolic. Furthermore restrictions of g_{k+3} to invariant projective subspaces have the same properties as above. See Section 2.3 for details.

2.1. S_{k+2} acts on $\mathbf{P}^k$.

An action of the $(k+2)$-th symmetric group S_{k+2} on $\mathbf{P}^k$ is induced by the permutation action of S_{k+2} on $\mathbf{C}^{k+2}$ for each $k \geq 1$. The transposition (i, j) in S_{k+2} corresponds with the transposition "$u_i \leftrightarrow u_j$" on $\mathbf{C}_u^{k+2}$, which pointwise fixes the hyperplane $\{u_i = u_j\} = \{u \in \mathbf{C}_u^{k+2} \mid u_i = u_j\}$. Here $\mathbf{C}^{k+2} = \mathbf{C}_u^{k+2} = \{u = (u_1, u_2, \ldots, u_{k+2}) \mid u_i \in \mathbf{C} \text{ for } i = 1, \ldots, k+2\}$.

The action of S_{k+2} preserves a hyperplane H in $\mathbf{C}_u^{k+2}$, which is identified with $\mathbf{C}_x^{k+1}$ by projection $A\colon \mathbf{C}_u^{k+2} \to \mathbf{C}_x^{k+1}$,

$$H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \overset{A}{\simeq} \mathbf{C}_x^{k+1} \text{ and } A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.$$

Here $\mathbf{C}^{k+1} = \mathbf{C}_x^{k+1} = \{x = (x_1, x_2, \ldots, x_{k+1}) \mid x_i \in \mathbf{C} \text{ for } i = 1, \ldots, k+1\}$.

Thus the permutation action of S_{k+2} on $\mathbf{C}_u^{k+2}$ induces an action of "S_{k+2}" on $\mathbf{C}_x^{k+1}$. Here "S_{k+2}" is generated by the permutation action S_{k+1} on $\mathbf{C}_x^{k+1}$ and a $(k+1, k+1)$-matrix T which corresponds to the transposition $(1, k+2)$ in S_{k+2},

$$T = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \ldots & 1 \end{pmatrix}.$$

Hence the hyperplane corresponding to $\{u_i = u_j\}$ is $\{x_i = x_j\}$ for $1 \leq i < j \leq k+1$. The hyperplane corresponding to $\{u_i = u_{k+2}\}$ is $\{x_i = 0\}$ for $1 \leq i \leq k+1$. Each element in "S_{k+2}" which corresponds to some transposition in S_{k+2} pointwise fixes one of these hyperplanes in $\mathbf{C}_x^{k+1}$.

The action of "S_{k+2}" on $\mathbf{C}^{k+1}$ projects naturally to the action of "S_{k+2}" on $\mathbf{P}^k$. These hyperplanes on $\mathbf{C}^{k+1}$ projects naturally to projective hyperplanes on $\mathbf{P}^k$. Here $\mathbf{P}^k = \{x = [x_1 : x_2 : \cdots : x_{k+1}] \mid (x_1, x_2, \ldots, x_{k+1}) \in \mathbf{C}^{k+1} \setminus \{\mathbf{0}\}\}$. Each element in the action of "S_{k+2}" on $\mathbf{P}^k$ which corresponds to some transposition in S_{k+2} pointwise fixes one of these projective hyperplanes. We denote "S_{k+2}" also by S_{k+2} and call these projective hyperplanes transposition hyperplanes.

2.2. Existence of our maps.

One way to get S_{k+2}-*equivariant* maps on $\mathbf{P}^k$ which are *critically finite* is to make S_{k+2}-*equivariant* maps whose critical sets coincide with the union of the transposition hyperplanes.

Theorem 1 ([**3**]). *For each $k \geq 1$, g_{k+3} defined below is the unique* S_{k+2}-equivariant *holomorphic map of degree $k+3$ which is doubly critical on each transposition hyperplane.*

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}] \colon \mathbf{P}^k \to \mathbf{P}^k,$$

$$\text{where } g_{k+3,l}(x) = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s+1}{s+3} x_l^s A_{k-s}, \quad A_0 = 1,$$

and A_{k-s} is the elementary symmetric function of degree $k-s$ in $\mathbf{C}^{k+1}$.

Then the critical set of g coincides with the union of the transposition hyperplanes. Since g is S_{k+2}-*equivariant* and each transposition hyperplane is pointwise fixed by some element in S_{k+2}, g preserves each transposition hyperplane. In particular g is *critically finite*. Although Crass [**3**] used this explicit formula to prove Theorem 1, we shall only use properties of the S_{k+2}-*equivariant* maps described below.

2.3. Properties of our maps.

Let us look at properties of the S_{k+2}-*equivariant* map g on $\mathbf{P}^k$ for a fixed k, which is proved in [**3**] and shall be used to prove our results. Let L^{k-1} denote one of the transposition hyperplanes, which is isomorphic to $\mathbf{P}^{k-1}$. Let L^m denote one of the intersections of $(k-m)$ or more distinct transposition hyperplanes which is isomorphic to $\mathbf{P}^m$ for $m = 0, 1, \ldots, k-1$.

First, let us look at properties of g itself. The critical set of g consists of the union of the transposition hyperplanes. By S_{k+2}-*equivariance*,

g preserves each transposition hyperplane. Furthermore the complement of the critical set of g is Kobayashi hyperbolic.

Next, let us look at properties of g restricted to L^m for $m = 1, 2, \ldots, k-1$. Let us fix any m. Since g preserves each L^m, we can also consider the dynamics of g restricted to any L^m. Each restricted map has the same properties as above. Let us fix any L^m and denote by $g|_{L^m}$ the restricted map of g to the L^m. The critical set of $g|_{L^m}$ consists of the union of intersections of the L^m and another L^{k-1} which does not include the L^m. We denote it by L^{m-1}, which is an irreducible component of the critical set of $g|_{L^m}$. By S_{k+2}-*equivariance*, $g|_{L^m}$ preserves each irreducible component of the critical set of $g|_{L^m}$. Furthermore the complement of the critical set of $g|_{L^m}$ in L^m is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of g. The set of superattracting points, where the derivative of g vanishes for all directions, coincides with the set of L^0's.

Remark 1. For every $k \geq 1$ and every m, $1 \leq m \leq k$, a restricted map of g_{k+3} to any L^m is not conjugate to g_{m+3}.

2.4. Examples for $k = 1$ and 2.

Let us see transposition hyperplanes of the S_3-*equivariant* function g_4 and the S_4-*equivariant* map g_5 to make clear what L^m is. In [**3**] one can find explicit formulas and figures of dynamics of S_{k+2}-equivariant maps in low-dimensions .

2.4.1. S_3-equivariant function g_4 in $\mathbf{P}^1$.

$$g_3([x_1 : x_2]) = [x_1^3(-x_1 + 2x_2) : x_2^3(2x_1 - x_2)] \colon \mathbf{P}^1 \to \mathbf{P}^1,$$
$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = \{0, 1, \infty\} \text{ in } \mathbf{P}^1.$$

In this case "transposition hyperplanes" are points in $\mathbf{P}^1$ and L^0 denotes one of three superattracting fixed points of g_3.

2.4.2. S_4-equivariant map g_5 in $\mathbf{P}^2$.

$$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\} \text{ in } \mathbf{P}^2.$$

In this case L^1 denotes one of six transposition hyperplanes in $\mathbf{P}^2$, which is an irreducible component of $C(g_5)$. For example, let us fix a transposition hyperplane $\{x_1 = 0\}$. Since g_5 preserves each transposition hyperplane, we can also consider the dynamics of g_5 restricted to $\{x_1 = 0\}$.

We denote by $g_5|_{\{x_1=0\}}$ the restricted map of g_5 to $\{x_1 = 0\}$. The critical set of $g_5|_{\{x_1=0\}}$ in $\{x_1 = 0\} \simeq \mathbf{P}^1$ is

$$C(g_5|_{\{x_1=0\}}) = \{[0:1:0], [0:0:1], [0:1:1]\}.$$

When we use L^0 after we fix $\{x_1 = 0\}$, L^0 denotes one of intersections of $\{x_1 = 0\}$ and another transposition hyperplane, which is a superattracting fixed point of $g_5|_{\{x_1=0\}}$ in $\mathbf{P}^1$. The set of superattracting fixed points of g_5 in $\mathbf{P}^2$ is

$$\{[1:0:0], [0:1:0], [0:0:1], [1:1:1], [1:1:0], [1:0:1], [0:1:1]\}.$$

In general L^0 denotes one of intersections of two or more transposition hyperplanes, which is a superattracting fixed point of g_5 in $\mathbf{P}^2$.

3. The Fatou sets of the S_{k+2}-equivariant maps

3.1. Definitions and preliminaries.

Let us recall theorems about *critically finite* holomorphic maps. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. The Fatou set of f is defined to be the maximal open subset where the iterates $\{f^n\}_{n\geq 0}$ is a normal family. The Julia set of f is defined to be the complement of the Fatou set of f. Each connected component of the Fatou set is called a Fatou component. Let U be a Fatou component of f. A holomorphic map h is said to be a limit map on U if there is a subsequence $\{f^{n_s}|_U\}_{s\geq 0}$ which locally converges to h on U. We say that a point q is a Fatou limit point if there is a limit map h on a Fatou component U such that $q \in h(U)$. The set of all Fatou limit points is called the Fatou limit set. We define the ω-limit set $E(f)$ of the critical points by

$$E(f) = \bigcap_{j=1}^{\infty} \overline{\bigcup_{n=j}^{\infty} f^n(C)}.$$

Theorem 2 ([**10**, Proposition 5.1])**.** *If f is a critically finite holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$, then the Fatou limit set is contained in the ω-limit set $E(f)$.*

Let us recall the notion of Kobayashi metrics. Let M be a complex manifold and $K_M(x, v)$ the Kobayashi quasimetric on M,

$$\inf\left\{|a| \mid \varphi\colon \mathbf{D} \to M\colon \text{holomorphic}, \varphi(0)=x,\ D\varphi\left(a\left(\frac{\partial}{\partial z}\right)_0\right)=v,\ a\in\mathbf{C}\right\}$$

for $x \in M$, $v \in T_xM$, $z \in \mathbf{D}$, where $\mathbf{D}$ is the unit disk in $\mathbf{C}$. We say that M is Kobayashi hyperbolic if K_M becomes a metric. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for $k = 1$ and 2.

Theorem 3 (a basic result whose former statement can be found in [**8**, Corollary 14.5])**.** *If f is a critically finite holomorphic function from $\mathbf{P}^1$ to $\mathbf{P}^1$, then the only Fatou components of f are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $\mathbf{P}^1$.*

Theorem 4 ([**5**, theorem 7.7])**.** *If f is a critically finite holomorphic map from $\mathbf{P}^2$ to $\mathbf{P}^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of f are attractive components of superattracting points.*

3.2. Our first result.

Let us fix any k and $g = g_{k+3}$. For every m, $2 \leq m \leq k$, we can apply an argument in [**5**] to a restricted map of g to any L^m because every L^{m-1} is smooth and because every $L^m \backslash C(g|_{L^m})$ is Kobayashi hyperbolic. We shall use this argument in Lemma 1, which is used to prove Proposition 1.

Proposition 1. *For any Fatou component U which is disjoint from $C(g)$, there exists an integer n such that $g^n(U)$ intersects with $C(g)$.*

Proof: We suppose that $g^n(U)$ is disjoint from $C(g)$ for any n and derive a contradiction by using Lemma 1 and Remark 3 below. Take any point $x_0 \in U$. Since $E(g)$ coincides with $C(g)$, $g^n(x_0)$ accumulates to $C(g)$ as n tends to ∞ from Theorem 2. Since $C(g)$ is the union of the transposition hyperplanes, there exists a smallest integer m_1 such that $g^n(x_0)$ accumulates to some L^{m_1}. Let h_1 be a limit map on U such that $h_1(x_0)$ belongs to the L^{m_1}. From Lemma 1 below, the intersection of $h_1(U)$ and the L^{m_1} is an open set in the L^{m_1} and is contained in the Fatou set of $g|_{L^{m_1}}$.

We next consider the dynamics of $g|_{L^{m_1}}$. If there exists an integer n_2 such that $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with $C(g|_{L^{m_1}})$, then $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with some L^{m_1-1}. In this case we can consider the dynamics of $g|_{L^{m_1-1}}$. On the other hand, if there does not exist such n_2, then there exists an integer m_2 and a limit map h_2 on $h_1(U) \cap L^{m_1}$ such that the intersection of $h_2(h_1(U) \cap L^{m_1})$ and some L^{m_2} is an open set in the L^{m_2} from Remark 3 below. Thus it is contained in the Fatou set of $g|_{L^{m_2}}$. Here m_2 is smaller than m_1. In this case we can consider the dynamics of $g|_{L^{m_2}}$.

We continue the same argument above. These reductions finally come to some L^1 and we use Theorem 3. One can find a similar reduction argument in the proof of Theorem 5. Consequently $g^n(x_0)$ accumulates to some superattracting point L^0. So there exists an integer s such that g^s sends U to the attractive Fatou component which contains the superattracting point L^0. Thus $g^s(U)$ intersects with $C(g)$, which is a contradiction. □

Remark 2. Even if a Fatou component U intersects with some L^m and is disjoint from any L^{m-1}, then the similar thing as above holds for the dynamics in the L^m. In this case $U \cap L^m$ is contained in the Fatou set of $g|_{L^m}$ and there exists an integer n such that $g^n(U \cap L^m)$ intersects with $C(g|_{L^m})$.

Lemma 1. *For any Fatou component U which is disjoint from $C(g)$ and any point $x_0 \in U$, let h be a limit map on U such that $h(x_0)$ belongs to some L^m and does not belong to any L^{m-1}. If $g^n(U)$ is disjoint from $C(g)$ for every $n \geq 1$, then the intersection of $h(U)$ and the L^m is an open set in the L^m.*

Proof: Let B be the complement of $C(g)$. Since B is Kobayashi hyperbolic and B includes $g^{-1}(B)$, $g^{-1}(B)$ is Kobayashi hyperbolic, too. So we can use Kobayashi metrics K_B and $K_{g^{-1}(B)}$. Since B includes $g^{-1}(B)$,

$$K_B(x,v) \leq K_{g^{-1}(B)}(x,v) \text{ for all } x \in g^{-1}(B),\ v \in T_x\mathbf{P}^k.$$

In addition, since g is an unbranched covering from $g^{-1}(B)$ to B,

$$K_{g^{-1}(B)}(x,v) = K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B),\ v \in T_x\mathbf{P}^k.$$

From these two inequalities we have the following inequality

$$K_B(x,v) \leq K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B),\ v \in T_x\mathbf{P}^k.$$

Since the same argument holds for any g^n from $g^{-n}(B)$ to B,

$$K_B(x,v) \leq K_B(g^n(x), Dg^n(v)) \text{ for all } x \in g^{-n}(B),\ v \in T_x\mathbf{P}^k.$$

Since g^n is an unbranched covering from U to $g^n(U)$ and B includes $g^n(U)$ for every n, a sequence $\{K_B(g^n(x), Dg^n(v))\}_{n\geq 0}$ is bounded for all $x \in U$, $v \in T_x\mathbf{P}^k$. Hence we have the following inequality for any unit vectors v_n in $T_{x_0}U$ with respect to the Fubini-Study metric in $\mathbf{P}^k$,

$$(1)\quad 0 < \inf_{|v|=1} K_B(x_0, v) \leq K_B(x_0, v_n) \leq K_B(g^n(x_0), Dg^n(x_0)v_n) < \infty.$$

That is, the sequence $\{K_B(g^n(x_0), Dg^n(x_0)v_n)\}_{n\geq 0}$ is bounded away from 0 and ∞ uniformly.

We shall choose v_n so that $Dg^n(x_0)v_n$ keeps parallel to the L^m and claim that $Dh(x_0)v \neq \mathbf{0}$ for any accumulation vector v of v_n. Let $h = \lim_{n\to\infty} g^n$ for simplicity. Let V be a neighborhood of $h(x_0)$ and ψ a local coordinate on V so that $\psi(h(x_0)) = \mathbf{0}$ and $\psi(L^m \cap V) \subset \{y = (y_1, y_2, \ldots, y_k) \mid y_1 = \cdots = y_{k-m} = 0\}$. In this chart there exists a constant $r > 0$ such that a polydisk $P(\mathbf{0}, 2r)$ does not intersect with any images of transposition hyperplanes which do not include the L^m. Since $\psi(g^n(x_0))$ converges to $\mathbf{0}$ as n tends to ∞, we may assume that $\psi(g^n(x_0))$ belongs to $P(\mathbf{0}, r)$ for large n. Let $\{v_n\}_{n\geq 0}$ be unit vectors in $T_{x_0}\mathbf{P}^k$ and $\{w_n\}_{n\geq 0}$ vectors in $T_{\psi(g^n(x_0))}\mathbf{C}^k$ so that w_n keep parallel to $\psi(L^m)$ with a same direction and

$$Dg^n(x_0)v_n = |Dg^n(x_0)v_n| \; D\psi^{-1}(w_n).$$

So we may assume that the length of w_n is almost unit for large n. We define holomorphic maps φ_n from $\mathbf{D}$ to $P(\mathbf{0}, 2r)$ as

$$\varphi_n(z) = \psi(g^n(x_0)) + rzw_n \text{ for } z \in \mathbf{D}$$

and consider holomorphic maps $\psi^{-1} \circ \varphi_n$ from $\mathbf{D}$ to B for large n. Then

$$(\psi^{-1} \circ \varphi_n)(0) = g^n(x_0),$$

$$D(\psi^{-1} \circ \varphi_n)\left(\frac{|Dg^n(x_0)v_n|}{r}\left(\frac{\partial}{\partial z}\right)_0\right) = Dg^n(x_0)v_n.$$

Suppose $Dh(x_0)v = \mathbf{0}$, then $Dg^n(x_0)v$ converges to $\mathbf{0}$ as n tends to ∞ and so does $Dg^n(x_0)v_n$. By the definition of Kobayashi metric we have that

$$K_B(g^n(x_0), Dg^n(x_0)v_n) \leq \frac{|Dg^n(x_0)v_n|}{r} \to 0 \text{ as } n \to \infty.$$

Since this contradicts (1), we have $Dh(x_0)v \neq \mathbf{0}$. This holds for all directions which are parallel to $\psi(L^m)$. Consequently the intersection of $h(U)$ and the L^m is an open set in L^m. □

Remark 3. The similar thing as above holds for the dynamics of any restricted map. Thus even if a Fatou component $g^n(U)$ intersects with $C(g)$ for some n, the same result as above holds. Because one can consider the dynamics in the L^m when $g^n(U)$ intersects with some L^m.

Theorem 5. *For each $k \geq 1$, the Fatou set of the S_{k+2}-equivariant map g consists of attractive basins of superattracting fixed points which are intersections of k or more distinct transposition hyperplanes.*

Proof: This theorem follows from Proposition 1 and Remark 2 immediately. Let us describe details. Take any Fatou component U. From Proposition 1 there exists an integer n_k such that $g^{n_k}(U)$ intersects with $C(g)$. Since $C(g)$ is the union of the transposition hyperplanes, $g^{n_k}(U)$ intersects with some L^{k-1}. By doing the same thing as above for the dynamics of g restricted to the L^{k-1}, there exists an integer n_{k-1} such that $g^{n_k+n_{k-1}}(U)$ intersects with some L^{k-2} from Remark 2. We again do the same thing as above for the dynamics of g restricted to the L^{k-2}.

These reductions finally come to some L^1. That is, there exists integers $n_{k-2}, \dots, n_2$ such that $g^{n_k+n_{k-1}+\cdots+n_2}(U)$ intersects with some L^1. From Theorem 3 there exists an integer n_1 such that $g^{n_1}(g^{n_k+n_{k-1}+\cdots+n_2}(U))$ contains some L^0. Hence $g^{n_k+n_{k-1}+\cdots+n_1}$ sends U to the attractive Fatou component which contains the superattracting fixed point L^0 in $\mathbf{P}^k$. □

4. Axiom A and the S_{k+2}-equivariant maps

4.1. Definitions and preliminaries.

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [**6**] for details. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$ and K a compact subset such that $f(K) = K$. Let $\widehat{K}$ be the set of histories in K and $\widehat{f}$ the induced homeomorphism on $\widehat{K}$. We say that f is hyperbolic on K if there exists a continuous decomposition $T_{\widehat{K}} = E^u + E^s$ of the tangent bundle such that $D\widehat{f}(E^{u/s}_{\widehat{x}}) \subset E^{u/s}_{\widehat{f}(\widehat{x})}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\widehat{f}^n(v)| \geq c\lambda^n |v| \qquad \text{for all } v \in E^u \quad \text{and}$$
$$|D\widehat{f}^n(v)| \leq c^{-1}\lambda^{-n} |v| \quad \text{for all } v \in E^s.$$

Here $|\cdot|$ denotes the Fubini-Study metric on $\mathbf{P}^k$. If a decomposition and inequalities above hold for f and K, then it also holds for $\widehat{f}$ and $\widehat{K}$. In particular we say that f is expanding on K if f is hyperbolic on K with unstable dimension k. Let Ω be the non-wandering set of f, i.e., the set of points for any neighborhood U of which there exists an integer n such that $f^n(U)$ intersects with U. By definition, Ω is compact and $f(\Omega) = \Omega$. We say that f satisfies Axiom A if f is hyperbolic on Ω and periodic points are dense in Ω.

Let us introduce a theorem which deals with repelling part of dynamics. Let f be a holomorphic map from $\mathbf{P}^k$ to $\mathbf{P}^k$. We define the k-th Julia set J_k of f to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set J_1 coincides with the Julia set J. Let K be a compact subset such that $f(K) = K$. We say that K is a repeller if f is expanding on K.

Theorem 6 ([**7**]). *Let f be a holomorphic map on $\mathbf{P}^k$ of degree at least 2 such that the ω-limit set $E(f)$ is pluripolar. Then any repeller for f intersects J_k. In particular,*

$$J_k = \overline{\{\textit{repelling periodic points of } f\}}.$$

If f is critically finite, then $E(f)$ is pluripolar. We need the following corollary to prove our second result.

Corollary 1 ([**7**]). *Let f be the same as above. Suppose that J_k is a repeller. Then any repeller for f is a subset of J_k.*

4.2. Our second result.

Theorem 7. *For each $k \geq 1$, the S_{k+2}-equivariant map g satisfies Axiom A.*

Proof: We only need to consider the S_{k+2}-*equivariant* map g for a fixed k, because argument for any k is similar as the following one. Let us show the statement above for a fixed k by induction. A restricted map of g to any L^1 satisfies Axiom A by using the theorem of *critically finite* functions (see [**8**, Theorem 19.1]). We only need to show that a restricted map of g to a fixed L^2 satisfies Axiom A. Then a restricted map of g to any L^2 satisfies Axiom A by symmetry. Argument for a restricted map of g to any L^m, $3 \leq m \leq k$, is similar as for a restricted map of g to the L^2. Let us denote $g|_{L^2}$, $\Omega(g|_{L^2})$, and L^2 by g, Ω, and $\mathbf{P}^2$ for simplicity.

We want to show that $g|_{L^2}$ is hyperbolic on $\Omega(g|_{L^2})$ by using Kobayashi metrics. If g is hyperbolic on Ω, then Ω has a decomposition to S_i,

$$\Omega = S_0 \cup S_1 \cup S_2,$$

where $i = 0, 1, 2$ indicate the unstable dimensions. Since $C(g)$ attracts all nearby points, S_0 includes all the L^0's and S_1 includes all the Julia sets of $g|_{L^1}$. We denote by $J(g|_{L^1})$ the Julia set of $g|_{L^1}$. Then g is contracting in all directions at L^0 and is contracting in the normal direction and expanding in an L^1-direction on $J(g|_{L^1})$. Let us consider a compact, completely invariant subset in $\mathbf{P}^2 \setminus C$,

$$S = \{x \in \mathbf{P}^2 \mid \operatorname{dist}(g^n(x), C) \nrightarrow 0 \text{ as } n \to \infty\}.$$

By definition, we have $J_2 \subset S_2 \subset S$. If g is expanding on S, then it follow that $S_0 = \cup L^0$, $S_1 = \cup J(g|_{L^1})$. Moreover $J_2 = S_2 = S$ holds from Corollary 1 (see Remark 4 below). Since periodic points are dense in $J(g|_{L^1})$ and J_2, expansion of g on S implies Axiom A of g.

Let us show that g is expanding on S. Because f is attracting on C and preserves C, there exists a neighborhood V of C such that V is relatively compact in $g^{-1}(V)$ and the complement of V is connected. We assume one of L^1's to be the line at infinity of $\mathbf{P}^2$. By letting B be $\mathbf{P}^2 \setminus V$ and U one of connected components of $g^{-1}(\mathbf{P}^2 \setminus V)$, we have the following inclusion relations,

$$U \subset g^{-1}(B) \Subset B \subset \mathbf{C}^2 = \mathbf{P}^2 \setminus L^1.$$

Because B and U are in a local chart, there exists a constant $\rho < 1$ such that

$$K_B(x, v) \leq \rho K_U(x, v) \text{ for all } x \in U,\ v \in T_x\mathbf{C}^2.$$

In addition, since the map g from U to B is an unbranched covering,

$$K_U(x, v) = K_B(g(x), Dg(v)) \text{ for all } x \in U,\ v \in T_x\mathbf{C}^2.$$

From these two inequalities we have the following inequality

$$K_B(x, v) \leq \rho K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B),\ v \in T_x\mathbf{C}^2.$$

Since g preserves S, which is contained in $g^{-n}(B)$ for every $n \geq 1$,

$$K_B(x, v) \leq \rho^n K_B(g^n(x), Dg^n(v)) \text{ for all } x \in S,\ v \in T_x\mathbf{C}^2.$$

Consequently we have the following inequality for $\lambda = \rho^{-1} > 1$,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x, v) \text{ for all } x \in S,\ v \in T_x\mathbf{C}^2.$$

Since $K_B(x, v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x, v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n|v| \text{ for all } x \in S,\ v \in T_x\mathbf{C}^2.$$

Thus g is expanding on S and satisfies Axiom A. □

Remark 4. Unlike the case when $k = 1$, it does not seem obvious that S being a repeller implies $J_k = S$ when $k \geq 2$.

Remark 5. From [**1**, Theorem 4.11] and [**9**], it follows that the Fatou set of the S_{k+2}-equivariant map g has full measure in $\mathbf{P}^k$ for each $k \geq 1$.

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Graduate School of Human and Environmental Studies
Kyoto University
Yoshida-Nihonmatsu-cho, Sakyo-ku
Kyoto 606-8501
Japan
E-mail address: `ueno@math.h.kyoto-u.ac.jp`

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