REMARK ON THE AVERAGES OF REAL FUNCTIONS

R. E. CHAMBERLIN

1. Introduction. Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. It is known that if for each x in the open interval (a, b) there is a positive number t such that

$$[x-t,\ x+t] \subset (a,\ b) \quad \text{and} \quad f(x) = \frac{f(x-t)+f(x+t)}{2}$$

then $f(x)$ is linear (see [2, p. 253]). The same method of proof shows that if there is such a t for each $x \in (a, b)$ with

$$f(x) = \frac{1}{2t}\int_{x-t}^{x+t} f(s)\, ds$$

then $f(x)$ is linear. Suppose $f(x)$ is such that for each $x \in (a, b)$ there exists a t with $[x-t, x+t] \subset (a, b)$ and

$$(1) \qquad \frac{f(x+t)+f(x-t)}{2} = \frac{1}{2t}\int_{x-t}^{x+t} f(s) ds\,.$$

Is $f(x)$ necessarily linear? On page 231 of [1] it is shown that if (1) holds for each x and all t such that $[x-t, x+t] \subset (a, b)$ then $f(x)$ is linear. The question arises whether or not one can relax the requirement that (1) holds for all t in the above intervals and still conclude that $f(x)$ is linear.

In this note a continuously differentiable non-linear function $f(x)$ is given which satisfies (1) for every $x \in (a, b)$ and an infinity of t's. The values of t depend on x but they may be chosen arbitrarily small for each x. Conditions which together with (1) make $f(x)$ linear are given and the note is concluded with some remarks on the approximation to a function by its averages

$$f(x,\ t) = \frac{1}{2t}\int_{x-t}^{x+t} f(s)\, ds\,.$$

DEFINITION. A continuous function $f(x)$ on $[a, b]$ will be said to have property (1) if for each $x \in (a, b)$ there are arbitrarily small values of t for which (1) is true.

2. An example. We give an example of a continuously differentiable function having property (1) which is not linear. Let

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$$f(x)=\sum_{n=1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2\cdot 10^{2n}} \tag{2}$$

It is clear that $f(x)$ is not linear and is continuously differentiable. To show that $f(x)$ has property (1) we begin with the following

LEMMA. *For every x,*

$$\overline{\lim_{n\to\infty}}\,|\cos 10^{2n}\pi x|\geq 10^{-3}.$$

Since the functions $\cos 10^{2n}\pi x$ $(n\geq 1)$ all have 1 as a period it is clear we need only consider $x\in[0, 1]$ in the proof of this lemma. Since there is no loss in generality we assume hereafter that we are dealing with the interval $[0, 1]$ and x is in this interval.

Let the decimal expansion of x be $.a_1a_2\cdots$. Then

$$10^{2n}x=a_1a_2\cdots a_{2n}+.a_{2n+1}a_{2n+2}\cdots \text{ and } |\cos 10^{2n}\pi x|=|\cos(.a_{2n+1}a_{2n+2}\cdots)\pi|.$$

Suppose $|\cos 10^{2n}\pi x|<10^{-3}$. Set $.a_{2n+1}a_{2n+2}\cdots=.5+r_n$ where $|r_n|<.5$. Then

$$10^{-3}\geq|\cos(.a_{2n+1}a_{2n+2}\cdots)\pi|=|\sin r_n\pi|=\sin|r_n\pi|\geq\frac{2}{\pi}|r_n\pi|,$$

that is $\frac{1}{2\cdot 10^3}\geq|r_n|$. Hence there is an integer b with $0\leq b\leq 5$ such that $|r_n|=.000b\cdots$. Therefore,

$$|\cos 10^{2(n+1)}\pi x|=|\cos(.0b\cdots)\pi|\geq\left(1-\frac{(.1\pi)^2}{2}\right)>.9.$$

Thus for every x and every n_0 there are integers $n>n_0$ such that $|\cos 10^{2n}\pi x|\geq 10^{-3}$. This proves the lemma.

For the function (2) we have

$$\begin{aligned} g(x, t)&=\frac{1}{2}[f(x+t)+f(x-t)]-\frac{1}{2t}\int_{x-t}^{x+t}f(s)ds \\ &=\sum_{n=1}^{\infty}\frac{1}{2\cdot 10^{2n}\cdot n^2}[\cos 10^{2n}\pi(x+t)+\cos 10^{2n}\pi(x-t)] \\ &\quad-\frac{1}{2t}\sum_{n=1}^{\infty}\frac{\sin 10^{2n}\pi(x+t)-\sin 10^{2n}\pi(-t)}{10^{2n}\cdot n^3\cdot 10^{2n}\pi}. \end{aligned} \tag{3}$$

From elementary trigonometric identities we now obtain

$$g(x, t)=\sum_{n=1}^{\infty}\frac{1}{10^{2n}n^2}\cos 10^{2n}\pi x\left[\cos 10^{2n}\pi t-\frac{\sin 10^{2n}\pi t}{10^{2n}\pi t}\right].$$

We investigate in detail the last expression for $g(x, t)$ in (3).

Given x, let $\overline{\lim_{n\to\infty}}|\cos 10^{2n}\pi x|=d$. From the lemma it is clear there are an infinity of integers k with the following properties:

(a) $|\cos 10^{2k}\pi x|>.99d$.

(b) $|\cos 10^{2n}\pi x|<1.01\ d$ for $n\geq[k/3]$

(c) $k\geq 10$.

For these values of k we show that the sign of $g(x, t)$ in (3) is determined by the sign of the k-th term in its series expansion if t is chosen properly. We assume hereafter that k is subject to conditions (a), (b) and (c).

For the given x and subject to conditions (a), (b) and (c) pick k large enough so that for $t=2\cdot 10^{-2k}$, $[x-t, x+t]\subset[0, 1]$. Then

$$(4)\qquad g(x, 10^{-2k})=\sum_{n=1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left[\cos 10^{2(n-k)}\pi-\frac{\sin 10^{2(n-k)}\pi}{10^{2(n-k)}\pi}\right]$$

$$=\sum_{n=1}^{k-1}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left(-\frac{\pi^2}{6}10^{4(n-k)}+\theta_n\cdot 10^{6(n-k)}\right)+(-1)\cdot\frac{\cos 10^{2k}\pi x}{k^2 10^{2k}}+\sum_{n=k+1}^{\infty}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}$$

where $|\theta_n|<2$. Now

$$(5)\qquad \left|\sum_{n=1}^{k-1}\frac{\cos 10^{2n}\pi x}{n^2 10^{2n}}\left(-\frac{\pi^2}{6}10^{4(n-k)}+\theta_n 10^{6(n-k)}\right)\right|$$

$$\leq\frac{10}{3}\cdot\frac{1}{10^{2k}}\cdot\sum_{n=1}^{k-1}\frac{|\cos 10^{2n}\pi x|}{n^2}10^{2(n-k)}$$

$$\leq\frac{10}{3}10^{-2k}\left(\sum_{n=1}^{[k/3]-1}\frac{1}{n^2}10^{2(n-k)}\right)\cdot 10^3 d+\left(\frac{10}{3}\right)10^{-2k}\left(\sum_{n=[k/3]}^{k-1}\frac{1}{n^2}10^{2(n-k)}\right)1.01\ d$$

where we have used the lemma and property (b) of k to get the last inequality.

For the first sum in the last inequality of (5) we have

$$(6)\qquad \sum_{n=1}^{[k/3]-1}\frac{1}{n^2}10^{2(n-k)}<\sum_{n=1}^{[k/3]-1}10^{2(n-k)}\leq 10^{-4/3(k-1)}\frac{1-(10^{-2})^{(k/3)}}{1-10^{-2}}$$

$$<(1.01)10^{-4/3(k-1)}.$$

To get an estimate on the second part of the last inequality of (5), recall that if $s_n=\sum_{i=1}^{n}\alpha_i$

then

$$\sum_{n=r}^{m}\alpha_n\beta_n=\sum_{n=r}^{m}s_n(\beta_n-\beta_{n+1})-s_{r-1}\beta_r+s_m\beta_{m+1}.$$

Letting $\alpha_n=10^{2(n-k)}$, $\beta_n=1/n^2$ we get

$$(7)\quad \sum_{n=[k/3]}^{k-1}\frac{10^{2(n-k)}}{n^2}=\sum_{n=[k/3]}^{k-1}10^{-2(k-n)}\left(\frac{1-10^{-2n}}{1-10^{-2}}\right)\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$
$$-10^{-2(k-[k/3]-1)}\left(\frac{1-10^{-2[k/3]}}{1-10^{-2}}\right)\cdot 1/[k/3]^2+10^{-2}\left(\frac{1-10^{-2(k-1)}}{1-10^{-2}}\right)\cdot\frac{1}{k^2}<\frac{2}{10^2}\frac{1}{k^2}$$

at least for $k\gtrsim 10$. Using the estimates obtained in (6) and (7) we get

$$(8)\quad \frac{10}{3}\cdot 10^{-2k}\sum_{n=1}^{k-1}\frac{|\cos 10^{2n}\pi x|}{n^2 10^{2n}}\leq\frac{d}{10^{2k}}\left[\frac{1.01}{3}10^{-4/3(k-1)+4}+\frac{10}{3}\cdot(1.01)\cdot\frac{2}{10^2 k^2}\right]$$
$$<\frac{2}{10}d\cdot\frac{1}{k^2 10^{2k}}\quad\text{for}\quad k\geq 10\,.$$

Furthermore

$$(9)\quad \left|\sum_{n=k+1}^{\infty}\frac{\cos 10^{2n}\pi x}{10^{2n}n^2}\left[\cos 10^{2(n-k)}\pi-\frac{\sin 10^{2(n-k)}\pi}{10^{2(n-k)}\pi}\right]\right|$$
$$\leq 1.01\,d\sum_{n=k+1}^{\infty}\frac{1}{10^{2n}n^2}<\frac{1.01\,d}{(k+1)^2}\cdot\frac{1}{10^{2(k+1)}}\cdot\frac{1}{1-10^{-2}}<\frac{1}{10}\frac{d}{k^2 10^{2k}}\,.$$

From (8) and (9) we see that the k-th term of the series for $g(x,\,10^{-2k})$ is greater in absolute value than the sum of the remaining terms. Hence the signs of $g(x,\,10^{-2k})$ and $-10^{-2k}k^{-2}\cos 10^{2k}\pi x$ are the same. For $t=2\cdot 10^{-2k}$ the k-th term of the series for $g(x,\,2\cdot 10^{-2k})$ is $10^{-2k}k^{-2}\cos 10^{2k}\pi x$ and in the same manner as above one can show that the signs of $g(x,\,2\cdot 10^{-2k})$ and the k-th term are the same. Since $10^{-2k}k^{-2}\cos 10^{2k}\pi x$ and $-10^{-2k}k^{-2}\cos 10^{2k}\pi x$ are of opposite signs, $g(x,\,t)$ vanishes for some $t\in(10^{-2k},2\cdot 10^{-2k})$. But for $g(x,\,t)$ to vanish means that $f(x)$ satisfies (1). Since for each x there are an infinity of k's satisfying (a), (b) and (c), there are (for each x) arbitrarily small values of t for which the $f(x)$ of (2) satisfies (1). Hence this $f(x)$ has the property (1).

3. Sufficient conditions for a function to be linear.

LEMMA 1. *If $f(x)$ is continuously differentiable and $f''(x_0)\neq 0$, then $g(x_0,\,t)$ is of one sign for some open interval $(0,\,t_0)$ $(t_0>0)$.*

Under the stated conditions we may represent $f(x)$ by

$$(10)\quad f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2+o[(x-x_0)^2].$$

Using (10) and the definition of $g(x,\,t)$ gives

$$(11)\quad g(x_0, t)=\frac{f(x_0+t)+f(x_0-t)}{2}-\frac{1}{2t}\int_{x_0-t}^{x_0+t} f(u)du$$
$$=\left\{f(x_0)+\frac{f''(x_0)}{2}t^2+o(t^2)\right\}-\frac{1}{2t}\int_{x_0-t}^{x_0+t}\left\{f(x_0)+f'(x_0)(u-x_0)\right.$$
$$\left.+\frac{f'(x_0)}{2}(u-x_0)^2+o[(u-x_0)^2]\right\}du=\frac{1}{3}f''(x_0)t^2+o(t^2).$$

Thus if $f''(x_0)\neq 0$, it is clear $g(x_0, t)$ is one-signed for sufficiently small values of t.

THEOREM 1. *If $f(x)$ has property* (1) *and $f'(x)$ is absolutely continuous then $f(x)$ is linear.*

For $f''(x)$ exists almost everywhere and by Lemma 1 it is zero everywhere it exists because $f(x)$ has property (1). Hence $f'(x)$ is a constant and $f(x)$ is linear.

THEOREM 2. *If $f'(x)$ is continuous, monotone increasing and not constant in any sufficiently small symmetric interval about x_0 then $g(x_0, t)$ is one-signed in an interval* $(0, t_0)$.

One has

$$f(x_0+t)=f(x_0-t)+\int_{x_0-t}^{x_0+t} f'(u)\,du$$

and for any $x\in(x_0-t,\ x_0+t)$ we get

$$(12)\quad f(x)\leq f(x_0-t)+f'(x)(x-x_0+t),\ f(x_0+t)\geq f(x)+f'(x)(x_0+t-x)$$

where at least one of the inequalities is strict by the hypothesis of Theorem 2. From (12) one obtains

$$(13)\qquad \frac{(x-x_0+t)f(x_0+t)+(x_0-x+t)f(x_0-t)}{2t}>f(x)\,.$$

It is obvious from (13) that $g(x_0, t)$ is positive. Clearly this result with the inequality reversed holds if $f'(x)$ is monotone decreasing.

We do not know if property (1) and bounded variation of $f'(x)$ imply linearity for $f(x)$. In view of the two preceding theorems it seems quite likely.

4. Remarks on the approximation of a function by its averages.

Suppose $f(x)$ is a continuous function defined on the interval $(a-\delta,\ b+\delta)$ $(\delta>0)$. We make some remarks on the approximation to

$f(x)$ by its averages

$$f(x,\ t)=\frac{1}{2t}\int_{x-t}^{x+t} f(u)du \qquad (0<t<\delta)\ ,\ x\varepsilon[a,\ b]\,.$$

If $f(x)$ is linear then $f(x,\ t)\equiv f(x)$. If $f(x)$ is not linear in any subinterval then there is an everywhere dense subset of points x at which the approximating functions are all either above or below $f(x)$. Otherwise the conditions of the theorem of [2, p. 253] are met and $f(x)$ would be linear.

One might ask if there are necessarily points at which $f(x,\ t)$ approaches $f(x)$ monotonely. From the results of § 2 above this can be seen to be false. For $t>0$, $f(x,\ t)$ is continuously differentiable function of t and

$$f_t(x,\ t)=\frac{1}{t}\left\{\frac{f(x+t)+f(x-t)}{2}-\frac{1}{2t}\int_{x-t}^{x+t} f(u)du\right\}=\frac{1}{t}\,g(x,\ t)\,.$$

From this it is clear the function of § 2 gives an example of a continuously differentiable function which at no point is approximated monotonely by its averages.

REFERENCES

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UNIVERSITY OF UTAH