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K- AND L-KERNELS ON AN ARBITRARY RIEMANN SURFACE

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The l-kernel which was first considered by Schiffer for plane regions is extended to arbitrary open Riemann surfaces for a number of significant subspaces of the space of square integrable harmonic differentials Γ_h. The l-kernel for each of the subspaces considered is expressed in terms of the principal functions. Thus if W is an open Riemann surface and p and q the L_1 principal functions of W with singularities Re $1/z$ and Im $1/z$ respectively, then the following result is proved.

THEOREM. **The differential $dp - dq^*$ is an l-kernel for the space Γ_h.**

The l-kernel and another kernel function called the k-kernel by Schiffer are applied to the solution of some well known extremal problems on open Riemann surfaces.

It should be noted here that these problems have also been considered by G. Weill [9]. Finally, the properties of the kernel functions are used to obtain a test for when a surface is of class 0_{AD}.

M. Schiffer in [7] defined the k- and l-kernels for plane regions G. The k-kernel reproduces the value of every square integrable analytic function on G at a prescribed point while the l-kernel is orthogonal to the space of square integrable analytic functions on G with Dirichlet metric. Schiffer showed that these kernel functions can be expressed in terms of the Green's function thus enabling one to actually construct them for a given region.

An important question is whether the k- and l-kernels can be generalized to arbitrary open Riemann surfaces and, if so, whether they can be expressed in terms of functions depending only on the surface as in the case of place regions. We shall answer this question in the affirmative for a number of significant subspaces of the space of square integrable harmonic differentials. In addition, we shall see that these generalized k- and l-kernels have important extremal properties.

1. Principal functions

2. In this section we shall briefly review certain results on principal functions (see cf. [1] pp. 148–186) that will be needed later on.

Let W be the interior of a compact bordered Riemann surface $\bar{W}$

with border β. Consider a regularly imbedded boundary neighborhood $W' \subset W$ with compact complement and with relative boundary α, and a continuous function f on α. Then $L_0 f$ is defined to be the harmonic function on W' with boundary values f on α and with vanishing normal derivative on β. $L_0 f$ is unique.

Let P be a partition of the contours of W. That is, $\beta = \cup \beta_i$, where the β_i are disjoint unions of contours. We then define $(P)L_1 f$ to be the harmonic function on W' with boundary values f on α and constant values on each β_1 such that the flux across β_i vanishes. L_0 and $(P)L_1$ are called the principal operators.

Let W now be an arbitrary open Riemann surface, and W' a fixed regularly imbedded subregion $\subset W$ with compact complement and relative boundary α, negatively oriented with respect to W'. We can extend the definitions of the principal operators to W' as follows. Let $\{\Omega_n\}$, with borders $\{\alpha \cup \beta_n\}$, be a regular exhaustion of W'. $L_0(\Omega_n)$ and $(P)L_1(\Omega_n)$, the principal operators on Ω_n, are well defined and by taking the limit as Ω_n tends to W', we obtain the principal operators on W'. For brevity, we shall used L_1 for $(P)L_1$.

3. The principal functions p_0 and p_1 of an arbitrary open Riemann surface W are defined as follows.

DEFINITION. Suppose at a finite number of points $\zeta_j \in W$, there are given singularities of the form

$$\mathrm{Re} \sum_{n=1}^{\infty} b_n^{(j)}(z - \zeta_j)^{n-1} + c^{(j)} \tag{3.1}$$

where the $c^{(j)}$ are real and subject to the condition $\sum_j c^{(j)} = 0$. The principal functions p_0 and p_1 are by definition harmonic on W, except for the singularities (3.1), such that $L_0 p_0 = p_0$ and $L_1 p_1 = p_1$ in W'.

These functions are unique and independent of W', save for an additive constant (cf. [1] p. 169).

2. Harmonic differentials on Riemann surfaces

4. We shall consider here the space of square integrable harmonic differentials Γ_h and some significant subspaces of Γ_h. The main tool to be used is the general formula for partial integration which states that if f is of class C^1, and if x is a 2- chain, then

$$\int_x (df)\omega = \int_{\partial x} f\omega - \int_x f d\omega \tag{4.1}$$

holds for all differentials $\omega \in C^1$ and thus in particular for all $\omega \in \Gamma_h$.

The k-kernel of any closed subspace $\Gamma_{h'}$ of Γ_h is defined as follows.

DEFINITION. A differential ψ is said to a k-kernel of $\Gamma_{h'}$ provided

$$\psi \in \Gamma_{h'} \tag{4.2}$$

$$\text{for all } \omega \in \Gamma_{h'},\ (\omega, \psi) = \frac{\partial}{\partial x} u(\zeta) \tag{4.3}$$

where $\omega = du$ near ζ.

The l-kernel of Γ_h is defined as follows.

DEFINITION. A harmonic differential θ on $W - \{\zeta\}$ with a harmonic pole at ζ is said to be an l-kernel of Γ_h provided

(4.4) $(\omega, \theta) = 0$ for all $\omega \in \Gamma_h$ where the inner product is taken in the Cauchy sense.

The k-kernel is easily seen to be unique while the l-kernel depends on the singularity. However the following result on the l-kernel is valid.

THEOREM 1. *If θ_1 and θ_2 are two l-kernels of Γ_h with the same singularity, then $\theta_1 = \theta_2$.*

Proof. Since θ_1 and θ_2 have the same singularity, $\theta_1 - \theta_2 \in \Gamma_h$. By property 4.4, $(\theta_1 - \theta_2,\ \theta_1 - \theta_2) = 0$ and thus $\theta_1 = \theta_2$.

Consequently, we can say that the l-kernel of Γ_h is unique up to a singularity.

We shall now relate the k- and l-kernels of Γ_h to the principal functions and thus obtain constructive proofs of the existence of the k- and l-kernels.

Let W be an arbitrary open Riemann surface with ideal boundary β, and denote by r, the L_1 principal function of W with respect to the identity partition of β and for any singularity. The following auxiliary result is valid.

LEMMA 1 (Rodin [6]). If $\omega \in \Gamma_h$, then

$$\int_\beta r\omega = 0\ . \tag{5.1}$$

Proof. Let Ω denote a canonical subregion of W with border β_Ω, and r_Ω the L_1 principal function of Ω with respect to the identity partition of β_Ω. Since r_Ω is constant on β_Ω, it follows that

$$\int_{\beta_\Omega} r_\Omega \omega = 0\ . \tag{5.2}$$

By 4.1, the right hand side of the above equation is dominated by $|(dr - dr_\Omega, \bar{\omega}^*)_\Omega|$. Consequently, using Schwarz's inequality, we find that

$$(5.3) \qquad \left|\int_{\beta_\Omega} r\omega\right| \leq \| dr - dr_\Omega \|_\Omega \| \omega \|_\Omega$$

In Ahlfors and Sario [1], it is shown that $\lim_{\Omega \to w} \| dr - dr_\Omega \| = 0$. Hence the right hand side of 5.3 tends to 0 as Ω tends to W which implies the result of the lemma.

Let ζ denote an arbitrary point on W and p and q the L_1 principal functions of W with respect to the identity partition of β and singularities Re $1/z$ and Im $1/z$ respectively at ζ. We can now state the following result.

THEOREM 2. *The differential $dp - dq^*$ is an l-kernel of the Hilbert space Γ_h. Explicitly*

$$(5.4) \qquad (\omega, dp - dq^*) = 0$$

for all $\omega \in \Gamma_h$.

Proof. Let Δ denote a parametric disc with center at ζ and corresponding to $\{z : |z| \leq r < 1\}$. In Δ, $p = \mathrm{Re}(1/z) + h(z)$ and $q = \mathrm{Im}(1/z) + v(z)$ where $h(z)$ and $v(z)$ are harmonic. From the definition of the inner product, it follows that

$$(5.5) \qquad (\omega, dp - dq^*)_{W-\Delta} = \iint_{W-\Delta} (dp)\omega^* - \iint_{W-\Delta} (dq)\omega \,.$$

By 4.1 and Lemma 1, we may rewrite 5.5 as follows.

$$(5.6) \qquad (\omega, dp - dq^*)_{W-\Delta} = \int_{|z|=r} q\omega - p\omega^* \,.$$

By the linearity of the scalar product, we may assume without loss of generality that ω is real. Therefore in Δ, we can write $\omega = du$ where u is harmonic. Since Re $(1/z) = $ Im $(i/\bar{z})$ and Im $(1/z) = $ Im $(1|\bar{z})$, we obtain from 5.6, the following result.

$$(5.7) \qquad (\omega, dp - dq^*)_{W-\Delta} = \mathrm{Im} \int_{|z|=r} \frac{du + idu^*}{\bar{z}} + \int_{|z|=r} hdu^* + vdu \,.$$

On the circle $|z| = r$, $\bar{z} = r^2/z$ and therefore

$$(5.8) \qquad \mathrm{Im} \int_{|z|=r} \frac{du + idu^*}{\bar{z}} = 0 \,.$$

Since u and u^* are of bounded variation on $|z|=r$, it follows that

$$\int_{|z|=r} hdu^* + vdu = o(1) . \tag{5.9}$$

Consequently,

$$(\omega, dp - dq^*)_{W-\Delta} = o(1) . \tag{5.10}$$

Letting r tend to 0, we obtain the desired conclusion.

It was shown by Rodin [6] in his doctoral dissertation that the corresponding k-kernel of Γ_h is $-(1/2\pi)(dp + dq^*)$.

6. We shall now consider the space of square integrable harmonic exact differentials Γ_{he}. The l-kernel of Γ_{he} is defined as follows.

DEFINITION. A harmonic exact differential θ on $W - \{\zeta\}$ with a harmonic pole at ζ is said to be an l-kernel of Γ_{he} provided

(6.1) $(\omega, \theta) = 0$ for all $\omega \in \Gamma_{he}$ where the inner product is again taken in the Cauchy sense.

We shall now prove the following lemma.

LEMMA 2. *Let W be an arbitrary open Riemann surface with border β, and let p denote the L_0 principal function of W for any singularity. Then*

$$\int_{\beta} udp^* = 0 \tag{6.2}$$

for all $du \in \Gamma_{he}$.

Proof. As before Ω will denote a canonical subregion of W with border β_Ω. Since p_Ω, the L_0 principal function of Ω with the same singularity as p, has vanishing normal derivative on β_Ω, it follows that

$$\left|\int_{\beta_\Omega} udp^*\right| = \left|\int_{\beta_\Omega} u(dp^* - dp_\Omega^*)\right| \tag{6.3}$$

The rest of the proof now follows in the same manner as that of Lemma 1.

Let p_{1I_s} denote the L_1 principal function of W with respect to the identity partition and singularity $s = \mathrm{Re}\,(1/z)$, and p_{0s} denote the L_0 principal function of W with singularity s. Applying Lemma 2, we obtain in a manner similar to the proof of Theorem 2 the following result.

THEOREM 3. *The differential* $dp_{1Is} + dp_{os}$ *is an l-kernel of* Γ_{he}.

It should be noted that the l-kernel of Γ_{he} is also unique up to a singularity. This remark follows in the same manner as that of Theorem 1.

The corresponding k-kernel [6] of Γ_{he} is $-(1/2\pi)(dp_{1Is} - dp_{os})$.

Let us now turn to the space of square integrable harmonic semiexact differentials Γ_{hse}. We recall that a differential (not necessarily square integrable) is said to be harmonic semiexact on an arbitrary open Riemann surface W if it is harmonic with vanishing periods along all dividing cycles of W. This leads us to the following definition.

DEFINITION. A harmonic semiexact differential θ on $W - \{\zeta\}$ with a harmonic pole at ζ is said to be an l-kernel of Γ_{hse} provided

$$(\omega, \theta) = 0 \text{ for all } \omega \in \Gamma_{hse} . \tag{6.4}$$

By a method of proof similar to that of Lemma 1, we obtain the following result.

LEMMA 3. *If* p *denotes the* L_1 *principal function of* W *with respect to the canonical partition for any singularity, then*

$$\int_\beta p\omega = 0 \tag{6.3}$$

for all $\omega \in \Gamma_{hse}$.

Denote by p_{1Ot} the L_1 principal function of W with respect to the canonical partition and with singularity $t = \text{Im}\,(1/z)$. Applying the result of Lemma 3, we can establish in a manner similar to the proof of Theorem 2 the following.

THEOREM 4. *The differential* $(dp_{1Is} - dp^*_{1Ot})$ *is an l-kernel of* Γ_{hse}.

We again note that the l-kernel of Γ_{hse} is unique up to a singularity. The corresponding k-kernel {6] is $-(1/2\pi)(dp_{1Is} + dp^*_{1Ot})$.

7. The k-kernel of Γ_{he} can also be characterized in terms of a complete orthonormal set of square integrable harmonic exact differentials on any surface $W \notin 0_{HD}$. In fact, if $\{du_\nu\}$, $\nu = 1, 2, \cdots$, is such a complete orthonormal set, then as an immediate consequence of the Riesz Fischer theorem we obtain the following result.

THEOREM 5.

$$-\frac{1}{2\pi}(dp_{1Is}-dp_{0s})=\sum_{\nu=1}^{\infty}\frac{\partial u(\zeta)}{dx}\,du_{\nu}\,.$$

3. Analytic differentials on Riemann surfaces

8. We shall now consider some important subspaces of the space of square integrable analytic differentials Γ_a. The l-kernel of Γ_a is defined as follows.

DEFINITION. An analytic differential θ on $W-\{\zeta\}$ with a pole at ζ is said to be an l-kernel of Γ_a provided

(8.1) $(\omega,\theta)=0$ for all $\omega\in\Gamma_a$ where the inner product is taken in the Cauchy sense.

With the same notation as before, we obtain the following result.

THEOREM 6. *The differential* $dp-dq^*+i(dp^*+dq)$ *is an l-kernel for* Γ_a.

The proof is similar to that of Theorem 2. Of course, the l-kernel is again unique up to a singularity.

Rodin [6] has shown that the k-kernel of Γ_a is

$$-\frac{1}{4\pi}(dp+dq^*+i(dp^*-dq))\,.$$

Let us now consider the space of square integrable analytic semiexact differentials Γ_{ase}. If in the definition of the l-kernel of Γ_a we replace the word analytic by analytic semiexact and the space Γ_a by the space Γ_{ase}, we obtain the definition of the l-kernel of Γ_{ase}.

Denote by p_{1Qs} the L_1 principal function of W with singularity $s=\mathrm{Re}\,(1/z)$ at ζ with respect to the canonical partition. As a consequence of Lemma 3, we have the following result.

THEOREM 7. *The differential* $dp_{1Qs}-dp^*_{1Qt}+i(dp^*_{1Qs}+dp_{1Qt})$ *is an l-kernel* of Γ_{ase}.

The corresponding k-kernel [6] of Γ_{ase} is

$$-\frac{1}{4\pi}(dp_{1Qs}+dp^*_{1Qt}+i(dp^*_{1Qs}-dp_{1Qt}))\,.$$

We shall now consider the space of square integrable analytic exact differentials. If in the definition of the l-kernel of Γ_a we replace the word analytic by analytic exact and Γ_a by Γ_{ae}, we obtain the definition of the l-kernel of the space of square integrable exact differentials Γ_{ae}.

Let us now consider a planar Riemann surface W. On a planar surface every cycle is dividing and consequently $P_1 = p_{1\sigma s} + ip^*_{1\sigma s}$ and $P_0 = p_{0\sigma s} + ip^*_{0\sigma s}$ become single valued meromorphic functions. Thus we obtain the following result.

COROLLARY 1. *On a planar surface, the differential* $dP_1 + dP_0$ *is an l-kernel of* Γ_{ae}.

The corresponding k-kernel [6] is $-(1/4\pi)(dP_1 - dP_0)$.

9. The k-kernel of Γ_a can also be characterized in terms of a complete orthonormal set of square integrable analytic differentials. To be precise, if $W \notin 0_{AD}$ and if $a_\nu(z)dz, \nu = 1, 2, \cdots,$ is such a complete orthonormal set, then the following result is valid.

THEOREM 8.

$$-\frac{1}{4\pi}(dp + dq^* + i(dp^* - dq)) = \sum_{\nu=1}^{\infty} a_\nu(\zeta)a_\nu(z)dz \ .$$

The proof is similar to that of Theorem 5.

4. Extremal problems on Riemann surfaces

The properties of the k- and l-kernels of the space Γ_{ae} can be used to solve certain extremal problems for harmonic functions on an arbitrary open Riemann surface. Thus if we let $B(p) = \int_\beta dp^*$, we obtain the following result.

THEOREM 9. *The function* $(1/2)(p_{1Is} + p_{os})$ *minimizes the expression* $B(p)$ *in the class of all harmonic functions* p *with singularity* s.

Proof. Since $dp_{1Is} + dp_{os}$ is an l-kernel or Γ_{he}, it follows that

$$(9.1) \qquad 0 \leqq \left\| d\left(p - \frac{1}{2}(p_{1Is} + p_{os})\right)\right\|^2_{W-\Delta} = \| dp \|^2_{W-\Delta}$$

$$- (dp, dp_{1Is} + dp_{os})_{W-\Delta} + \frac{1}{4}\| dp_{1Is} + dp_{os} \|^2_{W-\Delta}$$

and that

$$\left\| d\left(p - \frac{1}{2}(p_{1Is} + p_{os})\right)\right\|^2_{W-\Delta} = \| dp \|^2_{W-\Delta} - \frac{1}{2}(dp, dp_{1Is} + dp_{os})_{W-\Delta} + o(1) . \tag{9.2}$$

Consequently

$$(dp, dp_{1Is} + dp_{os})_{W-\Delta} = \frac{1}{2} \| dp_{1Is} + dp_{os} \|^2_{W-\Delta} + o(1) , \tag{9.3}$$

and therefore

$$\frac{1}{4} \| dp_{1Is} + dp_{os} \|^2_{W-\Delta} \leqq \| dp \|^2_{W-\Delta} + o(1) . \tag{9.4}$$

Applying 4.1 to 9.4, we obtain the following.

$$B\left(\frac{1}{2}(p_{1Is} + p_{os})\right) - \int_{|z|=r} \frac{1}{2}(p_{1Is} + p_{os})d\left(\frac{1}{2}(p_{1Is} + p_{os})\right)^* = B(p) - \int_{|z|=r} pdp^* + o(1) . \tag{9.5}$$

Letting r tend to 0, we obtain the desired result.

Theorem 9 was first proved by Sario [9] by another method. The proof presented here is considerably shorter thus showing the power of the l-kernel concept.

We also have the following extremal property of the k-kernel of the space Γ_{he}.

THEOREM 10. *In the class of all harmonic functions u, the expression*

$$\| du \|^2 - 4\pi \frac{\partial u(\zeta)}{\partial x} \tag{9.6}$$

is minimized by the function $p_{os} - p_{1Is}$.

Proof. By the reproducing property of the k-kernel, it follows that

$$\| du \|^2 - 4\pi \frac{\partial u(\zeta)}{\partial x} - \| du - (dp_{os} - dp_{1Is}) \|^2 - \| dp_{os} - dp_{1Is} \|^2 . \tag{9.7}$$

However

$$(9.8)\quad \| du - (dp_{os} - dp_{1Is}) \|^2 - \| dp_{os} - dp_{1Is} \|^2 \\ \leqq - \| dp_{os} - dp_{1Is} \|^2 = \| dp_{os} - dp_{1Is} \|^2 - 4 \frac{\partial(p_{os} - p_{1Is})(\zeta)}{\partial x} .$$

Actually in the course of the proof we have shown more, namely that the value of the minimum is $-\| dp_{os} - dp_{1Is} \|^2$. Again we see that the proof is very easy once the reproducing property of

$$-\frac{1}{2\pi}(dp_{1Is} - dp_{os})$$

is established. For another proof of this theorem the reader is referred to [9].

10. The k-kernel of the space Γ_{ae} also possesses an extremal property with respect to the class of analytic functions. Let $dk(z, \zeta)$ denote the k-kernel of Γ_{ae}. Then the following result is valid.

THEOREM 11. *The function $4\pi k(z, \zeta)$ minimizes the expression $\| df \|^2 - 4\pi \operatorname{Re} a(f)$ in the class of all analytic functions f on W where $a(f)$ is the coefficient of $z - \zeta$ in the Taylor expansion of f about ζ. Moreover $a(4\pi k(z, \zeta))$ is nonnegative and $2\pi a(4\pi k(z, \zeta) = D(4\pi k(z, \zeta))$ where in general $D(f)$ denotes the Dirichlet integral of f.*

The proof follows from the reproducing property of the k-kernel and is similar to that of Theorem 10. The existances of the k-kernel of Γ_{ae} follows from the existence of the k-kernel of Γ_a and the orthogonal decomposition $\Gamma_a = \Gamma_{ae} + \Gamma_{as}$ where Γ_{as} denotes the space of all square integrable analytic Schottky differentials.

If we call $a(4\pi k(z, \zeta))$ the span, then by Theorem 11, we have the following result.

THEOREM 12. *An arbitrary open Riemann surface is of class 0_{AD} if and only if the span vanishes for all choices of ζ.*

The span $a(4\pi k(z, \zeta))$ is a generalization of the well known Schiffer span which was defined by him for planar surfaces.

BIBLIOGRAPHY

1. L. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton University Press, 1960.
2. N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337-404.
3. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys No. V, Amer. Math. Soc., 1950.

4. ———, *Ueber die Kernfunktion eines Bereiches und ihr Verhatten am Rande*, J. Reine Angew. Math. **169** (1930), 1-42.
5. R. Courant, *Dirichlet's Principle*, Academic Press Inc., 1950.
6. B. Rodin, *Reproducing formulas on Riemann Surfaces*, Technical Report No. 26, Contract No. DA-04-495-ORD-1959, OOR Project No. 1517.
7. M. Schiffer, *The kernel function of an orthonormal system*, Duke Math. J. **13** (1946), 529-540.
8. M. Schiffer and D. Spencer, *Functionals of Finite Riemann Surfaces*, Princeton University Press, 1954.
9. G. Weill, *Some extremal properties of linear combinations of kernels on Riemann surfaces*, Pacific J. Math. **12** (1962), 1459-1465.

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