FOURIER ANALYSIS AND DIFFERENTIATION OVER REAL SEPARABLE HILBERT SPACE

F. H. BROWNELL

1. Introduction. Let l_2 denote as usual the space of square summable real sequences, the prototype of real separable Hilbert space. It is well known that l_2 possesses no non-trivial, translation invariant Borel measures. However, l_2 does have infinitely many subspaces X, locally compact in the l_2 norm relative topology, which we may call translation spaces and for which such measures φ exist [2]. Here the spaces X are not groups under l_2 vector addition, so the notion of translation invariance must be appropriately modified. For any such X we may of course use the corresponding φ to define over $z \in l_2$ a Fourier transform F of $f \in L_1(X, \mathscr{B}, \varphi)$ by

$$F(z) = \int_X f(x) e^{i(z,x)} d\varphi(x) .$$

However, in order to get the expected inverse formula, it seems necessary to be able to make X into a group—roughly speaking to define a vector in X corresponding to $x+y$ when this l_2 vector sum $\notin X$. This is a severe restriction on our translation spaces X, and the only natural ones still available seem to be essentially modifications of Jessen's infinite torus [9]. With orthogonal coordinates this is the space X_0 defined below, a modified Hilbert cube.

Since X_0 is a locally compact abelian topological group, Fourier analysis upon it becomes standard procedure. We are able to extend some standard one-variable theorems (see [1]), relating Fourier transforms and the operation of differentiation, to the situation here, which seems new. In a summary at the end we discuss the significance of these results as related to the work in functional analysis of Fréchet, Gâteaux, Lévy, Hille, Zorn, Cameron and Martin, and Friedrichs.

2. Fourier integrals on X_0. Let

$$X_0 = \{x \in l_2 \mid -h_n < x_n \leq h_n \text{ for integer } n \geq 1\}$$

where the fixed sequence of extended real h_n, $0 < h_n \leq +\infty$, has

$$\sum_{n=N+1}^{\infty} h_n^2 < +\infty$$

for some fixed integer $N \geq 0$. For simplicity we assume $h_n = +\infty$ for

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$1 \leq n \leq N$ if $N \geq 1$. Define $+'$ addition as l_2 vector addition modulo the subgroup $I_0 = \{x \in l_2 \mid x_n = 0 \text{ for } n \leq N,\ x_n/2h_n = m_n, \text{ an integer, for } n \geq N+1\}$. Define $P(x)$ for $x \in l_2$ as the unique element of X_0 in the coset $x + I_0$; thus clearly $x +' y = P(x+y) \in X_0$ for x and $y \in X_0$. After defining the inverse $-'x = P(-x)$ for $x \in X_0$, we see that X_0 becomes a group under $+'$ and $-'$. However, the operation $+'$ is not continuous under the metric $\|x-y\|$ defined by the l_2 norm

$$\|x\| = \left[\sum_{n=1}^{\infty} x_n^2\right]^{\frac{1}{2}}.$$

Thus, following Gelfand [5], we introduce the modified norm $|||x||| = \|P(x)\|$ for $x \in l_2$. That $+'$ and $-'$ are continuous under the resulting metric $|||x-y|||$ is clear from the easily verified statements

$$\begin{aligned} |||(\tilde{x} +' \tilde{y}) - (x +' y)||| &= \|P(\tilde{x} - x + \tilde{y} - y)\| \leq \|P(\tilde{x} - x) + P(\tilde{y} - y)\| \\ &\leq |||\tilde{x} - x||| + |||\tilde{y} - y||| \quad \text{and} \quad |||(-'y) - (-'x)||| = |||y - x|||. \end{aligned}$$

Thus X_0 is a topological group under the metric topology of the modified norm. Note that $P(x)$ is continuous from l_2 onto X_0 under the appropriate l_2 and modified norm metrics, since

$$|||P(x) - P(y)||| = \|P(x-y)\| \leq \|x-y\|.$$

We can easily verify that the as yet unused condition

$$\sum_{n=N+1}^{\infty} h_n^2 < +\infty$$

is necessary and sufficient for X_0 to be locally compact under either the l_2 norm or modified norm metric topologies. Thus X_0, under the latter topology, possesses a regular Haar measure φ defined over $\mathscr{B}$, the Borel subsets of X_0; and φ is unique up to constant factors. Hence φ is non-trivial and invariant under $+'$, though, as we remarked above, this φ could be constructed for $+$ alone without making X_0 into a group, (see [2]). To fix φ, let

$$V_1 = \{x \in X_0 \mid |x_n| < \tfrac{1}{2} \text{ for } n \leq N\};$$

thus V_1, being non-void and open with compact closure, must satisfy $0 < \varphi(V_1) < +\infty$. We specify φ uniquely by requiring $\varphi(V_1) = 1$.

In order to get Fourier analysis on X_0 following Godement [6] or Weil [11], we need to determine the continuous characters on X_0, that is all continuous complex valued functions $\psi(x)$ on $x \in X_0$ with $|\psi(x)| = 1$ and $\psi(x +' y) = \psi(x)\psi(y)$. Here let

$$Z_0 = \left\{z \in l_2 \mid z_n = \frac{\pi p_n}{h_n} \text{ with } p_n \text{ an integer for } n \geq N+1\right\}.$$

Note that since $\sum_{n=N+1}^{\infty} h_n^2 < +\infty$ and $z \in l_2$ make $h_n \to 0$ and $z_n \to 0$ as $n \to \infty$, each $z \in Z_0$ must have $p_n \equiv 0$ and thus $z_n \equiv 0$ for sufficiently large n. Let

$$(x, y) = \sum_{n=1}^{\infty} x_n y_n$$

denote the l_2 inner product.

LEMMA 1. *The group of characters $\tilde{X}_0$ is isomorphic with Z_0, each character having the form $\psi(x) = e^{i(z,x)}$ with $z \in Z_0$.*

Proof. Let $\exp[i\Phi(x)] = \psi(P(x))$ for any $\psi \in \tilde{X}_0$, with $\Phi(0) = 0$ and $\Phi(x)$ defined uniquely by requiring continuity. Thus $\Phi(x)$ is a continuous linear functional over l_2, so $\Phi(x) = (x, z) = (z, x)$ for some unique $z \in l_2$. For $h_n < +\infty$, taking $x_j = 2h_n$ if $j = n$ and $x_j = 0$ if not, we see that $P(x) = 0$. Hence $2\pi p_n = \Phi(x) = (z, x)$ makes $z_n = \pi p_n / h_n$, so $z \in Z_0$.

Let $Z_0 \subseteq l_2$ be topologized relatively from l_2. Clearly this topology is equivalent to the product of the euclidean E_N topology with the discrete topology on the part $n > N$, where $z_n = \pi p_n / h_n$ and $h_n \to 0$. Z_0 so topologized forms a locally compact abelian topological group under l_2 vector addition, η denoting its Haar measure. Clearly this topology on Z_0 is equivalent to the Hausdorff space topology with neighborhoods as finite intersections of sets of the form

$$N_{\rho,F}(z_0) = \{z \in Z_0 \mid |(z - z_0, x)| < \rho \text{ for } x \in F\} ,$$

$\rho > 0$ and F a norm bounded subset of X_0. Equivalently on $\tilde{X}_0$ this topology is given by

$$\tilde{N}_{\delta,F}(\psi_0) = \{\psi \in X_0 \mid |\psi(x) - \psi_0(x)| < \delta \text{ for } x \in F\} .$$

Now $(X, \mathscr{B}, \varphi)$ is a σ-finite measure space, so $L_\infty(X_0, \mathscr{B}, \varphi)$ is the conjugate space of $L_1(X_0, \mathscr{B}, \varphi)$. Thus the argument of Godement, [6, p. 87], is valid and Z_0 is homeomorphic to $\tilde{X}_0 \subseteq L_\infty(X_0, \mathscr{B}, \varphi)$ under the weak topology defined by $L_1(X_0, \mathscr{B}, \varphi)$.

We may normalize η uniquely by requiring the Fourier inversion formula (2.2), which must hold as stated in Lemmas 2 and 3 following. The formulae are:

$$F(z) = \int_{X_0} e^{i(z,x)} f(x) d\varphi(x) . \tag{2.1}$$

$$f(x)=\int_{Z_0} e^{-i(z,x)}F(z)d\eta(z)\ . \tag{2.2}$$

Here we note that any $f\in L_1(X_0, \mathscr{B}, \varphi)$ has its Fourier transform $F(z)$ defined and continuous on Z_0 by (2.1); and if such $F\in L_1(Z_0, \mathscr{B}', \eta)$, $\mathscr{B}'$ being the Borel subsets of Z_0, then the right side of (2.2) also exists and is continuous. For Lemmas 2 and 3 let $\mathscr{M}$ be the class of all convolutions

$$[u*v](x)=\int_{X_0} u(x-'y)v(y)d\varphi(y)$$

of continuous functions $u(x)$ and $v(x)$ vanishing outside compact subsets of X_0. (For proof of these following well-known lemmas see [6, p. 90-94]. The density of $\mathscr{M}$ in Lemma 2 follows from the regularity of φ.)

LEMMA 2. *$\mathscr{M}$ is dense in $L_1(X_0, \mathscr{B}, \varphi)$ and $L_2(X_0, \mathscr{B}, \varphi)$, and each $f\in\mathscr{M}$ has its Fourier transform $F\in L_1(Z_0, \mathscr{B}', \eta)$ with* (2.2) *holding at each $x\in X_0$ for the inverse transformation.*

LEMMA 3. *If $f\in L_2(X_0, \mathscr{B}, \varphi)$, then there exists a unique Plancherel transform $F\in L_2(Z_0, \mathscr{B}', \eta)$ such that every sequence $\{f_k\}\subseteq\mathscr{M}$ with the L_2 norm $\|f-f_k\|_2\to 0$ also has $\|F-F_k\|_2\to 0$. Moreover, every sequence $\{f_k\}\subseteq\mathscr{M}$ with $\|F-F_k\|_2\to 0$ also has $\|f-f_k\|_2\to 0$. This Plancherel transformation takes $L_2(X_0, \mathscr{B}, \varphi)$ onto $L_2(Z_0, \mathscr{B}', \eta)$ as a Hilbert space isomorphism,*

$$\int_{X_0} f(x)\overline{g(x)}d\varphi(x)=\int_{Z_0} F(z)\overline{G(z)}d\eta(z)\ , \qquad f,\ g\in L_2\ . \tag{2.3}$$

In order to determine η explicitly, let S be the set of all integer valued sequences $\zeta=\{p_n\}$ over $n>N$ such that $p_n=0$ for large enough n for each sequence; thus S is countable. Let $z=(\omega\,;\,\zeta)$ be defined for $\omega\in E_N$, $\zeta\in S$ by $z_n=\omega_n$ for $n\leq N$ and $z_n=\pi p_n/h_n$ for $n>N$. Letting $\chi_A(z)$ be the characteristic function of any $A\in\mathscr{B}'$, with μ_N Lebesgue measure on E_N,

$$\eta(A)=\left(\frac{1}{2\pi}\right)^N \sum_{\zeta\in S}\left\{\int_{E_N}\chi_A(\omega\,;\,\zeta)\,d\mu_N(\omega)\right\}=\int_{E_N}\left\{\sum_{\zeta\in S}\chi_A(\omega\,;\,\zeta)\right\}\frac{d\mu_N(\omega)}{(2\pi)^N} \tag{2.4}$$

follows, by applying Lemma 3 to the Gaussian

$$f(x)=\exp\left(-\frac{1}{2}\sum_{n=1}^{N}x_n^2\right)$$

to determine the normalization.

3. Fourier transforms and X_0 differentiation. Here let X_n denote X_0 with the nth coordinate omitted, φ_n the corresponding measure over the σ-algebra $\mathscr{B}_n$ of Borel subsets of X_n, and $\mathscr{G}_1$ the Borel σ-algebra of E_1 if $n \leq N$, of $(-h_n, h_n)$ if $n > N$. Then [7, p. 222], we see that $\mathscr{B} = \mathscr{B}_n \times \mathscr{G}_1$ as the uncompleted product; also, using the uniqueness of Haar measure, $\varphi = \varphi_n \times \mu_1$ or $= \varphi_n \times (\mu_1/2h_n)$ according as $n \leq N$ or $> N$. Now consider $f \in L_1(X_0, \mathscr{B}, \varphi)$, let $\tilde{x}$ denote x with the nth coordinate omitted, and define $K_n(t, x_n) = 1$ if $-h_n < t \leq x_n$, $K(t, x_n) = 0$ if not. Clearly $K_n(t, x_n) f(x_1, \cdots, x_{n-1}, t, x_{n+1}, \cdots)$ is measurable $(\mathscr{B}_n \times \mathscr{G}_1 \times \mathscr{G}_1 = (\mathscr{B} \times \mathscr{G}_1)$ over $(\tilde{x}, x_n, t) \in X_n \times E_1 \times E_1$ if $n \leq N$, or $X_n \times (-h_n, h_n] \times (-h_n, h_n]$ if $n > N$. Thus if we define

$$\int f(x)\, dx_n = \int_{-\infty}^{\infty} K_n(t, x_n) f(x, t) dt,$$

then the Fubini theorem makes $\int f(x)\, dx_n \in L_1(X_n \times I, \mathscr{B}, \varphi)$ for any finite x_n interval I.

For the following theorems we will say that $f(x)$ is x_n absolutely continuous if for all $\tilde{x} \in X_n - A$, where A is some set $\in \mathscr{B}_n$ having $\varphi_n(A) = 0$, we have $f(P(\tilde{x}, x_n))$ absolutely continuous as a function of x_n over every finite interval of E_1.

THEOREM 4. *If $f \in L_1(X_0, \mathscr{B}, \varphi)$, if f is x_n absolutely continuous, and if f'_n, the resulting x_n first partial, is $\in L_1(X_0, \mathscr{B}, \varphi)$ also, then the* (2.1) *defined Fourier transforms F_n and F of f'_n and f have $F_n(z) = -iz_n F(z)$ over $z \in Z_0$.*

Proof. Consider first $h_n < +\infty$, so we know almost everywhere (φ) on X_0 that

$$f(x) = \int f'_n(x)\, dx_n + f(P(\tilde{x}, -h_n)) = \int f'_n(x) dx_n + f(\tilde{x}, h_n).$$

Now

$$\int_{-h_n}^{h_n} e^{iz_n t} dt = 0 \qquad \text{for } z_n \neq 0,$$

so

$$F(z) = \int_{X_n} \int_{-h_n}^{h_n} e^{i(z,x)} \left\{ \int f'_n(x_n)\, dx_n \right\} \frac{dx_n}{2h_n} d\varphi_n(\tilde{x}) .$$

But

$$\int_{-h_n}^{h_n} e^{iz_n s} \left\{ \int_{-h_n}^{s} f'_n(\tilde{x}, t)\, dt \right\} ds$$
$$= \frac{e^{iz_n h_n}}{iz_n} \int_{-h_n}^{h_n} f'_n(\tilde{x}, t)\, dt - \frac{1}{iz_n} \int_{-h_n}^{h_n} e^{iz_n s} f'_n(\tilde{x}, s)\, ds$$

by integrating by parts, and

$$\int_{-h_n}^{h_n} f'_n(\tilde{x},\, t)dt = f(P(\tilde{x},\, h_n)) - f(P(\tilde{x},\, -h_n)) = 0\ .$$

Thus $F(z) = -(1/iz_n)F_n(z)$ for $z_n \neq 0$. If $z_n = 0$, then

$$\int_{-h_n}^{h_n} f'_n(\tilde{x},\, t)dt = 0$$

makes $F_n(z) = 0$, so $F_n(z) = -iz_n F(z)$ for all $z \in Z_0$.

Secondly if $h_n = +\infty$, we know

$$f(x) = \int f'_n(x)dx_n + C(\tilde{x})$$

almost everywhere (φ_n) over $\tilde{x} \in X_n$. Thus $f(\tilde{x},\, x_n) \to C(\tilde{x})$ as $x_n \to -\infty$, so $f(\tilde{x},\, x_n) \in L_1(E_1)$ in x_n almost everywhere (φ_n) requires $C(\tilde{x}) = 0$, $f(x) = \int f'_n(x)dx_n$, and similarly $\int_{-\infty}^{\infty} f'_n(\tilde{x},\, t)dt = 0$ almost everywhere (φ_n). Thus

$$\begin{aligned}\int_{-\infty}^{\infty} e^{iz_n s} f(\tilde{x},\, s)\,ds &= \int_{-\infty}^{\infty} e^{iz_n s}\left\{\int_{-\infty}^{s} f'_n(\tilde{x},\, t)\,dt\right\}ds \\ &= \lim_{a,b\to\infty}\left[\frac{e^{iz_n s}}{iz_n}\int_{-\infty}^{s} f'_n(\tilde{x},\, t)\,dt\right]_{-a}^{b} - \frac{1}{iz_n}\int_{-\infty}^{\infty} e^{iz_n s} f'_n(\tilde{x},\, s)ds \\ &= \frac{-1}{iz_n}\int_{-\infty}^{\infty} e^{iz_n s} f'_n(\tilde{x},\, s)ds, \quad \text{so} \quad F(z) = -\frac{1}{iz_n}F_n(z) \qquad \text{for } z_n \neq 0.\end{aligned}$$

If $z_n = 0$, then $\int_{-\infty}^{\infty} f_n'(\tilde{x},\, t)dt = 0$ makes $F_n(z) = 0$, so $F_n(z) = -iz_n F(z)$ for all $z \in Z_0$.

For the next lemma we need to remark that $T(x;\, y) = (x;\, y - 'x)$ is a homeomorphism of $X_0 \times X_0$ into itself, and hence leaves unchanged the Borel class $\mathscr{B} \times \mathscr{B}$, [7, p. 257]. Thus $A \in \mathscr{B}$ has $T(X_0 \times A) \in \mathscr{B} \times \mathscr{B}$, so clearly any $f(x)$ measurable $(\mathscr{B})$ has $f(x + 'y)$ measurable $(\mathscr{B} \times \mathscr{B})$. Let ${}_n e \in l_2$ be defined by ${}_n e_k = \delta_{n,k}$, and we then easily see, using

$$\{(x\,;\, y) \in X_0 \times X_0 \mid y_k = 0 \text{ for } k \neq n\} \in \mathscr{B} \times \mathscr{B},$$

that such f also have $f(x + 't_n e)$ measurable $(\mathscr{B} \times \mathscr{G}_1)$ over $x \in X_0$ and t real.

Lemma 5. *If $f \in L_r(X_0,\, \mathscr{B},\, \varphi)$ with real $r \geq 1$, if f is x_n absolutely continuous, and if the resulting $f'_n \in L_r(X_0,\, \mathscr{B},\, \varphi)$, then defining*

$$_nf_h(x)=\frac{1}{h}\{f(x+'h\,_ne)-f(x)\}$$

over real $h\neq 0$ *yields*

$$\lim_{h\to 0}\|_nf_h-f'_n\|_r=0\ .$$

Proof. Since $x+'h\,_ne=P(x+h\,_ne)$, we know that

$$_nf_h(x)-f'_n(x)=\frac{1}{h}\int_0^h\{f'_n(x+'t\,_ne)-f'_n(x)\}dt$$

almost everywhere (φ_n) over $\tilde{x}\in X_n$. With $1/r'=1-1/r$ if $r>1$, $1/r'$ replaced by 0 if $r=1$. The Schwarz-Hölder inequality thus yields

$$|_nf_h(x)-f'_n(x)|\leq|h|^{1/r'-1}\left|\int_0^h|f'_n(x+'t\,_ne)-f'_n(x)|^r\,dt\right|^{1/r}.$$

Then by the Fubini theorem

$$\begin{aligned}\|_nf_h-f'_n\|_r^r&\leq\frac{1}{|h|}\left|\int_0^h\left\{\int_{X_0}|f'_n(x+'t\,_ne)-f'_n(x)|^r\,d\varphi(x)\right\}dt\right|\\&\leq\sup_{|t|\leq|h|}\|g_t-g\|_r^r\end{aligned}$$

where $g(x)=f'_n(x)\in L_r$ and $g_t(x)=g(x+'t\,_ne)$. The functions $u(x)$, continuous on X_0 under the modified norm topology and vanishing outside compact subsets of X_0, are L_r norm dense in $L_r(X_0,\mathscr{B},\varphi)$ by the regularity of φ; and such u have $\|u_t-u\|_r\to 0$ as $t\to 0$ by their uniform continuity. Also $\|g_t-u_t\|_r=\|g-u\|_r$ by φ invariance, so

$$\|_nf_h-f'_n\|_r\leq 2\|g-u\|_r+\sup_{|t|\leq|h|}\|u_t-u\|_r$$

and hence $\|_nf_h-f'_n\|_r\to 0$ as $h\to 0$.

We also have the following converse for $r=2$.

LEMMA 6. *If* f *and* $g\in L_2(X_0,\mathscr{B},\varphi)$ *and if* $\lim_{h\to 0}\|_nf_h-g\|_2=0$, *then* $f(x)=\tilde{f}(x)$ *almost everywhere* (φ) *for some* $\tilde{f}(x)$ *measurable* $(\mathscr{B})$ *which is* x_n *absolutely continuous and has its derivative* $\tilde{f}'_n(x)=g(x)$ *almost everywhere* (φ).

Proof.

$$\|_nf_h-g\|_2^2=K\int_{X_n}\left\{\int_{-h_n}^{h_n}|_nf_h(x)-g(x)|^2dx_n\right\}d\varphi_n(\tilde{x})$$

by the Fubini theorem, so using a Riesz-Fischer subsequence $h=t_k\to 0$

we have

$$\lim_{t_k\to 0}\int_{-h_n}^{h_n} |{}_nf_{t_k}(x)-g(x)|^2dx_n=0$$

for almost (φ_n) all $\tilde{x}\in X_n$. This reduces our statement to the one real variable analogue, where the result is well known (see for example Bochner, [1, p. 131], if $h_n=+\infty$). Since we may take

$$\tilde{f}(x)=\int_0^{x_n} g(\tilde{x},\, t)\,dt+\tilde{f}(\tilde{x},\, 0)$$

almost everywhere (φ_n) with

$$\tilde{f}(\tilde{x},\, 0)=\frac{1}{a}\int_0^a \left\{f(\tilde{x},\, s)-\int_0^s g(\tilde{x},\, t)\,dt\right\}ds$$

for $0<a<h_n$, clearly $\tilde{f}(x)$ may be taken measurable $(\mathscr{B})$.

The L_2 counterpart of Theorem 4 now follows.

THEOREM 7. *If $f\in L_2(X_0,\, \mathscr{B},\, \varphi)$, if f is x_n absolutely continuous, and if the resulting $f_n'\in L_2(X_0,\, \mathscr{B},\, \varphi)$ too, then the Plancherel transforms F and F_n of f and f_n' satisfy $F_n(z)=-iz_nF(z)$ almost everywhere (η).*

Proof. Using the Fubini theorem in (2.1) and the translation invariance of φ, we have

$${}_nF_h(z)=\frac{1}{h}(e^{-iz_nh}-1)F(z)$$

for the transform of ${}_nf_h$ in case $f\in L_1\cap L_2$, and hence for all $f\in L_2$ by the Plancherel Lemma 3 with $L_1\cap L_2$ dense in L_2. Since

$$\lim_{h\to 0}\frac{1}{h}(e^{-iz_nh}-1)=-iz_n$$

and since $\|{}_nF_h-F_n\|_2\to 0$ as $h\to 0$ by Lemma 5 and (2.3), the Riesz-Fischer theorem yields $F_n(z)=-iz_nF(z)$ as desired.

It is easy to get an extended converse of Theorem 7.

THEOREM 8. *If f and $g\in L_2(X_0,\, \mathscr{B},\, \varphi)$ and have transforms F and G satisfying $G(z)=(-iz_n)^kF(z)$ for integer $k>0$, then $f(x)=\tilde{f}(x)$ almost everywhere (φ) for some $\tilde{f}(x)$ measurable $(\mathscr{B})$ such that $\tilde{f}(x)$ possesses everywhere up to $(k-1)$st order x_n partials which are $\in L_2(X_0,\, \mathscr{B},\, \varphi)$,* *the* $(k-1)$*st* $\tilde{f}^{(k-1)}_{n,n,\cdots,n}(x)$ *is* x_n *absolutely continuous, and*

$$\tilde{f}^{(k)}_{n,n,\cdots,n}(x)=g(x)$$

almost everywhere (φ).

Proof. From $(-iz_n)^k F(z)$ and $F(z)\in L_2(Z_0, \mathscr{B}', \eta)$ clearly $(-iz_n)^p F(z) \in L_2$ also for $p=0, 1, \cdots, k-1$, and by taking inverse Plancherel transforms we get ${}_p g \in L_2(X_0, \mathscr{B}, \varphi)$ transforming into $(-iz_n)^p F(z)$. As we have seen before the difference quotient ${}_p g_h$ of ${}_p g$ will have the transform

$$\frac{1}{h}(e^{-iz_n h}-1)(-iz_n)^p F(z)=\left\{\int_0^1 e^{-iz_n ht}dt\right\}(-iz_n)^{p+1}F(z) .$$

Since $|\{\ \ \}|\leq 1$ and $\{\ \ \}\to 1$, this transform $\to(-iz_n)^{p+1}F(z)$ in L_2 norm as $h\to 0$. Hence $\|{}_p g_h - {}_{p+1}g\|_2\to 0$ as $h\to 0$ by the Plancherel lemma, and so Lemma 6 with ${}_0 g=f$ and ${}_k g=g$ gives the result.

The following converse of Theorem 8 is considerably deeper than Theorem 7. We remark that if f and $g\in L_1(X_0, \mathscr{B}, \varphi)$, then $f*g\in L_1$ also and has the Fourier transform $F(z)G(z)$, where

$$[f*g](x)=\int_{X_0} f(x-'y)g(y)d\varphi(y)$$

exists almost everywhere (φ). More important for us, if f and $g\in L_2(X_0, \mathscr{B}, \varphi)$, then $f*g$ is the inverse Fourier transform of $F(z)G(z)\in L_1(Z_0, \mathscr{B}, \eta)$, defined pointwise by (2.2), and hence also the inverse Plancherel transform if $FG\in L_2$. This follows by noting that $e^{i(z,x)}\overline{F(z)}$ is the transform of $\overline{f(x-'y)}$ as a function of y and by using (2.3).

THEOREM 9. *If* $f\in L_2(X_0, \mathscr{B}, \varphi)$ *and possesses everywhere up to* $(k-1)$*st order* x_n *partials, if* $f^{(k-1)}_{n,\cdots,n}(x)$ *is* x_n *absolutely continuous, and if* $f^{(k)}_{n,\cdots,n}(x)\in L_2(X_0, \mathscr{B}, \varphi)$, *then also* $f^{(p)}_{n,\cdots,n}(x)\in L_2(X_0, \mathscr{B}, \varphi)$ *for* $p=1, 2, \cdots, k$, *and such* $f^{(p)}_{n,\cdots,n}$ *have the transforms* $(-iz_n)^p F(z)$.

Proof. First we construct rather arbitrarily a smoothing transform

$$G(z)=\exp\left(-\frac{1}{2}\sum_{j=1}^{N}\omega_j^2-\frac{1}{2}\gamma_n z_n^2\right)\rho(\zeta)$$

for $z=(\omega\,;\,\zeta)$ of $\omega\in E_N$ and $\zeta\in S$ using the notation of (2.4), where $\gamma_n=0$ if $n\leq N$ and $\gamma_n=1$ if $n>N$. S being countable we may set $S=\{{}_k\zeta\}$ and define $\rho(\zeta)$ on S by setting $\rho({}_k\zeta)=e^{-k}$. We see clearly from (2.4) for each integer $p\geq 0$ that $(-iz_n)^p G(z)\in L_1\cap L_2\cap L_\infty$ for the measure space $(Z_0, \mathscr{B}', \eta)$, since

$$|z_n|^p e^{-\frac{1}{2}z_n^2}$$

is bounded and $O(e^{-|z_n|})$ as $|z_n|\to\infty$. Also $G(z)>0$ everywhere on Z_0, these two conditions being all we really need. Take g as the unique element of $L_2(X_0, \mathscr{B}, \varphi)$ transforming into G, and by Theorem 8 we may take $g(x)$ to possess all order derivatives in x_n with $g^{(p)}_{n,\cdots,n}\in L_2$ transforming into $(-iz_n)^p G(z)$.

Now for $h_n<+\infty$ and $0\leq p\leq k$, by integrating by parts we see that

$$\int_{-h_n}^{h_n} g^{(p)}_{n,\cdots,n}(x-'y)f(y)\,dy_n=\int_{-h_n}^{h_n} g(x-'y)f^{(p)}_{n,\cdots,n}(y)dy_n$$

existent finite for almost (φ_n) all $\tilde{y}\in X_n$ for each $x\in X_0$, using the periodicity of $g(P(x-y))$ and $f(P(y))$ at the endpoints $y_n=\pm h_n$. If $h_n=+\infty$ we still get the same result by a slightly different argument. Here we know $f^{(k)}_{n,\cdots,n}(x)\in L_2(-\infty, \infty)$ over x_n for almost (φ_n) all $\tilde{x}\in X_n$, so by the Schwarz inequality follows

$$f^{(k-1)}_{n,\cdots,n}(x)=O(|x_n|)$$

as $|x_n|\to\infty$ for such $\tilde{x}$. Thus by further integration

$$f^{(p)}_{n,\cdots,n}(x)=O(|x_n|^k)$$

as $|x_n|\to\infty$ for such $\tilde{x}$, $0\leq p\leq k-1$. Now clearly

$$g(x)=e^{-\frac{1}{2}x_n^2}g_1(\tilde{x}) ,$$

so

$$g^{(p)}_{n,\cdots,n}(x)=O(e^{-|x_n|})$$

as $|x_n|\to\infty$. These two estimates are enough to make the endpoint terms vanish in integration by parts, so

$$\int_{-\infty}^{\infty} g^{(p)}_{n,\cdots,n}(x-'y)f(y)\,dy_n=\int_{-\infty}^{\infty} g(x-'y)f^{(p)}_{n,\cdots,n}(y)dy_n.$$

Thus with $K=1$ or $1/2h_n$ we have

$$[g^{(p)}_{n,\cdots,n}*f](x)=K\int_{X_n}\int_{-h_n}^{h_n} g(x-'y)f^{(p)}_{n,\cdots,n}(y)dy_n d\varphi_n(\tilde{y})$$

existent finite in the order written for $0\leq p\leq k$ and all $x\in X_0$.

Now for $p=k$ we are given $f^{(k)}_{n,\cdots,n}\in L_2$, so the Schwarz inequality shows $g(x-'y)f^{(k)}_{n,\cdots,n}(y)$ to be $\in L_1$. Thus by the Fubini theorem

$$[g^{(k)}_{n,\dots,n} * f](x) = [g * f^{(k)}_{n,\dots,n}](x)$$

at all $x \in X_0$. By our remarks preceding this theorem, since $(-iz_n)^k G(z)$ and $G(z) \in L_\infty$ make $(-iz_n)^k G(z) F(z)$ and $G(z) F_k(z) \in L_2$, for the Plancherel transforms we have $[(-iz_n)^k G(z)] F(z) = G(z) F_k(z)$. Thus since $G(z) > 0$ everywhere, $F_k(z) = (-iz_n)^k F(z)$ with $F_k \in L_2$ the transform of $f^{(k)}_{n,\dots,n} \in L_2$. Thus Theorem 8 gives the result.

THEOREM 10. *If f and $g \in L_2(X_0, \mathscr{B}, \varphi)$ and if their transforms F and G satisfy*

$$G(z) = -\left(\sum_{n=1}^{\infty} z_n^2\right) F(z) ,$$

then there exists a sequence of functions ${}_nf(x)$ measurable ($\mathscr{B}$) such that ${}_nf(x) = f(x)$ almost everywhere (φ), ${}_nf(x)$ is x_n absolutely continuous as well as its everywhere existent first x_n derivative ${}_nf'_n(x)$, ${}_nf'_n$ and ${}_nf''_{nn} \in L_2(X_0, \mathscr{B}, \varphi)$, and $\sum_{n=1}^{M} {}_nf''_{nn}$ converges in L_2 norm to g as $M \to \infty$.

Proof. Let $g_n \in L_2(X_0, \mathscr{B}, \varphi)$ be defined uniquely by requiring $G_n(z) = -z_n^2 F(z)$, since $|z_n^2 F(z)| \leq |G(z)|$ makes $z_n^2 F(z) \in L_2(Z_0, \mathscr{B}', \eta)$. Now $\sum_{n=1}^{\infty} z_n^2$ is actually a finite sum for each $z \in Z_0$, and also

$$\left| \sum_{n=1}^{M} z_n^2 \right| |F(z)| \leq |G(z)| \in L_2 ,$$

so by dominated convergence $\sum_{n=1}^{M} G_n(z) \to G(z)$ in L_2 norm as $M \to \infty$, and hence also $\sum_{n=1}^{M} g_n \to g$ in L_2 norm. Finally Theorem 8 for each n gives the desired ${}_nf(x) = f(x)$ almost everywhere (φ), ${}_nf'_n$ and ${}_nf''_{nn} \in L_2$, and ${}_nf''_{nn}(x) = g_n(x)$ almost everywhere (φ) as desired.

THEOREM 11. *Let f and $g \in L_2(X_0, \mathscr{B}, \varphi)$ and let a sequence of functions ${}_nf(x)$ measurable ($\mathscr{B}$) satisfy the conditions: ${}_nf(x) = f(x)$ almost everywhere (φ); ${}_nf$ everywhere possesses a first x_n partial ${}_nf'_n$ which is x_n absolutely continuous; ${}_nf''_{nn} \in L_2(X_0, \mathscr{B}, \varphi)$; and $\sum_{n=1}^{M} {}_nf''_{nn} \to g$ in $L_2(X_0, \mathscr{B}, \varphi)$ norm as $M \to \infty$. Then the transforms F and G satisfy*

$$G(z) = -\left(\sum_{n=1}^{\infty} z_n^2\right) F(z)$$

almost everywhere (η).

Proof. By Theorem 9 we also have ${}_nf'_n \in L_2$ and ${}_nf''_{nn}$ has the transform $G_n(z) = -z_n^2 F(z)$. From $\sum_{n=1}^{M} {}_nf''_{nn} \to g$ in L_2 we thus know

$\sum_{n=1}^{M} G_n \to G$ in L_2 norm as $M \to \infty$, where

$$\sum_{n=1}^{M} G_n(z) = -\Big(\sum_{n=1}^{M} z_n^2\Big) F(z) .$$

Since $\sum_{n=1}^{\infty} z_n^2$ is actually a finite sum at each $z \in Z_0$, Riesz-Fischer subsequences yield

$$G(z) = -\Big(\sum_{n=1}^{\infty} z_n^2\Big) F(z)$$

as desired.

4. Significance of results. The main results of this paper are Theorems 7 through 11 relating Fourier transforms over X_0, a modification of the Hilbert cube, to the operations of differentiation in an L_2 sense. It is clear that Theorems 10 and 11 allow one to use Fourier transforms to define a generalized Laplace differential operator for scalar functions on X_0. This definition is in a global L_2 sense, which gives a pointwise definition only by using Riesz-Fischer subsequences. The ideas of pointwise infinite dimensional derivatives seem to go back to Fréchet and Gâteaux. Hille [8, pp. 71–90], Zorn [12], and others have developed a notion of analyticity from similar complex differentiability on complex Banach spaces.

To be precise, for real l_2 consider a real valued function $f(x)$ over $x \in l_2$ and define the gradient $\nabla f(x) = y$ at each x such that there exists $y \in l_2$ having over $u \in l_2$

$$\lim_{\|u\| \to 0} \|u\|^{-1} |f(x+u) - f(x) - (u, y)| = 0 , \tag{4.1}$$

such y being clearly unique. This is a Fréchet type definition. If we let $\{w_n\}$ be a complete orthonormal system in l_2, we have where $\nabla f(x)$ exists that

$$(w_n, \nabla f(x)) = \Big[\frac{d}{d\lambda} f(x + \lambda w_n)\Big]_{\lambda=0} . \tag{4.2}$$

This equation could also serve as a Gâteaux type definition of $\nabla f(x)$, possibly depending on $\{w_n\}$, wherever the squares of the right hand terms are summable. For the divergence, if $T(x) \in l_2$ for each $x \in l_2$, we may formulae the Gâteaux type definition

$$(\nabla, T(x)) = \sum_{n=1}^{\infty} (w_n, \nabla \psi_n(x)) \text{ for } \psi_n(x) = (T(x), w_n) , \tag{4.3}$$

which is independent of the choice of base $\{w_n\}$ if

$$\sum_{n=1}^{\infty} \|\nabla \psi_n(x)\| < +\infty \quad \text{and} \tag{4.4}$$

$$0 = \lim_{\|u\| \to 0} \|u\|^{-1} \left[\sum_{n=1}^{\infty} |\psi_n(x+u) - \psi_n(x) - (u, \nabla \psi_n(x))|^2 \right]^{\frac{1}{2}}.$$

Finally we can define the Laplacian $\nabla^2 f(x) = (\nabla, \nabla f(x))$, so that

$$(\nabla, \nabla f(x)) = \sum_{n=1}^{\infty} \left[\frac{d^2}{d\lambda^2} f(x + \lambda w_n) \right]_{\lambda=0}$$

shows this definition to agree pointwise with the expression in Theorems 10 and 11, $\sum_{n=1}^{\infty} f''_{nn}(x)$.

Lévy has also constructed a vector analysis for Hilbert space, though he is led to define

$$\lim_{M \to \infty} \frac{1}{M} \left\{ \sum_{n=1}^{M} f''_{nn}(x) \right\}$$

as the Laplacian, [5, p. 248]. He differs more seriously from our approach by using a development of mean values of functions instead of integration with respect to a non-trivial, translation invariant measure. Thus he has no need to reduce l_2 to X_0, though naturally his theory of mean values must pay for this by certain anomalous features. Cameron and Martin have also done a great deal of functional analysis in terms of Wiener measure on the continuous functions ([3] and others), but since this is not translation invariant, it has little contact with our work.

It seems that our results relating Fourier transforms and differentiation over real Hilbert space may be useful in a mathematical formulation of quantum radiation theory, just as finite dimensional differential operators are very conveniently defined self-adjointly in terms of Fourier transforms. Friedrichs has discussed such problems and is led to still a different method of integration over Hilbert space, [4, p. 60]. However, the set functions induced by his method are not σ-additive and apparently not translation invariant either.

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