ON THE APPROXIMATION OF A FUNCTION OF SEVERAL VARIABLES BY THE SUM OF FUNCTIONS OF FEWER VARIABLES

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1. The problems. Let R denote the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, and C_R the space of all continuous real-valued functions z defined on R, with norm $\| z \|$ defined by $\| z \| = \max_{(x,y) \epsilon R} |z|$. Let I_x and I_y denote respectively the unit intervals $0 \leq x \leq 1$ and $0 \leq y \leq 1$; and let C_x and C_y denote respectively the classes of all continuous functions on I_x and I_y. By an obvious identification C_x and C_y may be considered as subsets of C_R. Let C_S denote the subset of C_R composed of all functions $z \in C_R$ such that $z = f + g$ where $f \in C_x$ and $g \in C_y$; C_S is closed (under the above norm).

For $z \in C_R$, define the functional $\mu[z]$ by

$$\mu[z] = \text{dist}[z, C_S] = \inf_{w \epsilon C_s} \| z - w \|.$$

The following problem was posed by The RAND Corporation.

Problem (A): Given $z \in C_R$ and $\epsilon > 0$, give a method for evaluating $\mu[z]$ to within ϵ.*

Problem (B): Given $z \in C_R$ and $\epsilon > 0$, give a method for constructing functions $f \in C_x$ and $g \in C_y$ such that

$$\| z - f - g \| \leq \mu[z] + \epsilon.$$

It is our purpose in the present note to solve these problems and to establish certain generalizations.

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*Actually, this differs somewhat from the problem as formulated by RAND, which was: Given z and δ, give a method for determining whether $\mu[z] < \delta$. This is in all probability unsolvable when $\mu[z] = \delta$, since any computation of $\mu[z]$ which can be carried out in a finite number of steps will, in general, yield only an approximation.

2. The role of the minimizing sequence. We shall now define a few terms by means of which our procedure can be outlined conveniently.

We shall say that two functions z and $\bar{z}$ in C_R are *equivalent* if $z - \bar{z} \in C_S$, and shall denote the equivalence of z and $\bar{z}$ by $z \sim \bar{z}$. Clearly, $z \sim \bar{z}$ implies $\mu[z] = \mu[\bar{z}]$.

According to the definition of $\mu[z]$, there exists a sequence of functions $\{w_i\}$, $w_i \in C_S$, such that

$$\|z - w_i\| \longrightarrow \mu[z] .$$

Let us define $z_i = z - w_i$; then $z_i \sim z$ and $\|z_i\| \longrightarrow \mu[z]$. We shall call a sequence $\{z_i\}$, $z_i \in C_R$, a *minimizing sequence for* z if $z_i \sim z$ for all i and $\|z_i\| \longrightarrow \mu[z]$.

Clearly, both of the proposed problems will be solved once one has constructed a minimizing sequence.*

We shall introduce a "leveling process," which when applied to z and then iterated will produce a sequence of functions $\{z_i\}$ with the properties (1) $z_i \sim z$ and (2) $\|z_i\| \geq \|z_{i+1}\|$ for all i. Properties (1) and (2) imply

$$\lim_{i \to \infty} \|z_i\| = M \geq \mu[z] .$$

That $M = \mu[z]$, that is, that our "leveling sequence" is in fact a minimizing sequence, is the principal result of this paper.

This will be established by a "pincers" argument to obtain convergence—$\mu[z]$ is simultaneously approximated from above and below: For each path in the class of admissible paths L (defined below) we shall define a functional $\pi_l[z]$, over C_R, with the important property $\pi_l[z] \leq \mu[z]$. Let

$$\sup_{l \in L} |\pi_l[z]| = \pi[z] .$$

Then

$$\pi[z] \leq \mu[z] \leq M .$$

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*Given a sequence of real numbers $a_i \to a$, let us call the integer-valued function $N(\epsilon)$ of the real variable ϵ, defined for $\epsilon \geq 0$, a *modulus of convergence* for the sequence a_i, if $i \geq N(\epsilon)$ implies $|a_i - a| < \epsilon$.

While a method for constructing a minimizing sequence answers the questions, the finiteness of the procedure is satisfactory only when one has an estimate for the modulus of convergence. This will be discussed at the end of this paper.

Our proof is accomplished by showing that $\pi[z] = M$, thus implying also that $\pi[z] = \mu[z] = M$.

3. The main theorem. We shall say that a closed polygonal line is *permissible* if it lies entirely within the square $0 \le x \le 1$, $0 \le y \le 1$, and if each of its sides is parallel either to the x- or to the y-axis.

We enumerate the vertices of a permissible line by (x_j, y_j), $j = 1, 2, \cdots$, where

$$x_{2k-1} = x_{2k} , \quad y_{2k+1} = y_{2k} , \quad x_{k+2n} = x_k , \quad y_{k+2n} = y_k ; \quad k = 1, 2, \cdots .$$

To each permissible polygonal line l we can associate a functional $\pi_l[z]$ with

$$\pi_l[z] = \frac{1}{2n} \sum_{j=1}^{2n} (-1)^j z(x_j, y_j) .$$

LEMMA 1. *If $z \sim \bar{z}$, then $\pi_l[z] = \pi_l[\bar{z}]$.*

Proof. Let $\bar{z}(x, y) = z(x, y) + g(x) + h(y)$; then

$$\sum_{j=1}^{2n} (-1)^j \bar{z}(x_j, y_j) = \sum_{j=1}^{2n} (-1)^j z(x_j, y_j) + \sum_{j=1}^{2n} (-1)^j g(x_j) + \sum_{j=1}^{2n} (-1)^j h(y_j) .$$

But

$$\sum_{j=1}^{2n} (-1)^j g(x_j) = - \sum_{k=1}^{n} g(x_{2k-1}) + \sum_{k=1}^{n} g(x_{2k}) = 0 ,$$

$$\sum_{j=1}^{2n} (-1)^j h(y_j) = - \sum_{k=1}^{n} h(y_{2k+1}) + \sum_{k=1}^{n} h(y_{2k}) = 0 .$$

Hence

$$\sum_{j=1}^{2n} (-1)^j \bar{z}(x_j, y_j) = \sum_{j=1}^{2n} (-1)^j z(x_j, y_j) ;$$

that is, $\pi_l[\bar{z}] = \pi_l[z]$.

We remark that these invariants (under equivalence) $\pi_l[z]$ form a complete set of invariants. That is to say: If $\pi_l[\bar{z}] = \pi_l[z]$ for all permissible lines, then

$z \sim \bar{z}$. In fact the π_l based on rectangles alone form a complete set of invariants. In order to relate μ with the π_l we prove the following result.

LEMMA 2. *The functional* $\mu[z]$ *satisfies* $\mu[z] \geq |\pi_l[z]|$ *for all permissible lines.*

Proof. If we had $\mu[z] < |\pi_l[z]|$ then there would exist a function $\bar{z} \sim z$ such that $\|\bar{z}\| < |\pi_l[z]|$ and hence:

$$|\pi_l[z]| = |\pi_l[\bar{z}]| = \frac{1}{2n} \left| \sum_{j=1}^{2n} (-1)^j \bar{z}(x_j, y_j) \right|$$

$$\leq \frac{1}{2n} \sum_{j=1}^{2n} |\bar{z}(x_j, y_j)| \leq \|\bar{z}\| < |\pi_l[z]| ,$$

which is a contradiction.

Problem A will be solved once we establish the following theorem.

THEOREM 1. *The functional* $\mu[z]$ *satisfies* $\mu[z] = \sup |\pi_l[z]| = \pi[z]$, *where the* sup *is taken over all permissible lines.*

As a preliminary to the proof of this theorem we introduce the following *leveling process*:

Given $z \in C_R$, we define the sequences of functions $z_n(x, y)$, $g_n(x)$, $h_n(y)$ ($n = 1, 2, \cdots$) by the relations:

$$z_1 = z , \quad z_{2n} = z_{2n-1} - g_n , \quad z_{2n+1} = z_{2n} - h_n ,$$

$$g_n(x) = \frac{1}{2} \left[\max_{0 \leq y \leq 1} z_{2n-1}(x, y) + \min_{0 \leq y \leq 1} z_{2n-1}(x, y) \right] ,$$

$$h_n(y) = \frac{1}{2} \left[\max_{0 \leq x \leq 1} z_{2n}(x, y) + \min_{0 \leq x \leq 1} z_{2n}(x, y) \right] .$$

It is obvious that $z_n \sim z (n = 1, 2, \cdots)$. The passage from z_{2n-1} to z_{2n} reduces $\|z_{2n-1}\|$ by the maximal amount by which it can be reduced through the subtraction of a function of x, while the passage from z_{2n} to z_{2n+1} reduces $\|z_{2n}\|$ by the maximal amount by which it can be reduced through the subtraction of a function of y. Thus, if we let

$$M_n[z] = \|z_n\| ,$$

then the M_n form a nonincreasing sequence of nonnegative numbers, so that $\lim_{n \to \infty} M_n[z] = M[z]$ exists. We have the following obvious result.

LEMMA 3. *The functional $\mu[z]$ satisfies $\mu[z] \leq M[z]$.*

Our solution of problem B will be a consequence of the following theorem.

THEOREM 2. *The functional $\mu[z]$ satisfies $\mu[z] = M[z]$.*

This, incidentally, will establish the fact that the functional $M[z]$ is invariant under equivalence. The direct proof of this fact might prove somewhat cumbersome.

Keeping in mind the results of Lemmas 2 and 3 we see that both Theorems 1 and 2 are consequences of the following result.

THEOREM 3. *The functional $\pi[z]$ satisfies $\pi[z] = M[z]$.*

Proof. We shall call a function z *horizontally level* if

$$\max_{0 \leq x \leq 1} z(x,y) = - \min_{0 \leq x \leq 1} z(x,y)$$

for $0 \leq y \leq 1$, and we shall call it *vertically level* if

$$\max_{0 \leq y \leq 1} z(x,y) = - \min_{0 \leq y \leq 1} z(x,y)$$

for $0 \leq x \leq 1$. For the sake of brevity we shall use the symbol M instead of $M[z]$.

There exists a number N such that $M_{2N} < M + \epsilon$, where ϵ is a small positive number which is to be further determined later. We now perform the next $2n$ steps of the leveling process on the function z_{2N}.

There exists a point (x_1, y_1) such that

$$z_{2N+2n}(x_1, y_1) = M_{2N+2n} = M + \delta, \qquad\qquad \delta \geq 0;$$

and since z_{2N+2n} is vertically level there exists a point (x_2, y_2) with $x_2 = x_1$ such that

$$z_{2N+2n}(x_2, y_2) = -M - \delta.$$

Hence we have

$$z_{2N+2n-1}(x_1, y_1) - g_{N+n}(x_1) = M + \delta,$$

$$z_{2N+2n-1}(x_2, y_2) - g_{N+n}(x_2) = -M - \delta;$$

and since $M_{2N+2n-1} < M + \epsilon$ and $x_1 = x_2$ this implies

$$M + \epsilon - g_{N+n}(x_1) > M + \delta \quad \text{or} \quad g_{N+n}(x_1) < \epsilon - \delta \, ,$$

$$-M - \epsilon - g_{N+n}(x_1) < -M - \delta \quad \text{or} \quad g_{N+n}(x_1) > \delta - \epsilon \, .$$

We therefore have certainly

$$- \epsilon < g_{N+n}(x_1) < \epsilon \, .$$

Thus

$$z_{2N+2n-1}(x_1, y_1) = M + \delta + g_{N+n}(x_1) > M + \delta - \epsilon \, ,$$

$$z_{2N+2n-1}(x_2, y_2) = -M - \delta + g_{N+n}(x_1) < -M - \delta + \epsilon \, ;$$

and since $z_{2N+2n-1}$ is horizontally level there exists a point (x_3, y_3) with $y_3 = y_2$ such that

$$z_{2N+2n-1}(x_3, y_3) > M + \delta - \epsilon \, .$$

By the same process as we applied to $g_{N+n}(x_1)$ we can now show that

$$-2\epsilon < h_{N+n-1}(y_1) < 2\epsilon \, ,$$

$$-2\epsilon < h_{N+n-1}(y_2) < 2\epsilon \, ;$$

hence

$$z_{2N+2n-2}(x_1, y_1) > M + \delta - 3\epsilon \, ,$$

$$z_{2N+2n-2}(x_2, y_2) < -M - \delta + 3\epsilon \, ,$$

$$z_{2N+2n-2}(x_3, y_3) > M + \delta - 3\epsilon \, ;$$

further, because $z_{2N+2n-2}$ is vertically level, there exists a point (x_4, y_4) with $x_4 = x_3$ such that

$$z_{2N+2n-3}(x_4, x_4) < -M - \delta + 3\epsilon \, .$$

Repeating this process $2n$ times we finally obtain a sequence of points (x_1, y_1), $\cdots, (x_{2n+1}, y_{2n+1})$, such that $x_{2k} = x_{2k-1}$, $y_{2k+1} = y_{2k}$ $(k = 1, \cdots, n)$ and

$$z_{2N}(x_{2k-1}, y_{2k-1}) > M + \delta - (2^{2n-1} - 1)\epsilon \qquad\qquad (k = 1, \cdots, n + 1),$$

$$z_{2N}(x_{2k}, y_{2k}) < -M - \delta + (2^{2n-1} - 1)\epsilon \qquad\qquad (k = 1, \cdots, n).$$

We complete the above sequence of points to form a permissible line by adding the point (x_{2n+2}, y_{2n+2}) with $x_{2n+2} = x_{2n+1}, y_{2n+2} = y_1$. If we construct the functional π_l associated with this permissible line then we obtain

$$|\pi_l[z]| = |\pi_l[z_{2N}]| = \frac{1}{2n+2} \left| \sum_{j=1}^{2n+2} (-1)^j z_{2N}(x_j, y_j) \right|$$

$$> \frac{2n+1}{2n+2} [M + \delta - (2^{2n-1} - 1)\epsilon] - \frac{1}{2n+2} |z_{2N}(x_{2n+2}, y_{2n+2})|$$

$$> M + \delta - (2^{2n-1} - 1)\epsilon - \frac{2}{2n+2} (M + \epsilon).$$

Since the choice of ϵ was independent of n, we can choose ϵ so that $(2^{2n-1})\epsilon = \epsilon_1/2$ where ϵ_1 is an arbitrary small positive number. At the same time we can choose n so large that

$$\frac{M + \epsilon}{n + 1} < \frac{\epsilon_1}{2}.$$

Thus we have: For every $\epsilon_1 > 0$ there exists a permissible line such that

$$|\pi_l[z]| > M[z] - \epsilon_1,$$

or, in other words,

$$\pi[z] \geq M[z].$$

In conjunction with Lemmas 2 and 3, this proves Theorem 3.

4. The discontinuous case. Examining our method of proof we can make the following observations:

(1) No essential use was made of the continuity of any of the functions $z(x, y)$, $g(x)$, $h(y)$ involved in the definition of $\mu[z]$. Specifically we may define

$$\mu^*[z] = \inf_{g,h} \sup_{0 \leq x \leq 1, 0 \leq y \leq 1} |z(x, y) - g(x) - h(y)|,$$

where z is an arbitrary (bounded) function defined for $0 \le x \le 1$, $0 \le y \le 1$ and $g(x)$, $h(y)$ are arbitrary functions defined over $0 \le x \le 1$ and $0 \le y \le 1$, respectively. The definition of $\pi[z]$ remains valid for discontinuous z, while $M[z]$ can be extended to a functional $M^*[z]$ which is defined for discontinuous (bounded) z, simply by replacing all the max and min symbols in the leveling process by sup and inf symbols respectively. With very minor modifications of the proof of Theorem 3 we then obtain the following result.

THEOREM 3*. *The functional* $\pi[z]$ *satisfies* $\pi[z] = \mu^*[z] = M^*[z]$, *where (unless we wish to allow infinite values for these functionals) z is an arbitrary bounded function.*

Theorems 3 and 3* yield the following corollary.

COROLLARY. *If z is continuous, then*

$$\mu[z] = \mu^*[z].$$

In other words, the approximation of a continuous $z(x,y)$ cannot be improved by permitting discontinuous $g(x) + h(y)$.

(2) The functions $\pi_l[z]$ are continuous functionals in our metric; more specifically, we have the following result.

LEMMA 4. *If* $\|z - \bar{z}\| < \epsilon$, *then* $|\pi_l[z] - \pi_l[\bar{z}]| < \epsilon$ *for any permissible line.*

Proof. We have

$$\pi_l[z] = \frac{1}{2n} \sum_{j=1}^{2n} (-1)^j z(x_j, y_j),$$

$$\pi_l[\bar{z}] = \frac{1}{2n} \sum_{j=1}^{2n} (-1)^j \bar{z}(x_j, y_j).$$

Hence

$$|\pi_l[z] - \pi_l[\bar{z}]| = \frac{1}{2n} \left| \sum_{j=1}^{2n} (-1)^j [z(x_j, y_j) - \bar{z}(x_j, y_j)] \right|$$

$$\le \frac{1}{2n} \sum_{j=1}^{2n} \| z - \bar{z} \| < \frac{1}{2n} \cdot 2n\epsilon = \epsilon.$$

As a consequence, $\pi[z]$ is itself a continuous functional, as expressed in the following corollary.

COROLLARY. *If* $\|z - \bar{z}\| < \epsilon$, *where* z *and* $\bar{z}$ *are arbitrary bounded functions,* *then*

$$|\mu^*[z] - \mu^*[\bar{z}]| < \epsilon .$$

It is also easy to show that the functionals $M_n^*[z]$ and $M_n^*[z]$ which arise in the leveling process are continuous in a similar sense.

5. **The n-dimensional case.** There was nothing in our treatment which demanded that z be a function of two variables only, or even that the variables be numbers. Most generally we can say:

Let $S_1, S_2, \cdots, S_k$ be arbitrary point sets, and let $z(s)$ be a bounded function defined over the Cartesian product $S = S_1 \times S_2 \times \cdots \times S_k$. Let $T_1, T_2, \cdots, T_l$ be direct subproducts of the S_i such that $T_i \nsubseteq T_j$ unless $i = j$. Let $E_1, E_2, \cdots, E_l$ be the projections of S on $T_1, T_2, \cdots, T_l$ respectively. We now define

$$\mu[z] = \inf_{f_1, \cdots, f_l} \sup_{s \in S} |z(s) - f_1(E_1 s) - \cdots - f_l(E_l s)| ,$$

where f_i ranges over all functions defined on T_i.

Our permissible lines are now replaced by a rather complicated permissible array of points $t_{i_1, i_2, \cdots, i_m}$ defined as follows:

(a) t_1 is an arbitrary point of S;

(b) to every point $t_{i_1, \cdots, i_m}$ there exist l points $t_{i_1, \cdots, i_m 1}; t_{i_1, \cdots, i_m 2}; \cdots;$ $t_{i_1, \cdots, i_m, l}$ such that $E_j t_{i_1, \cdots, i_m, j} = E_j t_{i_1, \cdots, i_m}$;

(c) $\qquad t_{i_1, \cdots, i_m, j, j} = t_{i_1, \cdots, i_m}; t_{i_1, \cdots, j, i_m, j} = t_{i_1, \cdots, j}$;

(d) the number of points in the set is finite;

(e) if $t_{i_1, \cdots, i_m} = t_{j_1, \cdots, j_n}$ then $m \equiv n \pmod{2}$.

(This last condition is not really necessary but it serves to avoid confusion.)

In order to visualize these sets it might be well to consider the case where S is three-dimensional Euclidean space; that is, S_1 is the x-axis, S_2 the y-axis, and S_3 the z-axis. If we take T_1, T_2, T_3 as the three coordinate planes then the permissible point sets consist of the vertices of closed polyhedral surfaces whose edges are parallel to the coordinate axes. If we take T_1, T_2, T_3 as the three coordinate axes then the permissible point sets consist of the vertices of closed polyhedral surfaces whose edges are parallel to the coordinate planes.

To each permissible point set p we now associate the functional

$$\pi_p[z] = \frac{1}{N} \Sigma (-1)^m \, z(t_{i_1, \cdots, i_m}) \, ,$$

where the summation is extended over all the N different points of the permissible set. If we generalize the concept of equivalence so that $z \sim \bar{z}$ whenever

$$z(s) - \bar{z}(s) = f_1(E_1 s) + \cdots + f_l(E_l s) \, ,$$

then $\pi_l[\, z \,]$ is seen to be invariant under equivalence.

The leveling process consists in the construction of the sequences

$$z_n(s) , \; f_1^{(n)}(E_1 s) , \cdots , f_l^{(n)}(E_l s)$$

according to the following rules:

$$z_1(s) = z(s) , \quad z_{m l + j}(s) = z_{m l + j - 1}(s) - f_j^{(m)}(E_j \, s) \, ,$$

$$f_j^{(m)}(E_j \, s) = \frac{1}{2} \left[\sup_{E_j t = E_j s} z_{m l + j - 1}(t) + \inf_{E_j t = E_j s} z_{m l + j - 1}(t) \right]$$

$$(j = 1, \cdots, l \, ; n = 0, 1, \cdots) \, .$$

We can again define the nonincreasing sequence of nonnegative functionals

$$M_n[z] = \sup_{s \in S} |z_n(s)|$$

and

$$M[z] = \lim_{n \to \infty} M_n[z] \, .$$

All the above lemmas and theorems remain valid under these new definitions; and the proofs, while more difficult to state, contain essentially no new ideas. Probably the greatest deviation from the above proofs takes place in the construction of the permissible set through the leveling process in the proof of Theorem 3. We shall therefore describe that process in greater detail.

Choose N so large that $M_{Nl+1} < M + \epsilon$, and let $M_{Nl+nl+1} = M + \delta$; then there exist two points t_1 and t_4 in S such that $E_1 t_1 = E_1 t_{1,1}$, and

$$z_{Nl+nl+1}(t_1) > M - \epsilon \, ,$$

$$z_{Nl+nl+1}(t_{1,1}) < -M + \epsilon \, .$$

and that there exist points $t_{1,2}$ and $t_{1,1,2}$ in S such that $E_2 t_1 = E_2 t_{1,2}$; $E_2 t_{1,1}$ $= E_2 t_{1,1,2}$, and

$$z_{Nl+nl}(t_{1,2}) < -M + 3\epsilon \, ,$$

$$z_{Nl+nl}(t_{1,1,2}) > M - 3\epsilon \, .$$

The next step in the leveling process adds the points $t_{1,3}$, $t_{1,1,3}$, $t_{1,2,3}$, $t_{1,1,2,3}$, and so on. After $nl + 1$ steps we have the set $t_{1,i_1,\cdots,i_m}$ ($m = 0, 1, \cdots,$ nl; $i_j \in \{1, 2, \cdots, l\}$), where

(1)
$$z_{Nl}(t_{1,i_1,\cdots,i_{2m}}) > M - (2^{nl+2} - 1)\epsilon \, ,$$

$$z_{Nl}(t_{1,i_1,\cdots,i_{2m+1}}) < -M + (2^{nl+2} - 1)\epsilon \, .$$

In order to form a permissible point set we have to adjoin additional points so that condition (b) will be satisfied. Condition (b) is already satisfied for all points $t_{1,i_1,\cdots,i_m}$ with $m \le (n-1)l$. The number of points with $(n-1)l < m \le nl$ is $l!$, and is therefore independent of n. It is easy to see that by adding a fixed finite number of points (this number A depends on k and l but not on n) we can obtain a point set which satisfies condition (b). Thus the augmented point set satisfies conditions (a), (b), (d). Since no points of the form $t_{1,i_1,\cdots,i_m,j,j}$ or $t_{1,i_1,\cdots,}$ j,i_m,j were constructed in the leveling process we can satisfy condition (c) by definition. For the nonaugmented point set, condition (c) is an immediate consequence of (1) if ϵ is sufficiently small. The augmented part can be constructed so that (c) is satisfied. We denote the nonaugmented set by p', the augmenting set by p''.

Thus we have constructed a permissible point set; if we form the corresponding functional $\pi_p[z]$, then we have

$$|\pi_p[z]| = \frac{1}{B} \left| \sum_p (-1)^m z_{Nl}(t_{1,i_1,\cdots,i_m}) \right|$$

$$\ge \frac{1}{B} \left| \sum_{p'} (-1)^m z_{Nl}(t_{1,i_1,\cdots,i_m}) \right| - \frac{1}{B} \left| \sum_{p''} z_{Nl}(t_{1,i_1,\cdots,i_m}) \right|$$

$$\ge \frac{1}{B}(B-A)[M - (2^{nl+2} - 1)\epsilon] - \frac{1}{B}A(M+\epsilon)$$

$$= M - \frac{2A}{B} M + \frac{B-A}{B} \left[(2^{nl+2} - 1) - \frac{A}{B} \right] \epsilon \, ,$$

where $B = B(n)$ is the number of points in the permissible set, so that $B(n) \longrightarrow \infty$ with n. For a suitable choice of ϵ and n we have

$$|\pi_p[z]| \geq M - \epsilon \, .$$

Hence Theorem 3 is true in this generalized case.

6. Further discussion of the leveling process. While the leveling process gave rise to a sequence of functions $z_n \sim z$ with $\lim_{n \to \infty} \|z_n\| = \mu [z]$, we were unable to show the convergence of the functions z_n. In fact, we have not yet proved the existence of a function $\bar{z} \in C_R$ with $\bar{z} \sim z$ and $\|\bar{z}\| = \mu[z]$, nor did we investigate the rate of convergence of $\|z_n\|$. It is the purpose of this section to treat the last two questions.

In order to prove the existence of $\bar{z}$ we prove the equicontinuity of the sequence $\{z_n\}$ and thus insure the existence of a convergent subsequence with a continuous limit $\bar{z}$. To this end we first prove:

LEMMA 5. *If* $f_1(x), f_2(x) \in C_x$ *and* $\|f_1 - f_2\| < \epsilon$, *then*

$$\left| \max_{0 \leq x \leq 1} f_1(x) - \max_{0 \leq x \leq 1} f_2(x) \right| < \epsilon \quad and \quad \left| \min_{0 \leq x \leq 1} f_1(x) - \min_{0 \leq x \leq 1} f_2(x) \right| < \epsilon \, .$$

Proof. Since

$$\min_{0 \leq x \leq 1} f(x) = - \max_{0 \leq x \leq 1} \left[-f(x) \right] \, ,$$

it suffices to prove the first statement. Let

$$f_1(x_1) = \max_{0 \leq x \leq 1} f_1(x) \, ;$$

then

$$f_2(x_1) > f_1(x_1) - \epsilon = \max_{0 \leq x \leq 1} f_1(x) - \epsilon \, ,$$

and hence

$$\max_{0 \leq x \leq 1} f_2(x) > \max_{0 \leq x \leq 1} f_1(x) - \epsilon \, ,$$

or

$$\max_{0 \le x \le 1} f_1(x) - \max_{0 \le x \le 1} f_2(x) < \epsilon .$$

Similarly,

$$\max_{0 \le x \le 1} f_2(x) - \max_{0 \le x \le 1} f_1(x) < \epsilon .$$

We define

$$\bar{g}_n(x) = \sum_{k=1}^{n} g_k(x) ,$$

$$\bar{h}_n(y) = \sum_{k=1}^{n} h_k(y) ,$$

so that

$$z_{2n+1} = z - \bar{g}_n - \bar{h}_n .$$

The equicontinuity of $\{z_n\}$ will be the direct consequence of the following result.

THEOREM 4. *If for fixed y and Δy we have*

$$|z(x, y) - z(x, y + \Delta y)| < \epsilon \quad for \quad 0 \le x \le 1 ,$$

then we have

$$|\bar{h}_n(y) - \bar{h}_n(y + \Delta y)| < \epsilon \qquad\qquad (n = 1, 2, \cdots) .$$

Similarly, if for fixed x and Δx we have

$$|z(x, y) - z(x + \Delta x, y)| < \epsilon \quad for \quad 0 \le y \le 1 ,$$

then we have

$$|\bar{g}_n(x) - \bar{g}_n(x + \Delta x)| < \epsilon \qquad\qquad (n = 1, 2, \cdots) .$$

Proof. It obviously suffices to prove the first part of the theorem.
Let $z^* = z(x, y) - \bar{g}_n(x)$; then

$$|z^*(x, y) - z^*(x, y + \Delta y)| = |z(x, y) - z(x, y + \Delta y)| < \epsilon \quad for \quad 0 \le x \le 1 .$$

Hence, if in Lemma 5 we let $f_1(x) = z^*(x, y)$ and $f_2(x) = z^*(x, y + \Delta y)$, we obtain

$$\left| \max_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] - \max_{0 \leq x \leq 1} [z(x, y + \Delta y) - \bar{g}_n(x)] \right| < \epsilon,$$

$$\left| \min_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] - \min_{0 \leq x \leq 1} [z(x, y + \Delta y) - \bar{g}_n(x)] \right| < \epsilon.$$

Now $z_{2n+1} = z - \bar{g}_n - \bar{h}_n$ is vertically level. Hence

$$\bar{h}_n(y) = \frac{1}{2} \left\{ \max_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] + \min_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] \right\},$$

$$\bar{h}_n(y + \Delta y) = \frac{1}{2} \left\{ \max_{0 \leq x \leq 1} [z(x, y + \Delta y] - \bar{g}_n(x)] + \min_{0 \leq x \leq 1} [z(x, y + \Delta y) - \bar{g}_n(x)] \right\},$$

or

$$|\bar{h}_n(y) - \bar{h}_n(y + \Delta y)|$$

$$\leq \frac{1}{2} \left| \max_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] - \max_{0 \leq x \leq 1} [z(x, y + \Delta y) - \bar{g}_n(x)] \right|$$

$$+ \frac{1}{2} \left| \min_{0 \leq x \leq 1} [z(x, y) - \bar{g}_n(x)] - \min_{0 \leq x \leq 1} [z(x, y + \Delta y) - \bar{g}_n(x)] \right| < \epsilon.$$

The discussion so far has failed to settle the questions of the rate of convergence of $\|z_n\|$ and of the convergence of z_n. We were able to obtain only partial answers. At the suggestion of the referee we omit the proofs of most of the following statements; their sequence will have to indicate our derivation.

LEMMA 6. *For $n \geq 2$ we have*

$$\| g_n \| \geq \| h_n \| \geq \| g_{n+1} \| \geq \| h_{n+1} \| > \cdots .$$

LEMMA 7. *We have also*

$$\| z_{2n-1} \| \geq - \| z_{2n+2N-1} \| + (2^{2N+1} - 2) \| g_{N+n} \|$$

.

$$- \sum_{k=1}^{N} (2^{2N-2k+1} - 1) \| g_{N+n-k} \| - \sum_{k=1}^{N} (2^{2N-2k+2} - 1) \| h_{N+n-k} \| .$$

THEOREM 5. *The norm* $\| z \|$ *satisfies*

$$\| z \| \geq N \| g_n \| - (2^{2n} - 1)(\| g_n \| - \| g_{N+n} \|) .$$

THEOREM 6. *For every* $\epsilon > 0$ *there is an* n_0 *such that for all* $n > n_0$ *we have*

$$\| g_n \| < (2 + \epsilon) \| z \| / \log_2 n .$$

COROLLARY. *The following relations hold:*

$$\lim_{n \to \infty} \| g_n \| = \lim_{n \to \infty} \| h_n \| = 0 .$$

DEFINITION. A function $z(x, y)$ is *level* if it is both horizontally and vertically level.

THEOREM 7. *For every* $z \in C_R$ *there is a* $\bar{z} \in C_R$ *such that* $\bar{z}$ *is level,* $\bar{z} \sim z$, *and* $\mu[z] = \| \bar{z} \|$.

Proof. According to Theorem 4, the sequence $\{ z_n \}$ has a uniformly convergent subsequence $\{ z_{n_i} \}$. Let

$$\bar{z} = \lim_{i \to \infty} z_{n_i} .$$

According to Theorem 6 we have

$$\lim_{i \to \infty} \| z_{n_i + 1} - z_{n_i} \| = 0 ;$$

hence

$$\bar{z} = \lim_{i \to \infty} z_{n_i + 1} .$$

Of the two functions z_{n_i}, $z_{n_i + 1}$ one is horizontally level, while the other is vertically level; hence the common uniform limit is level.

Since $z_{n_i} \sim z$, we have $\bar{z} \sim z$ and

$$\| \bar{z} \| = \lim_{i \to \infty} \| z_{n_i} \| = \lim_{n \to \infty} \| z_n \| = \mu[z] .$$

We remark that the bound obtained for $\|g_n\|$ in Theorem 6 .does not seem to be the best possible. In fact in all the cases we have investigated we obtained $\|g_n\| < c\, 2^{-n}$. Such an estimate would of course settle the unsolved question of the convergence of the sequence $\{z_n\}$.

Another unsettled question is that of the existence of a minimizing function $\bar{z}$ equivalent to a discontinuous function z. While Theorems 4-7 remain valid with minor modifications for discontinuous z, Theorem 4 no longer implies the existence of a convergent subsequence of $\{z_n\}$.

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